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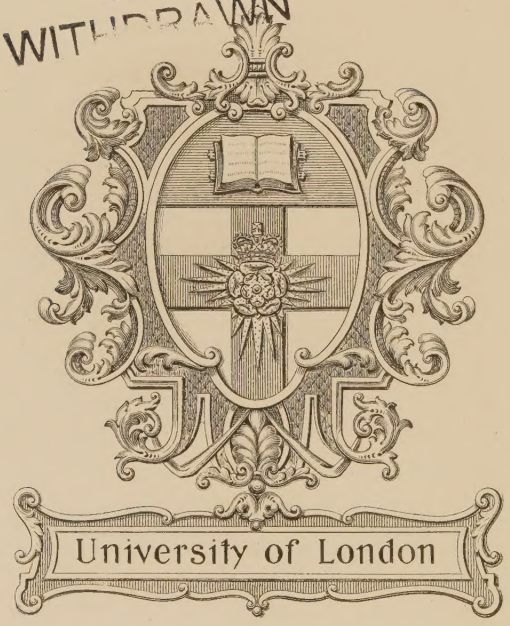

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PAPERS

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TRIGONOMETRIC INTEGRALS AND HARMONIC FUNCTIONS

By S. VERBLUNSKY.

[Received 10 February, 1933.—Read 16 February, 1933.]

1. If $$\sum_1^\infty A_n(x) \quad [A_n(x) = a_n \cos nx + b_n \sin nx] \quad \text{(i)}$$

is a Fourier series, say of the function $F(x)$, then, at a point x at which (i) is summable by Poisson's method to $F(x)$, there are certain relations of inequality† between the upper and lower limits, as $h \to 0$, of

$$\frac{F(x+h)+F(x-h)-2F(x)}{h}, \quad \frac{F(x+h)+F(x-h)-2F(x)}{h^2},$$

and the upper and lower limits respectively, as $r \to 1$, of

$$(1-r)\,P(r, x), \quad P(r, x),$$

where $$P(r, x) = -\sum_1^\infty n^2 A_n(x) r^n.$$

If we wish to find an analogous result for trigonometric integrals‡, and one which shall be applicable to a wide class of cases, we should not be misled by the verbal analogy between the terms " Fourier series " and

† S. Verblunsky, *Proc. London Math. Soc.* (2), 34 (1933), 441–456, Theorems I, II.

‡ The first investigation of this problem is due to M. E. Grimshaw, *Proc. London Math. Soc.* (2), 29 (1929), 243–256.

"Fourier integral". The class of integrals which can be called Fourier integrals (in any specified sense) has really no natural connection with the theorems which we seek to prove. The "right" class of integrals (which, I believe, is here introduced for the first time) is the class of integrals

$$\int_0^\infty \{a(t)\cos tx+b(t)\sin tx\}\,dt \tag{ii}$$

such that there is a function $f(x)$ for which

$$\int_0^\infty e^{-\eta t}\{a(t)\cos tx+b(t)\sin tx\}\,dt=\frac{1}{\pi}\int_{-\infty}^\infty \frac{\eta}{\eta^2+(x-t)^2}f(t)\,dt \tag{iii}$$

for all $\eta>0$. The first member in (iii) is the "Poisson function" of (ii), just as

$$\sum_1^\infty A_n(x)r^n \tag{iv}$$

is the Poisson function of (i); and the second member of (iii) is the analogue of

$$\frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\,\frac{1-r^2}{1-2r\cos(t-x)+r^2}\,dt,$$

which, when (i) is the Fourier series of $F(x)$, is equal to (iv).

In Theorems I and II we establish the desired relations between the upper and lower limits, as $h\to 0$, of

$$\frac{f(x+h)+f(x-h)-2f(x)}{h},\quad \frac{f(x+h)+f(x-h)-2f(x)}{h^2},$$

and the upper and lower limits respectively, as $\eta\to 0$, of

$$\eta P(\eta,x),\quad P(\eta,x),$$

where

$$P(\eta,x)=-\int_0^\infty e^{-\eta t}\{t^2a(t)\cos tx+t^2b(t)\sin tx\}\,dt.$$

We then apply these theorems to prove that for certain types of integrals (ii), when the upper and lower Poisson limits satisfy certain conditions, there is a function $f(x)$ say, such that, p.p. in t (*i.e.* almost everywhere in t), $t>0$,

$$a(t)=\frac{1}{\pi}\int_{-\infty}^\infty f(x)\cos xt\,dt,\quad b(t)=\frac{1}{\pi}\int_{-\infty}^\infty f(x)\sin xt\,dt, \tag{v}$$

where the integrals are taken in a conventional sense, which will actually be $(C, 1)$ or $(C, 2)$, *i.e.* as

$$\lim_{\omega\to\infty}\frac{1}{\omega}\int_0^\omega d\lambda\int_{-\lambda}^{\lambda}dt \quad \text{or} \quad \lim_{\omega\to\infty}\frac{2}{\omega^2}\int_0^\omega d\lambda\int_0^\lambda d\mu\int_{-\mu}^{\mu}dt.$$

Some restrictions other than those on the Poisson limits are requisite; namely, restrictions on $a(t)$, $b(t)$. Now, as regards the Poisson limits, it is unnecessary to attempt the widest possible generality in the enunciations, because the arguments involving them are the same as for trigonometric series†. We therefore content ourselves with a fairly general condition, namely that $\bar{P}(x)$ and $\underline{P}(x)$ are finite except at an enumerable set at which $\lim \eta P(\eta, x) = 0$, and integrable in every finite interval.

Concerning the restrictions on $a(t)$, $b(t)$, we show that any one of the following is sufficient.

(*a*) $a(t)/t^2$, $b(t)/t^2$ absolutely integrable in a neighbourhood of $t=\infty$.

(*b*) $|a(t)/t^2|^p$, $|b(t)/t^2|^p$ $(1<p\leqslant 2)$ integrable in a neighbourhood of $t=\infty$ and

$$\lim_{\eta\to 0}\int_X^\infty e^{-\eta t}\{a(t)\cos tx+b(t)\sin tx\}\,dt \tag{vi}$$

continuous. [Here X is some positive number which, without loss of generality, may be taken equal to 1. The neighbourhoods of $t=\infty$ in (*a*) and (*b*) and in all that follows will be taken to be $(1, \infty)$.]

(*c*) $$\int_\mu^{\mu+1}|a(t)|^p\,dt = o(\mu^p), \qquad \int_\mu^{\mu+1}|b(t)|^p\,dt = o(\mu^p) \quad (1<p\leqslant 2)$$

as $\mu\to\infty$.

(*d*) $$\int_1^\infty e^{-\eta t}\{a(t)\cos tx+b(t)\sin tx\}/t^2\,dt$$

bounded for $\eta>0$ and (vi) continuous.

The condition (*d*) is of peculiar interest. Consider the function

$$J(x, y)=\int_0^\infty e^{-yt}\{A(t)\cos tx+B(t)\sin tx\}\,dt \quad (y>0).$$

This is a harmonic function in $y>0$. Let it be bounded. We know that (iv) is a harmonic function in $r<1$, and that, if it is bounded, then (i) is

† S. Verblunsky, *Proc. London Math. Soc.* (2), 34 (1933), 457–491.

the Fourier series of a bounded function. No one will suppose that

$$\int_0^\infty \{A(t)\cos tx + B(t)\sin tx\}\,dt \qquad \text{(vii)}$$

is a Fourier integral, because, if $f(x)$ is bounded in $(-\infty, \infty)$, it does not follow that the integrals in (v) have a meaning. Nevertheless, if J is bounded, (vii) actually is a Fourier integral. This beautiful result is established by Theorem VII.

The problem arises of deciding, from a knowledge of $A(t)$, $B(t)$, when J is bounded. By a conformal representation of $y > 0$ on $r < 1$, J gives rise to a harmonic function K. Under wide conditions (in particular if A and B eventually become less in modulus than some power of t) the Fourier constants of K can be calculated. They are of the form

$$\alpha_n = \int_0^\infty e^{-t} A(t)\, p_n(t)\,dt, \qquad \beta_n = \int_0^\infty e^{-t} B(t)\, p_n(t)\,dt,$$

where $p_n(t)$ is a known polynomial of degree n in t. We thus reduce the problem to the corresponding one for trigonometric series. This corresponding problem admits of an elegant solution. The criterion given in Theorem XII can be deduced from the known theorems of Carathéodory and Fejér† on harmonic functions. An independent proof is given here; some merit is claimed for the striking enunciation, which I believe is new.

We further show that the problem of deciding when J is bounded and its limit continuous can be reduced to the analogous problem for trigonometric series, and that this analogous problem can be solved.

We finally consider the case in which $J(x, y) > 0$ for $y > 0$. When (iv) is positive in $r < 1$, it is easy to see that there is a non-decreasing function $g(x)$ such that

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx\, dg(x), \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx\, dg(x).$$

When the integral is positive, we show that there is a bounded non-decreasing function $G(x)$ such that p.p. in $t > 0$,

$$A(t) = \lim_{h\to 0} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-h|x|}\cos tx(1+x^2)\,dG(x)$$

† C. Carathéodory and L. Fejér, *Rend. di Palermo*, 32 (1911), 218–239 (226).

with a corresponding expression for $B(t)$, provided that $A(t) = o(t)$ and $B(t) = o(t)$, as $t \to \infty$, or else provided that $A(t)/(1+t^2)$ and $B(t)/(1+t^2)$ are integrable in $(0, \infty)$.

We assume as known (what is easy to prove) that, if $f(x)$ is defined in $(-\infty, \infty)$, and is absolutely integrable in every finite interval, and if

$$\lim_{M, N \to \infty} \int_{-M}^{N} \frac{f(t)}{1+t^2}\, dt$$

exists, then

$$\lim_{h \to 0} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{h}{h^2+(t-x)^2} f(t)\, dt = f(x) \quad (h > 0)$$

p.p. in x. It should be noticed that $f(t)/(1+t^2)$ is not required to be absolutely integrable in $(-\infty, \infty)$.

2. Let $a(t)$, $b(t)$ be two functions defined in $0 \leqslant t < \infty$, absolutely integrable in every finite interval in which they are defined, and such that $e^{-\eta t} a(t)$, $e^{-\eta t} b(t)$ are absolutely integrable in $(0, \infty)$ for all $\eta > 0$. The symbol

$$\int_0^{\infty} A(t, x)\, dt \quad [A(t, x) = a(t) \cos tx + b(t) \sin tx] \qquad (1)$$

will be called a trigonometric integral. For all given expressions of the form (1) which occur in this paper, we shall suppose the above conditions to be satisfied.

A function $F(x)$ defined in $(-\infty, \infty)$ will be said to be of class A if $F(x)/(1+x^2)$ is absolutely integrable in every finite interval, and integrable in $(-\infty, \infty)$. We say that (1) possesses the property (F), if there is an $\alpha > 0$ and an $F_\alpha(x)$ of class A, such that

$$-\int_\alpha^{\infty} e^{-\eta t} \frac{A(t, x)}{t^2}\, dt = \frac{1}{\pi} \int_{-\infty}^{\infty} F_\alpha(x+t) \frac{\eta}{\eta^2+t^2}\, dt \qquad (2)$$

for all $\eta > 0$. If (2) holds for some $\alpha > 0$, then it holds for all $\alpha > 0$. For let $\beta > 0$, and

$$\phi(x) = -\int_\beta^{\alpha} \frac{A(t, x)}{t^2}\, dt. \qquad (3)$$

Consider the functions

$$J(x, \eta) = -\int_\beta^{\alpha} e^{-\eta t} \frac{A(t, x)}{t^2}\, dt, \qquad K(x, \eta) = \frac{1}{\pi} \int_{-\infty}^{\infty} \phi(x+t) \frac{\eta}{\eta^2+t^2}\, dt.$$

We have

$$K = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\eta}{\eta^2+t^2}\,dt\int_{\beta}^{\alpha}\frac{a(y)\cos(t+x)y+b(y)\sin(t+x)y}{y^2}\,dy$$

$$= -\int_{\beta}^{\alpha}\frac{a(y)}{y^2}\,dy\,\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\eta}{\eta^2+t^2}\cos(t+x)y\,dt$$

$$-\int_{\beta}^{\alpha}\frac{b(y)}{y^2}\,dy\,\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\eta}{\eta^2+t^2}\sin(t+x)y\,dt. \quad (4)$$

We easily find (by the theory of residues) that

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\eta}{\eta^2+t^2}\,e^{i(t+x)y}\,dt = e^{-\eta y}e^{ixy}.$$

Hence (4) gives $J = K$. By adding J to the first member of (2) and K to the second member, we obtain the desired relation

$$-\int_{\beta}^{\infty}e^{-\eta t}\frac{A(t,x)}{t^2}\,dt = \frac{1}{\pi}\int_{-\infty}^{\infty}F_{\beta}(x+t)\frac{\eta}{\eta^2+t^2}\,dt$$

with $$F_{\beta}(x) = F_{\alpha}(x)+\phi(x).$$

This function belongs to class A since $F_{\alpha}(x)$ does so and $\phi(x)$ is bounded.

If the functions $a(t)$, $b(t)$ satisfy the further conditions that $a(t)/t^2$, $b(t)/t^2$ are absolutely integrable in every finite interval in which they are defined, and if there is an $F(x)$ of class A such that

$$-\int_{0}^{\infty}e^{-\eta t}\frac{A(t,x)}{t^2}\,dt = \frac{1}{\pi}\int_{-\infty}^{\infty}F(x+t)\frac{\eta}{\eta^2+t^2}\,dt \quad (5)$$

for all $\eta > 0$, then we say that (1) satisfies the condition (F^*). By the preceding argument, if (1) satisfies the condition (F^*), then it satisfies the condition (F).

We write

$$\Delta^2 F(x,h) = F(x+h)+F(x-h)-2F(x),$$

$$\bar{D}^2 F(x) = \overline{\lim_{h\to 0}}\,\Delta^2 F(x,h)/h^2, \quad \underline{D}^2 F(x) = \underline{\lim_{h\to 0}}\,\Delta^2 F(x,h)/h^2,$$

and $$D^2 F(x) = \lim \Delta^2 F(x,h)/h^2$$

when the limit exists. Further, we write

$$P(\eta,x) = \int_{0}^{\infty}e^{-\eta t}A(t,x)\,dt \quad (\eta > 0),$$

$$\bar{P}(x) = \overline{\lim_{\eta\to 0}}\,P(\eta,x), \quad \underline{P}(x) = \underline{\lim_{\eta\to 0}}\,P(\eta,x),$$

and $$P(x) = \lim_{\eta \to 0} P(\eta, x)$$

when the limit exists.

We shall use the following results. If $f(t)$ is absolutely integrable in $(-\infty, \infty)$, then

$$\int_0^\infty e^{-\eta t}\{\alpha(t)\cos tx + \beta(t)\sin tx\}\,dt = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t+x)\,P dt, \tag{6}$$

where

$$\alpha(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(y)\cos ty\,dy, \quad \beta(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(y)\sin ty\,dy,$$

$$P = \frac{\eta}{\delta}, \quad \delta = \eta^2 + t^2.$$

There are two positive constants c and C such that

$$c\eta^2 < \delta < C\eta^2 \quad (0 \leqslant t \leqslant \eta),$$

$$ct^2 < \delta < Ct^2 \quad (\eta \leqslant t < \infty).$$

3. Let a, b, and $\lambda > 0$ be arbitrarily assigned numbers. Consider the function

$$F(x) = ae^{-\lambda|x|} + b|x|e^{-\lambda|x|}. \tag{7}$$

Since $F(x)$ is absolutely integrable in $(-\infty, \infty)$, the formula (6) is applicable. This gives us the relation (5) with

$$-\frac{a(t)}{t^2} = \frac{2}{\pi}\left[\frac{a\lambda}{\lambda^2+t^2} - b\frac{\partial}{\partial\lambda}\left(\frac{\lambda}{\lambda^2+t^2}\right)\right], \quad -\frac{b(t)}{t^2} = 0.$$

We have

$$F(0) = a, \quad \lim \Delta^2 F(0, h)/h = -2a\lambda + 2b = A, \tag{8}$$

and, by suitable choice of a and b, the numbers a and A may be given any assigned finite values. Further,

$$\pi \lim \eta P(\eta, 0) = \pi \lim_{\eta \to 0} \eta \int_0^\infty e^{-\eta t} a(t)\,dt$$

$$= \lim \eta \int_{-\infty}^{\infty} F(t)\frac{d^2 P}{dt^2}\,dt. \tag{9}$$

Given $\epsilon > 0$, there is, by (8), a $\xi > 0$ such that

$$2\{F(t) - F(0)\}/t = A + \theta(t) \quad (0 < t \leqslant \xi)$$

with $|\theta| < \epsilon$. The expression (9) is manifestly equal to

$$\lim \eta \int_{-\xi}^{\xi} F(t)\, P''\, dt = \lim \eta \int_{0}^{\xi} 2\{F(t) - F(0)\}\, P''\, dt$$

$$= \lim \eta \int_{0}^{\xi} \{A + \theta(t)\}\, tP''\, dt.$$

Now
$$\lim \int_{0}^{\xi} \eta t P''\, dt = 1,$$

and, since
$$P'' = -\frac{2\eta}{\delta^2} + \frac{8\eta t^2}{\delta^2},$$

we have

$$\left| \eta \int_{0}^{\xi} \theta(t)\, tP''\, dt \right| < 8\epsilon \int_{0}^{\infty} \left(\frac{\eta^2 t}{\delta^2} + \frac{\eta^2 t^3}{\delta^3} \right) dt < K\epsilon,$$

where K is an absolute constant. Since ϵ is arbitrary, it follows that

$$\pi \lim \eta P(\eta, 0) = \lim \Delta^2 F(0, h)/h.$$

THEOREM I. *If* (1) *satisfies the condition* (F^*), *then, at a point* x *at which*

$$\lim_{\eta \to 0} - \int_{0}^{\infty} e^{-\eta t} \frac{A(t, x)}{t^2}\, dt = F(x),$$

we have

$$\overline{\lim_{h \to 0}}\, \Delta^2 F(x, h)/h \geqslant \pi \underline{\lim_{\eta \to 0}}\, \eta P(\eta, x), \quad \underline{\lim_{h \to 0}}\, \Delta^2 F(x, h)/h \leqslant \pi \overline{\lim_{\eta \to 0}}\, \eta P(\eta, x).$$

We may suppose that $x = 0$ and that $F(x)$ is even. It is sufficient to prove the second inequality and to suppose that $\underline{\lim}\, \Delta^2 F(0, h)/h > -\infty$. We must therefore show that $\underline{\lim}\, 2\{F(h) - F(0)\}/h > k$ implies that

$$\pi\, \overline{\lim}\, \eta P(\eta, 0) \geqslant k,$$

where k is a finite number. By adding a suitable function (7), we may suppose that $F(0) = 0$, and we need only show that

$$\underline{\lim}\, 2F(h)/h > 0 \tag{10}$$

implies that
$$\overline{\lim}\, \eta P(\eta, 0) \geqslant 0.$$

Let
$$A(\eta) = \frac{1}{\pi} \int_{-\infty}^{\infty} F(t)\, P dt.$$

Then
$$P(\eta, 0) = -\frac{d^2 A}{d\eta^2}.$$

If now $$\overline{\lim}\,\eta P(\eta, 0) < 0,$$

there is a $K > 0$ such that

$$\frac{d^2 A}{d\eta^2} > \frac{K}{\eta}$$

for sufficiently small π. Hence

$$\frac{dA}{d\eta} \to -\infty$$

as $\eta \to 0$, and so A increases strictly as η diminishes to zero. Since by hypothesis $F(0) = 0$, we have $\lim A(\eta) = 0$, and so $A(\eta) < 0$ for all sufficiently small η. Thus

$$\frac{A(\eta)}{\eta} = \frac{2}{\pi}\int_0^\infty F(t)\,\frac{dt}{\eta^2+t^2} < 0$$

for such η. This, however, is impossible. For, by (10), there is a $d > 0$ and a $\xi > 0$ such that

$$F(t) > dt \quad (0 < t \leqslant \xi).$$

Then $$\int_0^\infty F(t)\,\frac{dt}{\eta^2+t^2} > d\int_0^\xi \frac{t\,dt}{\eta^2+t^2} - \left|\int_\xi^\infty \frac{F\,dt}{\eta^2+t^2}\right|,$$

which tends to $+\infty$ as $\eta \to 0$.

4. Let a, b, and $\lambda > 0$ be arbitrarily assigned numbers. Consider the function

$$F(x) = ae^{-\lambda x^2} + bx^2 e^{-\lambda x^2}. \tag{11}$$

Since $F(x)$ is absolutely integrable in $(-\infty, \infty)$, the formula (6) is applicable. This gives us the relation (5) with

$$-\frac{a(t)}{t^2} = \frac{1}{\pi}\left[a\sqrt{\left(\frac{\pi}{\lambda}\right)}e^{-t^2/4\lambda} - b\,\frac{\partial}{\partial\lambda}\left\{\sqrt{\left(\frac{\pi}{\lambda}\right)}e^{-t^2/4\lambda}\right\}\right], \quad -\frac{b(t)}{t^2} = 0.$$

We have $$F(0) = a, \quad D^2 F(0) = -2a\lambda + 2b = A, \tag{12}$$

and, by suitable choice of a and b, the numbers a and A may be given any assigned finite values. Further,

$$\lim P(\eta, 0) = \lim \int_0^\infty e^{-\eta t} a(t)\,dt$$

$$= \lim \frac{1}{\pi}\int_{-\infty}^\infty F(t)\,\frac{d^2 P}{dt^2}\,dt. \tag{13}$$

Given $\epsilon > 0$, there is, by (12), a $\xi > 0$ such that

$$\frac{2\{F(t)-F(0)\}}{t^2} = A+\theta(t) \quad (0 < t \leqslant \xi)$$

with $|\theta| < \epsilon$. Then (13) is equal to

$$\lim \frac{1}{\pi}\int_{-\xi}^{\xi} F(t)\,P''\,dt = \lim \frac{1}{\pi}\int_0^{\xi} 2\{F(t)-F(0)\}\,P''\,dt$$

$$= \lim \frac{1}{\pi}\int_0^{\xi} \{A+\theta(t)\}\,t^2\,P''\,dt.$$

Now
$$\lim \frac{1}{\pi}\int_0^{\xi} t^2\,P''\,dt = 1,$$

and
$$\left|\int_0^{\xi} \theta(t)\,t^2\,P''\,dt\right| < 8\epsilon\int_0^{\infty}\left(\frac{\eta t^2}{\delta^2}+\frac{\eta^2 t^4}{\delta^3}\right)dt < K\epsilon,$$

where K is an absolute constant. Since ϵ is arbitrary, it follows that

$$\lim_{\eta\to 0} P(\eta,\,0) = D^2\,F(0).$$

THEOREM II. *If* (1) *satisfies the condition* (F^*), *then, at a point* x *at which*

$$\lim_{\eta\to 0}\int_0^{\infty} e^{-\eta t}\,\frac{A(t,\,x)}{t^2}\,dt = F(x),$$

we have
$$\bar{D}^2\,F(x) \geqslant \underline{P}\,(x), \quad \underline{D}^2\,F(x) \leqslant \bar{P}(x).$$

We may suppose that $x = 0$ and that $F(x)$ is even. It is sufficient to prove the second inequality and to suppose that $\underline{D}^2\,F(0) > -\infty$. We need, therefore, only show that $\underline{\lim}\, 2\{F(h)-F(0)\}/h^2 > k$ implies that $\bar{P}(0) \geqslant k$, where k is a finite number. By adding a suitable function (11), we may suppose that $F(0) = 0$, and we need only show that

$$\underline{\lim}\, 2F(h)/h^2 > 0 \tag{14}$$

implies that $\bar{P}(0) \geqslant 0$.

If now $\bar{P}(0) < 0$, we have

$$\frac{d^2 A}{d\eta^2} > 0 \quad (0 < \eta \leqslant \eta_0). \tag{15}$$

Since $A(0) = 0$, we have

$$\frac{A(\eta)}{\eta} = A'(\zeta) \quad (0 < \zeta < \eta).$$

Hence
$$\frac{A(\eta)}{\eta}-\frac{A(\zeta)}{\zeta}=A'(\zeta)-A'(\xi) \quad (0<\xi<\zeta<\eta)$$
$$>0,$$

by (15), for $\eta \leqslant \eta_0$. The theorem will now follow if we show that

$$\underline{\lim}\,\frac{d}{d\eta}\left(\frac{A(\eta)}{\eta}\right)<0. \tag{16}$$

We have
$$\frac{A(\eta)}{\eta}=\frac{1}{\pi}\int_{-\infty}^{\infty}F(t)\,\frac{dt}{\delta}=\frac{2}{\pi}\int_{0}^{\infty}F(t)\,\frac{dt}{\delta}$$

and
$$\frac{d}{d\eta}\left(\frac{A(\eta)}{\eta}\right)=-\frac{4}{\pi}\int_{0}^{\infty}F(t)\,\frac{\eta\,dt}{\delta^2}.$$

In virtue of (14), there is a $d>0$ and a $\xi>0$ such that

$$F(t)>dt^2 \quad (0<t\leqslant\xi).$$

Then
$$-\int_0^{\infty}F(t)\,\frac{\eta}{\delta^2}\,dt<-d\int_0^{\xi}\frac{\eta t^2}{\delta^2}+\eta\left|\int_{\xi}^{\infty}\frac{F(t)}{\delta^2}\,dt\right|,$$

and the limit of this expression is negative. This proves (16).

THEOREM III. *Under the conditions of the preceding theorem, $\bar{P}(x)$, $\underline{P}(x)$ lie between the upper and lower limits, as $h\to 0$, of*

$$\frac{3}{2h^3}\int_0^h[\Delta^2F(x,h)-\Delta^2F(x,u)]\,du.$$

We may suppose that $x=0$, $F(0)=0$, and that $F(x)$ is even. If we write

$$\phi(t)=F(t)-\frac{1}{t}\int_0^t F(u)\,du,$$

then it is sufficient to prove that

$$\underline{P}(0)\geqslant\underline{\lim}\,3\phi(t)/t^2.$$

We have
$$P(\eta,0)=\frac{1}{\pi}\int_{-\infty}^{\infty}F(t)\,P''\,dt$$
$$=\frac{2}{\pi}\int_0^{\infty}F(t)\left[\frac{8\eta t^2}{\delta^3}-\frac{2\eta}{\delta^2}\right]dt.$$

Let
$$\Phi(t)=\int_0^t F(u)\,du.$$

Since F belongs to class A, we have $\Phi(t) = O(t^2)$ as $t \to \infty$. Then

$$-\int_0^\infty F(t)\,\frac{2\eta}{\delta^2}\,dt = -\Big[\frac{\Phi(t)\,2\eta}{\delta^2}\Big]_0^\infty - \int_0^\infty \Phi(t)\,\frac{8\eta t}{\delta^3}\,dt,$$

and the integrated term is zero. Hence

$$\underline{P}(0) = \varliminf \frac{2}{\pi}\int_0^\infty \{F(t) - \Phi(t)/t\}\,\frac{8\eta t^2}{\delta^3}\,dt$$

$$\geqslant \varliminf \frac{\phi(t)}{t^2} \lim_{\eta\to 0} \frac{2}{\pi}\int_0^\infty \frac{8\eta t^4}{\delta^3}\,dt,$$

and the value of the second factor is 3.

COROLLARY 1. *If $D^2 F(x)$ exists, then $P(x) = D^2 F(x)$.*

COROLLARY 2. *If, in the neighbourhood of the point x, $F(t)$ is absolutely continuous and*

$$F(t) = \int^t f(u)\,du,$$

then $\overline{P}(x)$, $\underline{P}(x)$ lie between the upper and lower limits, as $h \to 0$, of

$$[f(x+h) - f(x-h)]/2h.$$

For

$$\int_0^h [\Delta^2 F(x, h) - \Delta^2 F(x, u)]\,du = \int_0^h u\{f(x+u) - f(x-u)\}\,du.$$

5. LEMMA 1. *If $A(t)$, $B(t)$ are absolutely integrable in $(0, \infty)$, then*

$$-\int_0^\infty e^{-\eta t}\{A(t)\cos xt + B(t)\sin xt\}\,dt = \frac{1}{\pi}\int_{-\infty}^\infty F(x+t)\,P\,dt, \qquad (17)$$

where (i) *$F(x)$ is bounded, continuous, and tends to zero as $x \to \pm\infty$; and*

$$\text{(ii)} \quad -A(t) = \frac{1}{\pi}\int_{-\infty}^\infty F(x)\cos xt\,dt, \qquad -B(t) = \frac{1}{\pi}\int_{-\infty}^\infty F(x)\sin xt\,dt \quad (C, 1) \qquad (18)$$

p.p. in $t > 0$.

The expression

$$-\int_0^\infty \{A(t)\cos xt + B(t)\sin xt\}\,dt$$

is bounded and continuous, and it tends to zero as $x \to \pm\infty$. We denote its value by $F(x)$. Then

$$\frac{1}{\pi}\int_{-\infty}^{\infty} F(x+t)\,P\,dt$$

$$= -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\eta}{\eta^2+t^2}\,dt \int_0^{\infty} \{A(y)\cos(x+t)y + B(y)\sin(x+t)y\}\,dy,$$

and (17) follows on inverting the order of integration as in § 2. It remains then to prove that the equations (18) hold for almost all $t > 0$. We have

$$F(x)+F(-x) = -2\int_0^{\infty} A(t)\cos xt\,dt. \tag{19}$$

Since $A(t)$ is absolutely integrable in $(0, \infty)$, it is equal p.p. in that interval to the sum $(C, 1)$ of its Fourier integral. Thus, by (19),

$$-A(t) = \frac{1}{\pi}\int_0^{\infty} \{F(x)+F(-x)\}\cos xt\,dx \quad (C, 1)$$

p.p. in $t > 0$. This is the first equation in (18), and the second is proved similarly.

THEOREM IV. *Let $a(t)/t^2$, $b(t)/t^2$ be absolutely integrable in $(1, \infty)$. Let $\bar{P}(x)$, $\underline{P}(x)$ be integrable in every finite interval, and finite except at the points of an enumerable set E at which*

$$\lim \eta\, P(\eta, x) = 0. \tag{20}$$

Then $\bar{P}(x) = \underline{P}(x) = f(x)$ p.p., and

$$a(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos tx\,dx, \quad b(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin tx\,dx \quad (C, 1) \tag{21}$$

for almost all $t > 0$.

Write

$$\phi(x) = \int_0^1 \{a(t)\cos xt + b(t)\sin xt\}\,dt,$$

$$\phi(\eta, x) = \int_0^1 e^{-\eta t} A(t, x)\,dt.$$

Then $\phi(x)$ is bounded and continuous, and it tends to zero as $x \to \pm\infty$. Let

$$p(\eta, x) = P(\eta, x) - \phi(\eta, x),$$

and let $\bar{p}(x), \underline{p}(x)$ denote the upper and lower limits of $p(\eta, x)$ as $\eta \to 0$. Then $\bar{p}(x), \underline{p}(x)$ are integrable in every finite interval, and finite except at the set E at which

$$\lim \eta p(\eta, x) = 0. \tag{22}$$

Let

$$\begin{aligned} a(t) &= a(t) \quad (t \geqslant 1) \\ &= 0 \qquad (0 \leqslant t < 1), \end{aligned}$$

and let $\beta(t)$ be defined in a similar way. Then $a(t)/t^2$, $\beta(t)/t^2$ are absolutely integrable in $(0, \infty)$. By Lemma 1, we have

$$-\int_0^\infty e^{-\eta t} \left(\frac{a(t)}{t^2} \cos xt + \frac{\beta(t)}{t^2} \sin xt\right) dt = \frac{1}{\pi} \int_{-\infty}^{\infty} G(x+t)\, P\, dt, \tag{23}$$

where (i) $G(x)$ is bounded and continuous, and tends to zero as $x \to \pm\infty$;

(ii) p.p. in $t > 0$,

$$-\frac{a(t)}{t^2} = \frac{1}{\pi} \int_{-\infty}^{\infty} G(x) \cos tx\, dx, \quad -\frac{\beta(t)}{t^2} = \frac{1}{\pi} \int_{\infty}^{\infty} G(x) \sin tx\, dx \quad (C, 1). \tag{24}$$

The trigonometric integral

$$\int_0^\infty \{a(t) \cos xt + \beta(t) \sin xt\}\, dt$$

satisfies the condition (F^*); and the first member of (23) has the limit $G(x)$ as $\eta \to 0$, since this is true of the second member. By Theorem II,

$$\bar{D}^2 G(x) \geqslant \underline{p}(x), \quad \underline{D}^2 G(x) \leqslant \bar{p}(x). \tag{25}$$

At the points of E we have, by Theorem I,

$$\overline{\lim}\, \Delta^2 G(x, h)/h \geqslant 0 \geqslant \underline{\lim}\, \Delta^2 G(x, h)/h. \tag{26}$$

Writing

$$g(x) = \min \{\bar{D}^2 G(x), \bar{p}(x)\},$$

we have, by (25),

$$\bar{D}^2 G(x) \geqslant g(x) \geqslant \underline{D}^2 G(x), \quad \bar{p}(x) \geqslant g(x) \geqslant \underline{p}(x).$$

Thus $g(x)$ is finite except at the points of E, and integrable in every finite interval. Hence†

$$G(x) = \int_0^x dy \int_0^y g(t)\, dt + px + q. \tag{27}$$

† C. de la Vallée Poussin, *Bull. de l'Acad. R. de Belg.* (1912), 701–707.

If we substitute in (24), and observe that $G(x)\to 0$ as $x\to\pm\infty$, we find that, p.p. in $t>0$,

$$\alpha(t)=\frac{1}{\pi}\int_{-\infty}^{\infty}g(x)\cos tx\,dx,\quad \beta(t)=\frac{1}{\pi}\int_{-\infty}^{\infty}g(x)\sin tx\,dx\quad (C,\,1).\tag{28}$$

We now consider the integral expression $\phi(x)$. Let

$$\begin{aligned}\alpha'(t)&=\alpha(t)\quad (0<t\leqslant 1)\\ &=0\qquad (t>1),\end{aligned}$$

and let $\beta'(t)$ be defined in a similar way. Then $\alpha'(t)$, $\beta'(t)$ are absolutely integrable in $(0,\,\infty)$. Hence, by Lemma 1,

$$\alpha'(t)=\frac{1}{\pi}\int_{-\infty}^{\infty}\phi(x)\cos tx\,dx,\quad \beta'(t)=\frac{1}{\pi}\int_{-\infty}^{\infty}\phi(x)\sin tx\,dx\quad (C,\,1)\tag{29}$$

p.p. in $t>0$. By adding (28) and (29) we find the equations (21) with $f(x)=g(x)+\phi(x)$. In virtue of (27), $D^2G(x)=g(x)$ p.p. By Theorem III, Cor. 1, $\bar{p}(x)=\underline{p}(x)=g(x)$ p.p. Hence

$$\bar{P}(x)=\bar{p}(x)+\phi(x)=f(x)$$

p.p., and similarly $\underline{P}(x)=f(x)$ p.p.

6. LEMMA 2. *If* $|A(t)|^p$, $|B(t)|^p$ $(1<p\leqslant 2)$ *are integrable in* $(0,\,\infty)$, *then* (17) *holds*; *and*

(i) $|F(x)|^{p/(p-1)}$ *is integrable in* $(-\infty,\,\infty)$;

(ii) *the equations* (18) *hold p.p. in* $t>0$.

It is known, by the theory of Fourier transforms†, that

$$-\int_0^{\infty}\{A(t)\cos xt+B(t)\sin xt\}\,dt$$

is summable $(C,\,1)$ p.p. to a function $F(x)$ with the property (i); and that the equations (18) hold p.p. in $t>0$. It is further known that, if $|f|^p$ and $|g|^{p/(p-1)}$ are integrable in $(\theta,\,\infty)$, and if F and G are their cosine transforms

$$F(x)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^{\infty}f(t)\cos xt\,dt,\quad G(x)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^{\infty}g(t)\cos xt\,dt\quad (C,\,1),$$

then

$$\int_0^{\infty}fG=\int_0^{\infty}Fg.\tag{30}$$

† *Cf.* E. W. Hobson, *Functions of a real variable*, 2 (1926), 742–752.

The cosine transform of $A(t)$ is

$$-\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{F(t)+F(-t)}{2}.$$

The cosine transform of $e^{-\eta t}\cos xt$ is

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty e^{-\eta\xi}\cos x\xi\cos\xi t\,d\xi = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \left[\frac{\eta}{\eta^2+(x-t)^2}+\frac{\eta}{\eta^2+(x+t)^2}\right].$$

Hence, by (30),

$$-\int_0^\infty e^{-\eta t}\cos xt\,A(t)\,dt = \frac{1}{2\pi}\int_0^\infty \{F(t)+F(-t)\}\left[\frac{\eta}{\eta^2+(x-t)^2}+\frac{\eta}{\eta^2+(x+t)^2}\right]dt.$$

By similar arguments about the sine transforms, we obtain

$$-\int_0^\infty e^{-\eta t}\sin xt\,B(t)\,dt = \frac{1}{2\pi}\int_0^\infty \{F(t)-F(-t)\}\left[\frac{\eta}{\eta^2+(x-t)^2}-\frac{\eta}{\eta^2+(x+t)^2}\,dt.\right]$$

By addition, we obtain the required relation (17).

THEOREM V. *Let* $|a(t)/t^2|^p$, $|b(t)/t^2|^p$ $(1<p\leqslant 2)$ *be integrable in* $(1,\infty)$, *and let*

$$\lim -\int_1^\infty e^{-\eta t}A(t,x)/t^2\,dt \tag{31}$$

be continuous. Let $\bar{P}(x)$, $\underline{P}(x)$ *be integrable in every finite interval, and finite except at the points of an enumerable set* E *at which* (20) *holds. Then*

$$a(t)=\frac{1}{\pi}\int_{-\infty}^\infty f(x)\cos tx\,dx,\quad b(t)=\frac{1}{\pi}\int_{-\infty}^\infty f(x)\sin tx\,dx \quad (C,\,2)$$

p.p. in $t>0$.

We use the notation of the preceding theorem. Since by hypothesis $G(x)$ is continuous, the equation (27) holds. Since, by Lemma 2, $|G|^{p/(p-1)}$ is integrable in $(-\infty,\infty)$, we have

$$\frac{1}{u}\int_{-u}^u |G(x)|\,dx\to 0 \tag{32}$$

as $u\to\infty$. We then find, by substituting (27) in (24) and taking account of (32), that the equations (28) hold $(C,\,2)$ p.p. in $t>0$. The equations (29) hold $(C,\,1)$, and therefore $(C,\,2)$, p.p. in $t>0$. By addition, we obtain the desired result.

7. THEOREM VI. *Let p be a number which satisfies $1 < p \leqslant 2$, and let*

$$\int_{\mu}^{\mu+1} |a(t)|^p \, dt = o(\mu^p), \quad \int_{\mu}^{\mu+1} |b(t)|^p \, dt = o(\mu^p)$$

as $\mu \to \infty$. If $\bar{P}(x)$, $\underline{P}(x)$ are integrable in every finite interval, and finite except at the points of an enumerable set E at which (20) *holds, then the conclusion of the preceding theorem is true.*

We use the notation of §§ 5, 6. The functions $|a(t)/t^2|^p$, $|b(t)/t^2|^p$ are integrable in a neighbourhood of $t = \infty$; this neighbourhood we may take to be $(1, \infty)$. If we show that (31) is continuous, the result will follow by Theorem V. We have, by Lemma 2, the relation

$$-\int_1^{\infty} e^{-\eta t} A(t, x)/t^2 \, dt = \frac{1}{\pi} \int_{-\infty}^{\infty} G(t+x) \, P \, dt, \tag{33}$$

where $|G|^{p/(p-1)}$ is integrable in $(-\infty, \infty)$. From this we infer that

$$\eta \int_1^{\infty} e^{-\eta t} \left(\frac{a(t)}{t} \sin xt - \frac{b(t)}{t} \cos xt \right) dt = -\frac{1}{\pi} \int_{-\infty}^{\infty} G(t+x) \, \frac{dP}{dt} \, dt.$$

Since
$$\int_{\mu}^{\mu+1} |a(t)| \, dt = o(\mu), \quad \int_{\mu}^{\mu+1} |b(t)| \, dt = o(\mu) \tag{34}$$

as $\mu \to \infty$, it follows that the limit of the first member is zero. But the second member tends, as $\eta \to 0$, to $(1/\pi)\{G(x+0) - G(x-0)\}$, provided that this expression has a meaning. Hence $G(x)$ cannot have an ordinary discontinuity.

Since $\bar{p}(x)$ and $\underline{p}(x)$ are finite except at the points of an enumerable set, given any perfect set Π, there is a† portion ϖ of Π and a constant $K = K(\varpi)$ such that

$$|p(\eta, x)| \leqslant K \quad (x \subset \varpi, \ \eta > 0).$$

Then

$$\left| \int_1^{\infty} e^{-\eta t} A(t, x)/t \, dt - \int_1^{\infty} e^{-\eta' t} A(t, x)/t \, dt \right| = \left| \int_{\eta}^{\eta'} p(\eta, x) \, d\eta \right|$$

$$\leqslant K |\eta' - \eta| \quad (x \subset \varpi).$$

Hence
$$\lim_{\eta \to 0} \int_1^{\infty} e^{-\eta t} A(t, x)/t \, dt = \pi(x)$$

† *Cf.* S. Verblunsky, *Proc. London Math. Soc.* (2), 34 (1933), 457–491 (460, Lemma 8).

is continuous on ϖ; applying the preceding argument to

$$\pi(\eta, x) = \int_1^\infty e^{-\eta t} A(t, x)/t \, dt,$$

we see that ϖ, and therefore Π, contains a portion on which the limit of the first member of (33) is continuous. Further, at any point x,

$$p(\eta, x) = o\left(\frac{1}{\eta}\right), \quad \pi(\eta, x) = o\left(\log \frac{1}{\eta}\right),$$

as $\eta \to 0$, whence the first member of (33) has a finite limit, as $\eta \to 0$, for all x. We may take $G(x)$ as the value of this limit.

Thus $G(x)$ is finite, and every perfect set contains a portion on which it is continuous. In an interval (a, b), on which $G(x)$ is continuous, (27) holds. Hence an isolated discontinuity of G can only be an ordinary discontinuity, which, as we have shown, cannot occur. The points of discontinuity of G, if existent, therefore form a non-dense set H, dense in itself, and therefore everywhere dense in a non-dense perfect set Π. By what precedes, there is a portion ϖ of Π on which G is continuous. Let a, b be the extremes of ϖ, and let (a_n, b_n) be a contiguous interval. Then

$$G(x) = \int_0^x dy \int_0^y g(t)\, dt + p_n x + q_n \quad (a_n < x < b_n). \tag{35}$$

Since

$$\int_\mu^{\mu+1} \left| \frac{A(t, x)}{t^2} \right| dt = o\left(\frac{1}{\mu}\right)$$

as $\mu \to \infty$, it follows, by the analogue of Tauber's theorem, that

$$G(x) = -\int_1^\infty \frac{A(t, x)}{t^2}\, dt,$$

the integral being convergent. Let

$$H(x) = \int_1^\infty \frac{B(t, x)}{t^3}\, dt \quad [B(t, x) = b(t) \cos tx - a(t) \sin tx].$$

Then H is continuous. For $h > 0$,

$$\frac{H(x+h) - H(x)}{h} = \frac{\Delta^2 H(x, h)}{2h} + \frac{H(x+h) - H(x-h)}{2h}$$

$$= \int_1^\infty \frac{B}{t} \frac{\sin^2 th}{t^2 h}\, dt + \int_1^\infty \frac{A}{t^2} \frac{\sin th}{th}\, dt. \tag{36}$$

By (34), the limit as $h \to 0$ of the first term in (36) is zero. For the second term, given $\epsilon > 0$, choose $N > 0$ so that

$$\left| G(x) - \int_1^N \frac{A}{t^2}\, dt \right| < \epsilon, \quad \left| \int_N^{N'} \frac{A}{t^2}\, dt \right| < \epsilon \ (N' > N), \quad \int_\mu^{\mu+1} \left| \frac{A}{t^2} \right| dt < \frac{\epsilon}{\mu} \ (\mu \geqslant N).$$

Then

$$\int_1^\infty \frac{A}{t^2} \frac{\sin th}{th}\, dt = \int_1^N + \int_N^{1/h} + \int_{1/h}^\infty = G + \theta\epsilon + \frac{\sin Nh}{Nh} \int_N^{N'} \frac{A}{t^2}\, dt + \frac{\theta\epsilon}{h} \sum_0^\infty (h^{-1}+k)^{-2}$$

$$(\theta^2 < 1).$$

Hence
$$\{H(x+h) - H(x)\}/h \to G(x)$$

as $h \to 0$. Similar reasoning holds when $h < 0$ tends to zero. Thus $G(x)$ is a finite differential coefficient, and therefore cannot have an ordinary discontinuity on one side. By (35), $G(a_n) = G(a_n + 0)$, $G(b_n) = G(b_n - 0)$, and so (35) holds for $a_n \leqslant x \leqslant b_n$. Thus

$$G(x) - \int_0^x dy \int_0^y g(t)\, dt$$

is linear in each closed contiguous interval of ϖ, and continuous on ϖ. It is therefore continuous in (a, b). So, then, is G. But H, if existent, is everywhere dense in ϖ. Hence H is non-existent, G is continuous, and this proves the theorem.

8. THEOREM VII. *If*

$$J(\eta, x) = \int_0^\infty e^{-\eta t} \{A(t) \cos tx + B(t) \sin tx\}\, dt \quad (\eta > 0) \tag{37}$$

is bounded in (η, x), *then*

$$\lim_{\eta \to 0} J(\eta, x) = F(x) \tag{38}$$

exists p.p., and

$$J(\eta, x) = \frac{1}{\pi} \int_{-\infty}^\infty F(t+x)\, P\, dt. \tag{39}$$

Further, for almost all $t > 0$,

$$A(t) = \frac{1}{\pi} \int_{-\infty}^\infty F(x) \cos tx\, dx, \quad B(t) = \frac{1}{\pi} \int_{-\infty}^\infty F(x) \sin tx\, dx \quad (C, 1). \tag{40}$$

We require the following lemma, due to Fatou†.

LEMMA 3. *If*

$$K(r, \theta) = \tfrac{1}{2}a_0 + \sum_1^\infty (a_n \cos n\theta + \beta_n \sin n\theta) r^n$$

is a bounded harmonic function in $r < 1$, *then there is a set* C *of* θ, *of measure* 2π, *with the following property. If* $\theta_0 \subset C$, *then* $\lim_{r\to 1} K(r, \theta_0) = K(\theta_0)$ *exists, and if* (r, θ) *tends to* $(1, \theta_0)$ *along a non-tangential path, then* $K(r, \theta) \to K(\theta_0)$.

By a non-tangential path, we mean a path which lies within a sector S whose vertex is θ_0, which has the radius ending in $(1, \theta_0)$ as an axis of symmetry, and whose angle is less than π.

The function $J(\eta, x)$ is a bounded harmonic function in $\eta > 0$. Let $= x + i\eta$. The conformal transformation

$$re^{i\theta} = -\frac{\zeta - i}{\zeta + i}$$

represents $\eta > 0$ on $r < 1$ and transforms J into a bounded harmonic function $K(r, \theta)$. The correspondence of the two domains is given by

$$x = \tan \tfrac{1}{2}\theta. \tag{41}$$

Thus to a set of x on $\eta = 0$ of measure zero, there corresponds a set of θ on $r = 1$ of measure zero, and conversely.

Let C be the set of θ of the lemma. Let C' be the corresponding set of x. Let $x_0 \subset C'$, and let θ_0 be the corresponding value of θ. The path η, x_0 $(\eta \to 0)$ corresponds to a path P_0 which ends in $(1, \theta_0)$ and which is non-tangential (it meets the unit circle normally). Since

$$\lim_{(P_0)} K(r, \theta) = K(\theta_0),$$

we have $\lim_{\eta\to 0} J(\eta, x_0) = K(\theta_0)$. We define $F(x)$ for almost all x by $F(x) = K(\theta)$, where x and θ correspond by (41). This proves (38).

The second member of (39) is a bounded harmonic function in $\eta > 0$ (since F is bounded), and

$$\lim_{\eta\to 0} \frac{1}{\pi} \int_{-\infty}^{\infty} F(t+x)\, P\, dt = F(x)$$

† P. Fatou, *Acta. Math.* 30 (1906), 335–400 (357); L. Tonelli, *Serie trigonometriche* (1928), 390.

p.p. in x. Hence

$$H(\eta, x) = J(\eta, x) - \frac{1}{\pi}\int_{-\infty}^{\infty} F(t+x)\, P\, dt$$

is a bounded harmonic function in $\eta > 0$ such that

$$\lim_{\eta \to 0} H(\eta, x) = 0 \tag{42}$$

p.p. in x. By the above conformal representation, we can write

$$H(\eta, x) = M(r, \theta),$$

where M is a bounded harmonic function in $r < 1$. Let C again denote the set of θ, of measure 2π, relative to M, whose existence is asserted by Lemma 3. Let K be the set of x for which (42) holds, and let K' be the set of θ which corresponds to K by (41). If $\theta_0 \subset K'C$, then $x_0 \subset K$. Let P_0' be the path η, x_0 $(\eta \to 0)$. Then, along the corresponding path P_0 in the (r, θ) plane (which is non-tangential), $M(r, \theta) \to 0$. But, since $\theta_0 \subset C$, $M(r, \theta) \to 0$ along any non-tangential path which ends in $(1, \theta_0)$, in particular

$$\lim_{r \to 1} M(r, \theta_0) = 0.$$

Since this is true for all $\theta_0 \subset K'C$, *i.e.* for almost all θ, and since M is bounded, it follows that $M \equiv 0$, and so $H \equiv 0$. This proves (39).

For a fixed η, $e^{-\eta t} A(t)$, $e^{-\eta t} B(t)$ are absolutely integrable in $(0, \infty)$. Hence, p.p. in $t > 0$, they are the sums $(C, 1)$ of their Fourier integrals; *i.e.*, by (39),

$$e^{-\eta t} A(t) = \frac{1}{\pi^2}\int_{-\infty}^{\infty} \cos tx\, dx \int_{-\infty}^{\infty} F(\xi)\, \frac{\eta}{\eta^2 + (\xi - x)^2}\, d\xi \quad (C, 1)$$

p.p. in $t > 0$, with a similar relation for $B(t)$. Consider a $t > 0$ for which the preceding equation is true. Then

$$e^{-\eta t} A(t) = \frac{1}{\pi^2} \lim_{\omega \to \infty} \frac{1}{\omega}\int_0^{\omega} d\lambda \int_{-\lambda}^{\lambda} \cos tx\, dx \int_{-\infty}^{\infty} F(\xi)\, \frac{\eta}{\eta^2 + (\xi - x)^2}\, d\xi. \tag{43}$$

We have

$$\int_{-\lambda}^{\lambda} \cos tx\, dx \int_{-\infty}^{\infty} F(\xi)\, \frac{\eta}{\eta^2 + (\xi - x)^2}\, d\xi = \int_{-\infty}^{\infty} F(\xi)\, d\xi \int_{-\lambda}^{\lambda} \cos tx\, \frac{\eta}{\eta^2 + (\xi - x)^2}\, dx. \tag{44}$$

Write

$$I = \int_{-\lambda}^{\lambda} \cos tx\, \frac{\eta}{\eta^2 + (\xi - x)^2}\, dx.$$

Then

$$\int_{-\infty}^{\infty} F(\xi)\, I\, d\xi = \int_{\lambda}^{\infty} \{F(\xi)+F(-\xi)\}\, I\, d\xi + \int_{-\lambda}^{\lambda} F(\xi) \left[\left(\int_{-\infty}^{\infty} - \int_{\lambda}^{\infty} - \int_{-\infty}^{-\lambda} \right) \cos tx \, \frac{\eta}{\eta^2+(\xi-x)^2}\, dx \right] d\xi.$$

But $$\int_{-\infty}^{\infty} \cos tx \, \frac{\eta}{\eta^2+(\xi-x)^2}\, dx = \pi \cos t\xi \, e^{-t\eta}.$$

Hence

$$\int_{-\infty}^{\infty} F(\xi)\, I\, d\xi = \pi \int_{-\lambda}^{\lambda} F(\xi) \cos t\xi \, e^{-t\eta}\, d\xi + \int_{\lambda}^{\infty} \{F(\xi)+F(-\xi)\}\, I\, d\xi - \int_{-\lambda}^{\lambda} F(\xi) \left[\left(\int_{\lambda}^{\infty} + \int_{-\infty}^{-\lambda} \right) \cos tx \, \frac{\eta}{\eta^2+(\xi-x)^2}\, dx \right] d\xi$$

$$= \pi \int_{-\lambda}^{\lambda} F(\xi) \cos t\xi \, e^{-t\eta}\, d\xi + J_0.$$

Thus (43) becomes

$$e^{-\eta t} A(t) = \frac{1}{\pi} \lim_{\omega\to\infty} \left[\frac{1}{\omega} \int_0^{\omega} d\lambda \int_{-\lambda}^{\lambda} F(\xi) \cos t\xi e^{-t\eta}\, d\xi + \frac{1}{\pi\omega} \int_0^{\omega} J_0\, d\lambda \right],$$

and, to establish the first equation in (40), all that we need to prove is that

$$\lim_{\eta\to 0} \overline{\lim_{\omega\to\infty}} \left| \frac{1}{\omega} \int_0^{\omega} J_0\, d\lambda \right| = 0,$$

which, since $$\overline{\lim_{\omega\to\infty}} \left| \frac{1}{\omega} \int_0^{\omega} J_0\, d\lambda \right| \leqslant \overline{\lim_{\lambda\to\infty}} |J_0|,$$

will follow if we show that

$$\lim_{\eta\to 0} \overline{\lim_{\lambda\to\infty}} |J_0| = 0. \tag{45}$$

We have

$$J_0 = \int_{\lambda}^{\infty} \{F(\xi)+F(-\xi)\}\, I\, d\xi - \int_{-\lambda}^{\lambda} F(\xi)\, d\xi \int_{\lambda}^{\infty} \cos tx \, \frac{\eta}{\eta^2+(\xi-x)^2}\, dx - \int_{-\lambda}^{\lambda} F(\xi)\, d\xi \int_{-\infty}^{-\lambda} \cos tx \, \frac{\eta}{\eta^2+(\xi-x)^2}\, dx$$

$$= J_1 - J_2 - J_3.$$

The function F is bounded. We denote by K an absolute constant which may be different at different occurrences. Then

$$|J_1| < K \int_\lambda^\infty |I|\, d\xi,$$

and

$$|I| < \left| \int_{-\lambda}^{0} \cos tx \, \frac{\eta}{\eta^2+(\xi-x)^2}\, dx \right| + \left| \int_0^\lambda \cos tx \, \frac{\eta}{\eta^2+(\xi-x)^2}\, dx \right| = |I_1|+|I_2|.$$

By an obvious change of variable, we have

$$|I_1| = \left| \int_0^\lambda \cos tx \, \frac{\eta}{\eta^2+(\xi+x)^2}\, dx \right| < \frac{2}{t} \, \frac{\eta}{\eta^2+\xi^2},$$

since the factor of $\cos tx$ is a decreasing function of x. Hence

$$\int_\lambda^\infty |I_1|\, d\xi < \frac{2}{t} \int_\lambda^\infty \frac{\eta}{\eta^2+\xi^2}\, d\xi$$

$$< \frac{2}{t} \left[\frac{\pi}{2} - \tan^{-1} \frac{\lambda}{\eta} \right],$$

which tends to zero as $\lambda \to \infty$. Next,

$$\int_\lambda^\infty |I_2|\, d\xi = \int_\lambda^{\lambda+\epsilon} + \int_{\lambda+\epsilon}^\infty \quad (\epsilon > 0)$$

$$= H_1 + H_2.$$

We have $$H_2 = \int_{\lambda+\epsilon}^\infty d\xi \left| \int_0^\lambda \cos tx \, \frac{\eta}{\eta^2+(\xi-x)^2}\, dx \right|.$$

For a fixed ξ in $(\lambda+\epsilon, \infty)$, the factor of $\cos tx$ is an increasing function of x. Hence

$$H_2 < \int_{\lambda+\epsilon}^\infty d\xi \, \frac{2}{t} \, \frac{\eta}{\eta^2+(\xi-\lambda)^2}$$

$$< \frac{2}{t} \int_\epsilon^\infty \frac{\eta}{\eta^2+\xi^2}\, d\xi$$

$$< \frac{2}{t} \left[\frac{\pi}{2} - \tan^{-1} \frac{\epsilon}{\eta} \right].$$

The operator $\lim_{\eta\to 0}\overline{\lim_{\lambda\to\infty}}$ annihilates this term. Further,

$$H_1=\int_\lambda^{\lambda+\epsilon}d\xi\left|\int_0^\lambda \cos tx\,\frac{\eta}{\eta^2+(\xi-x)^2}\,dx\right|$$

$$<\int_\lambda^{\lambda+\epsilon}d\xi\int_0^\lambda\frac{\eta}{\eta^2+(\xi-x)^2}\,dx$$

$$<\epsilon\max_{\lambda\leqslant\xi\leqslant\lambda+\epsilon}\int_0^\lambda\frac{\eta}{\eta^2+(\xi-x)^2}\,dx.$$

For any ξ,

$$\int_0^\lambda\frac{\eta}{\eta^2+(\xi-x)^2}\,dx<\int_{-\infty}^{\infty}\frac{\eta}{\eta^2+(\xi-x)^2}\,dx=\pi.$$

Thus $H_1<\epsilon\pi$. Since ϵ is arbitrary, and H_1+H_2 is independent of ϵ, it follows that H_1+H_2 is annihilated by the operator $\lim_{\eta\to 0}\overline{\lim_{\lambda\to\infty}}$. We have thus disposed of J_1.

We have

$$J_2=\int_0^\lambda F(\xi)\,d\xi\int_\lambda^\infty\cos tx\,\frac{\eta}{\eta^2+(\xi-x)^2}\,dx+\int_0^\lambda F(-\xi)\,d\xi\int_\lambda^\infty\cos tx\,\frac{\eta}{\eta^2+(\xi+x)^2}\,dx,$$

$$J_3=\int_0^\lambda F(-\xi)\,d\xi\int_\lambda^\infty\cos tx\,\frac{\eta}{\eta^2+(\xi-x)^2}\,dx+\int_0^\lambda F(\xi)\,d\xi\int_\lambda^\infty\cos tx\,\frac{\eta}{\eta^2+(\xi+x)^2}\,dx.$$

Thus $$J_2+J_3=L_1+L_2$$

with $$L_1=\int_0^\lambda\{F(\xi)+F(-\xi)\}\int_\lambda^\infty\cos tx\,\frac{\eta}{\eta^2+(\xi-x)^2}\,dx,$$

$$L_2=\int_0^\lambda\{F(\xi)+F(-\xi)\}\int_\lambda^\infty\cos tx\,\frac{\eta}{\eta^2+(\xi+x)^2}\,dx.$$

We have $$|L_1|<K\int_0^\lambda|A|\,d\xi$$

with

$$|A|=\left|\int_\lambda^\infty\cos tx\,\frac{\eta}{\eta^2+(x-\xi)^2}\,dx\right|\leqslant\left|\int_\lambda^{\lambda+\epsilon}\right|+\left|\int_{\lambda+\epsilon}^\infty\right|=|A_1|+|A_2|.$$

Now $$\int_0^\lambda|A_2|\,d\xi=\int_0^\lambda d\xi\left|\int_{\lambda+\epsilon}^\infty\cos tx\,\frac{\eta}{\eta^2+(x-\xi)^2}\,dx\right|.$$

For a fixed ξ in $(0,\lambda)$, the coefficient of $\cos tx$ is a decreasing function.

Hence the preceding expression does not exceed

$$\int_0^\lambda d\xi \, \frac{2}{t} \, \frac{\eta}{\eta^2+(\lambda+\epsilon-\xi)^2} = \frac{2}{t} \int_\epsilon^{\lambda+\epsilon} \frac{\eta}{\eta^2+\xi^2} \, d\xi$$

$$= \frac{2}{t} \left[\tan^{-1} \frac{\lambda+\epsilon}{\eta} - \tan^{-1} \frac{\epsilon}{\eta} \right].$$

The operator $\lim_{\eta \to 0} \overline{\lim_{\lambda \to \infty}}$ annihilates this expression. Further,

$$\int_0^\lambda |A_1| \, d\xi < \int_0^\lambda d\xi \int_\lambda^{\lambda+\epsilon} \frac{\eta}{\eta^2+(x-\xi)^2} \, dx. \tag{46}$$

The interior integral is less than ϵ times the maximum of the integrand in $\lambda \leqslant x \leqslant \lambda+\epsilon$; this maximum is attained for $x = \lambda$. Hence (46) does not exceed

$$\epsilon \int_0^\lambda \frac{\eta d\xi}{\eta^2+(\lambda-\xi)^2} < \epsilon \int_{-\infty}^{\infty} \frac{\eta d\xi}{\eta^2+(\lambda-\xi)^2} = \epsilon\pi.$$

Since ϵ is arbitrary, and L_1 is independent of ϵ, we have

$$\lim_{\eta \to 0} \overline{\lim_{\lambda \to \infty}} \, |L_1| = 0.$$

Finally,

$$|L_2| < K \int_0^\lambda d\xi \left| \int_\lambda^\infty \cos tx \, \frac{\eta}{\eta^2+(\xi+x)^2} \, dx \right|$$

$$< K \int_0^\lambda d\xi \, \frac{2}{t} \, \frac{\eta}{\eta^2+(\xi+\lambda)^2}$$

$$< \frac{2K}{t} \int_0^\infty \frac{\eta}{\eta^2+(\xi+\lambda)^2} \, d\xi$$

$$< \frac{2K}{t} \left[\frac{\pi}{2} - \tan^{-1} \frac{\lambda}{\eta} \right],$$

which is annihilated by $\overline{\lim_{\lambda \to \infty}}$. This completes the proof of the first equation in (40), and the other is proved similarly.

9. THEOREM VIII. *Let*

$$J(\eta, x) = -\int_1^\infty e^{-\eta t} \frac{A(t, x)}{t^2} \, dt \quad (\eta > 0)$$

be bounded, and let $G(x) = \lim_{\eta \to 0} J(\eta, x)$ *exist for all* x *and be continuous. Let*

$\bar{P}(x)$, $\underline{P}(x)$ be integrable in every finite interval, and finite except at the points of an enumerable set E at which (20) *holds. Then $\bar{P}(x) = \underline{P}(x) = f(x)$ p.p., and, p.p. in $t > 0$,*

$$a(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos tx\,dx, \qquad b(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin tx\,dx \quad (C,\,2).$$

By Theorem VII,

$$-\int_1^{\infty} e^{-\eta t}\frac{A(t,x)}{t^2}\,dt = \frac{1}{\pi}\int_{-\infty}^{\infty} G(x+t)\,P\,dt, \tag{47}$$

and, for almost all t,

$$(C,1)\ \frac{1}{\pi}\int_{-\infty}^{\infty} G(x)\cos tx\,dx = -\frac{a(t)}{t^2}, \quad (C,1)\ \frac{1}{\pi}\int_{-\infty}^{\infty} G(x)\sin tx\,dx = -\frac{b(t)}{t^2}$$

when $t > 1$; when, however, $0 < t < 1$, both of the right-hand sides must be replaced by zero.

In virtue of (47) and our hypotheses, we have, by the argument and with the notation of § 5,

$$G(x) = \int_0^x dy\int_0^y g(t)\,dt + px + q, \tag{48}$$

where $$g(x) = \min\{\bar{p}(x),\ \bar{D}^2 G(x)\}.$$

We require the following lemma.

LEMMA 4. *If* (48) *holds with a bounded G, then, at a point $t > 0$ at which the integrals*

$$\int_{-\infty}^{\infty} G(x)\cos tx\,dx, \qquad \int_{-\infty}^{\infty} G(x)\sin tx\,dx \tag{49}$$

exist $(C,\,1)$, *we have*

$$(C,\,2)\int_{-\infty}^{\infty} g(x)\cos tx\,dx = -t^2\ (C,\,1)\int_{-\infty}^{\infty} G(x)\cos tx\,dx,$$

$$(C,\,2)\int_{-\infty}^{\infty} g(x)\sin tx\,dx = -t^2\ (C,\,1)\int_{-\infty}^{\infty} G(x)\sin tx\,dx.$$

We prove the first relation. We must show that

$$\lim_{\lambda\to\infty}\frac{1}{\lambda^2}\int_0^{\lambda} d\mu\int_0^{\mu} d\nu\int_{-\nu}^{\nu}\{g(x)\cos tx + t^2 G(x)\cos tx\}\,dx = 0.$$

We have

$$\int_{-\nu}^{\nu} \{g(x)+t^2 G(x)\} \cos tx\, dx = \int_{-\nu}^{\nu} \{G''(x)+t^2 G(x)\} \cos tx\, dx$$

$$= \int_{-\nu}^{\nu} \frac{d}{dx} \{G'(x) \cos tx + tG(x) \sin tx\}\, dx$$

$$= \{G'(\nu)-G'(-\nu)\} \cos t\nu + \{G(\nu)+G(-\nu)\}\, t \sin t\nu.$$

Performing the operation $\int_0^{\mu} d\nu$, we get

$$\int_0^{\mu} \{G'(\nu)-G'(-\nu)\} \cos t\nu\, d\nu + \int_0^{\mu} \{G(\nu)+G(-\nu)\}\, t \sin t\nu\, d\nu$$

$$= \Big[\{G(\nu)+G(-\nu)\} \cos t\nu\Big]_0^{\mu} + 2t \int_{-\mu}^{\mu} G(\nu) \sin t\nu\, d\nu. \quad (50)$$

Since the second integral in (49) exists $(C, 1)$, we have

$$\int_0^{\lambda} d\mu \int_{-\mu}^{\mu} G(\nu) \sin t\nu\, d\nu = O(\lambda),$$

so that, on dividing by λ^2, this term tends to zero as $\lambda \to \infty$. We are left with

$$\frac{1}{\lambda^2} \int_{-\lambda}^{\lambda} G(\mu) \cos t\mu\, d\mu - \frac{2}{\lambda^2} \int_0^{\lambda} G(0)\, d\mu.$$

The second term tends to zero as $\lambda \to \infty$; and the first term is $O(1/\lambda)$ since G is bounded. This proves the lemma.

We thus have, p.p. in t,

$$(C, 2)\ \frac{1}{\pi} \int_{-\infty}^{\infty} g(x) \cos tx\, dx = a(t), \quad (C, 2)\ \frac{1}{\pi} \int_{-\infty}^{\infty} g(x) \sin tx\, dx = b(t) \quad (t > 1),$$

$$= 0, \qquad\qquad = 0$$

$$(0 < t < 1).$$

Further, as in § 5, p.p. in t,

$$(C, 1)\ \frac{1}{\pi} \int_{-\infty}^{\infty} \phi(x) \cos tx\, dx = 0, \quad (C, 1)\ \frac{1}{\pi} \int_{-\infty}^{\infty} \phi(x) \sin tx\, dx = 0 \quad (t > 1),$$

$$= a(t), \qquad\qquad = b(t)$$

$$(0 < t < 1).$$

The preceding relations hold $(C, 2)$. By addition, we obtain the desired result with $f = g+\phi$. That $\bar{P} = \underline{P} = f$ p.p. follows as for Theorem IV.

COROLLARY. *If, under the conditions of Theorem VIII,*

$$\lim_{u\to\infty}\frac{1}{u}\int_{-u}^{u}|G(x)|\,dx=0, \tag{51}$$

then the equations for $a(t)$, $b(t)$, *hold* $(C, 1)$.

We need only show that, in the result of Lemma 4, $(C, 2)$ may be replaced by $(C, 1)$. To prove this for the first relation of that lemma, we must show that, on dividing the second member of (50) by μ, the resulting expression tends to zero as $\mu\to\infty$. This, however, is manifestly the case in virtue of the boundedness of G, and of (51).

The condition (51) is, in virtue of the boundedness of G, easily seen to be the same as

$$\lim_{u\to\infty}\frac{1}{u}\int_{-u}^{u}G^2(x)\,dx=0. \tag{52}$$

10. We shall now analyse the conditions, (i) that $J(\eta, x)$ of Theorem VIII shall be bounded; (ii) that, in addition to (i), $G(x)=\lim J(\eta, x)$ shall be continuous; and (iii) that, in addition to (i), (52) shall be satisfied.

THEOREM IX. *Necessary and sufficient conditions for*

$$J(\xi,\eta)=\int_0^\infty e^{-\eta t}\{A(t)\cos\xi t+B(t)\sin\xi t\}\,dt$$

to satisfy

$$|J(\xi,\eta)|<1\quad(\eta>0) \tag{53}$$

are, with the notation

$$C(t)=\tfrac{1}{2}\{A(t)-iB(t)\},\qquad P_n(t)=\frac{(-1)^{n-1}}{(n-1)!}\int_0^{2t}e^x\frac{d^{n-1}}{dx^{n-1}}(x^{n-1}e^{-x})\,dx$$

$$(n=1, 2, \ldots),$$

$$\gamma_0=\tfrac{1}{2}\alpha_0=\int_0^\infty e^{-t}A(t)\,dt,\qquad \gamma_n=\tfrac{1}{2}(\alpha_n-i\beta_n)=\int_0^\infty e^{-t}C(t)\,P_n(t)\,dt$$

$$(n=1, 2, \ldots),$$

that (i)

$$H(r,\theta)=\tfrac{1}{2}\alpha_0+\sum_1^\infty(\alpha_n\cos n\theta+\beta_n\sin n\theta)r^n$$

satisfies

$$|H(r,\theta)|<1\quad(0\leqslant r<1), \tag{54}$$

and that (ii)

$$\int_0^\infty e^{-\eta t}e^{-t}C(t)\,P_n(t)\,dt\to\gamma_n$$

as $\eta\to 0$ *uniformly in* n.

The function $J(\xi, \eta)$ is harmonic in $\eta > 0$. Let $\zeta = \xi + i\eta$, and let

$$w = re^{i\theta} = -\frac{\zeta - i}{\zeta + i}. \tag{55}$$

This transforms $\eta > 0$ into the unit circle of the w-plane, and transforms J into $J'(r, \theta)$. We write $C(-t) = \bar{C}(t)$, the conjugate of $C(t)$, and then

$$J(\xi, \eta) = \int_{-\infty}^{\infty} e^{-\eta|t|} C(t)\, e^{i\xi t} dt.$$

To prove that the conditions are necessary, suppose that (53) holds. Then $|J'| < 1$, and

$$\lim_{r \to 1} J'(r, \theta) = f(\theta) \quad (|f| \leqslant 1) \tag{56}$$

boundedly, p.p. in θ. Further,

$$\lim_{\eta \to 0} J(\xi, \eta) = F(\xi) \quad (|F| \leqslant 1) \tag{57}$$

boundedly, p.p. in ξ, and $f(\theta) = F(\xi)$, where

$$\xi = \tan \tfrac{1}{2}\theta. \tag{58}$$

We can write

$$J'(r, \theta) = \tfrac{1}{2}a_0 + \sum_1^{\infty} (a_n \cos n\theta + b_n \sin n\theta)\, r^n, \tag{59}$$

where, in virtue of (56),

$$c_n = \tfrac{1}{2}(a_n - ib_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta \quad (n = 0, 1, \ldots). \tag{60}$$

In (60) we change the variable from θ to ξ by means of (58). We have

$$e^{-i\theta} = -\frac{\xi + i}{\xi - i}, \quad 2d\xi = (1 + \xi^2)\, d\theta,$$

and so

$$c_n = \frac{(-1)^n}{\pi} \int_{-\infty}^{\infty} F(\xi) \left(\frac{\xi + i}{\xi - i}\right)^n \frac{d\xi}{1 + \xi^2}.$$

In virtue of the bounded convergence in (57), we have

$$c_n = \lim_{\eta \to 0} \frac{(-1)^n}{\pi} \int_{-\infty}^{\infty} J(\xi, \eta) \left(\frac{\xi + i}{\xi - i}\right)^n \frac{d\xi}{1 + \xi^2} \quad (n),$$

where the symbol (n) denotes that approach to the limit $\eta \to 0$ is uniform in n. Thus

$$c_n = \lim_{\eta\to 0} \frac{(-1)^n}{\pi} \int_{-\infty}^{\infty} \left(\frac{\xi+i}{\xi-i}\right)^n \frac{d\xi}{1+\xi^2} \int_{-\infty}^{\infty} e^{-\eta|t|} C(t)\, e^{i\xi t}\, dt \quad (n)$$

$$= \lim_{\eta\to 0} \frac{(-1)^n}{\pi} \int_{-\infty}^{\infty} e^{-\eta|t|} C(t)\, dt \int_{-\infty}^{\infty} e^{i\xi t} \left(\frac{\xi+i}{\xi-i}\right)^n \frac{d\xi}{1+\xi^2} \quad (n).$$

We write $$\Pi_n(t) = (-1)^n \int_{-\infty}^{\infty} e^{i\xi t} \left(\frac{\xi+i}{\xi-i}\right)^n \frac{d\xi}{1+\xi^2} \quad (n = 0,\ 1,\ \ldots).$$

We have $$\Pi_0(t) = \pi e^{-|t|}.$$

For a fixed $n > 0$, we have, for $t > 0$ (by the theory of residues),

$$\Pi_n(t) = (-1)^n\, 2\pi i \operatorname*{res}_{\zeta=i} e^{i(\zeta-i)t} e^{-t} \frac{(\zeta+i)^{n-1}}{(\zeta-i)^{n+1}}.$$

We find that $$\Pi_n(t) = \pi e^{-t} P_n(t)$$

with

$$P_n(t) = \frac{(2t)^n}{n!} - (n-1)\frac{(2t)^{n-1}}{(n-1)!} + \frac{(n-1)(n-2)}{2!}\frac{(2t)^{n-2}}{(n-2)!} - \ldots + (-1)^{n-1} 2t.$$

Further, for $n > 0$ and $t < 0$, we find that $\Pi_n(t) = 0$. It is easy to see that $P_n(t)$ has the value given in the enunciation if we use the formula

$$e^x \frac{d^n}{dx^n}(x^n e^{-x}) = (-1)^n \left[x^n - \frac{n^2}{1!} x^{n-1} + \frac{n^2(n-1)^2}{2!} x^{n-2} - \ldots + (-1)^n n!\right].$$

Thus $$c_0 = \lim_{\eta\to 0} \int_{-\infty}^{\infty} e^{-\eta|t|} C(t) \pi e^{-|t|}\, dt = \int_0^{\infty} e^{-|t|} C(t)\, dt$$

$$= \int_0^{\infty} e^{-t} A(t)\, dt = \gamma_0;$$

and, for $n > 0$,

$$c_n = \lim_{\eta\to 0} \frac{1}{\pi} \int_0^{\infty} e^{-\eta t} C(t)\, \pi e^{-t} P_n(t)\, dt \quad (n) \qquad (61)$$

$$= \int_0^{\infty} e^{-t} C(t)\, P_n(t)\, dt = \gamma_n.$$

The relation (61) shows that the condition (ii) is satisfied, and further that $J' = H$. Since $|J'| < 1$, the condition (i) holds.

To prove the sufficiency, let (i) and (ii) be satisfied. Let (r, θ) be a fixed point of the unit circle in the w-plane, and let (ξ, η) be the corresponding point in the ζ-plane. We have, by the definition of γ_n,

$$H(r, \theta) = \int_0^\infty e^{-t} A(t)\, dt + \sum_{n=1}^{\infty} r^n e^{in\theta} \int_0^\infty e^{-t} C(t)\, P_n(t)\, dt + \sum_{n=1}^{\infty} r^n e^{-in\theta} \int_0^\infty e^{-t} \bar{C}(t)\, P_n(t)\, dt.$$

Since $$\int_0^\infty e^{-\eta t} e^{-t} C(t)\, P_n(t)\, dt \to \int_0^\infty e^{-t} C(t)\, P_n(t)\, dt$$

uniformly in n, given $\epsilon > 0$, there is a $y(\epsilon)$ such that, for $0 < y \leqslant y(\epsilon)$,

$$\left| \int_0^\infty e^{-t} C(t)\, P_n(t)\, dt - \int_0^\infty e^{-yt} e^{-t} C(t)\, P_n(t)\, dt \right| < \tfrac{1}{2}\epsilon \quad (n = 1, 2, \ldots). \tag{62}$$

Hence, also,

$$\left| \int_0^\infty e^{-t} \bar{C}(t)\, P_n(t)\, dt - \int_0^\infty e^{-yt} e^{-t} \bar{C}(t)\, P_n(t)\, dt \right| < \tfrac{1}{2}\epsilon \quad (n = 1, 2, \ldots). \tag{63}$$

Let

$$H(r, \theta, y) = \int_0^\infty e^{-t} A(t)\, dt + \sum_{n=1}^{\infty} r^n e^{in\theta} \int_0^\infty e^{-yt} e^{-t} C(t)\, P_n(t)\, dt + \sum_{n=1}^{\infty} r^n e^{-in\theta} \int_0^\infty e^{-yt} e^{-t} \bar{C}(t)\, P_n(t)\, dt \tag{64}$$

$$= \int_0^\infty e^{-t} A(t)\, dt + S_1(y) + S_2(y).$$

By (62) and (63),

$$|H(r, \theta) - H(r, \theta, y)| < \epsilon \sum_1^\infty r^n < \epsilon/(1-r). \tag{65}$$

We have

$$S_1(y) = \sum_{n=1}^{\infty} (-1)^n \int_0^\infty e^{-yt} e^{-t} P_n(t)\, C(t) \left(\frac{\zeta - i}{\zeta + i}\right)^n dt.$$

Now

$$|e^{-t} P_n(t)| = \left| \frac{1}{\pi} \int_{-\infty}^{\infty} e^{ixt} \left(\frac{x+i}{x-i}\right)^n \frac{dx}{1+x^2} \right| \leqslant 1. \tag{66}$$

Since $e^{-yt} C(t)$ is absolutely integrable in $(0, \infty)$, it follows that

$$S_1(y) = \int_0^\infty e^{-yt} e^{-t} C(t) \left\{ \sum_{n=1}^{\infty} (-1)^n P_n(t) \left(\frac{\zeta - i}{\zeta + i}\right)^n \right\} dt. \tag{67}$$

Now

$$\sum_{n=1}^{\infty}(-1)^n P_n(t)\left(\frac{\zeta-i}{\zeta+i}\right)^n$$

$$=\sum_{n=1}^{\infty}\left(\frac{\zeta-i}{\zeta+i}\right)^n\left[-2t+(n-1)\frac{(2t)^2}{2!}-\ldots+(-1)^n\frac{(2t)^n}{n!}\right]. \quad (68)$$

This series can be arranged in powers of t without altering its sum. For, if we replace each term by its modulus, we get

$$\sum_{1}^{\infty} r^n\left[2t+(n-1)\frac{(2t)^2}{2!}+\ldots\frac{(2t)^n}{n!}\right].$$

And if we arrange this series in powers of t, we get a convergent series, namely

$$\frac{2tr}{1-r}+\frac{(2t)^2}{2!}\frac{r^2}{(1-r)^2}+\frac{(2t)^3}{3!}\frac{r^3}{(1-r)^3}+\ldots=e^{2tr/(1-r)}-1.$$

If now we arrange (68) in powers of t, we get

$$i(\zeta-i)t+\frac{\{i(\zeta-i)t\}^2}{2!}+\ldots=e^{i(\zeta-i)t}-1.$$

Hence

$$S_1(y)=\int_0^\infty e^{-yt}e^{-t}C(t)\{e^{i(\zeta-i)t}-1\}dt$$

$$=\int_0^\infty e^{-yt}C(t)e^{i\zeta t}dt-\int_0^\infty e^{-yt}e^{-t}C(t)dt.$$

Further,

$$S_2(y)=\bar{S}_1(y)=\int_0^\infty e^{-yt}\bar{C}(t)e^{-i\bar{\zeta}t}dt-\int_0^\infty e^{-yt}e^{-t}\bar{C}(t)dt.$$

Hence

$$H(r,\theta,y)=\int_0^\infty(1-e^{-yt})e^{-t}A(t)dt+\int_0^\infty e^{-yt}[C(t)e^{i\zeta t}+\bar{C}(t)e^{-i\bar{\zeta}t}]dt. \quad (69)$$

Let $y\to 0$. Then

$$H(r,\theta)=\lim_{y\to 0}H(r,\theta,y) \quad \text{[by (65)]}$$

$$=\int_0^\infty[C(t)e^{i\zeta t}+\bar{C}(t)e^{-i\bar{\zeta}t}]dt \quad \text{[by (69)]}$$

$$=J(\xi,\eta).$$

Thus (54) implies (53). This completes the proof.

The significance of the condition (ii) of Theorem IX arises from its being a necessary condition. It can be replaced by a condition which is *prima facie* much less restrictive, and whose significance will arise from its being a sufficient condition. This is shown by

THEOREM X. *In the preceding theorem, the condition* (ii) *may be replaced by the condition*: (ii$a$) *given* $\epsilon > 0$, $\phi > 0$, *there is a* $y(\epsilon, \phi)$ *such that, for* $0 < y \leqslant y(\epsilon, \phi)$,

$$\left| \int_0^\infty (1-e^{-yt})\, e^{-t}\, C(t)\, P_n(t)\, dt \right| < \epsilon(1+\phi)^n \quad (n = 1, 2, \ldots). \quad (70)$$

The conditions (i) and (ii$a$) are necessary in order that the conclusion of Theorem IX shall hold, because (ii) implies (ii$a$). To prove that (i) and (ii$a$) are sufficient, we observe that (69) is true, and all we need show is that $H(r, \theta) = \lim H(r, \theta, y)$. Since $0 \leqslant r < 1$, we can choose $\phi > 0$ so that $(1+\phi)r < 1$. Then, for $0 \leqslant y \leqslant y(\epsilon, \phi)$,

$$|H(r, \theta) - H(r, \theta, y)| < 2\epsilon \sum_1^\infty (1+\phi)^n r^n$$

$$< 2\epsilon/\{1-r(1+\phi)\},$$

which can be made arbitrarily small by suitable choice of ϵ. This is the desired result.

This theorem will enable us to show that, in a large class of cases, the condition (ii$a$) may be omitted.

THEOREM XI. *If* $|C(t)| < Kt^\rho$ *as* $t \to \infty$ *for some* $\rho > 0$, *then the condition* (ii$a$) *is necessarily satisfied.*

Let $A \geqslant 0$ be a number such that the inequality of the enunciation holds for $t > A$. Then, by (66),

$$\left| \int_0^A (1-e^{-yt})\, e^{-t}\, P_n(t)\, C(t)\, dt \right| < (1-e^{-yA}) \int_0^A |C(t)|\, dt$$

$$< \epsilon$$

for $y < y(\epsilon, A)$, so that we need only consider the integrals in (70) over (A, ∞). This amounts to putting $C(t) = 0$ in $(0, A)$ in the relation (70) which is to be proved, so that we may suppose the inequality of the enunciation to hold for $t > 0$.

We have

$$P_n(t) = \sum_{r=0}^{n-1} (-1)^{n-1-r} \frac{(n-1)!}{r!\,(n-1-r)!} \frac{(2t)^{r+1}}{(r+1)!}. \tag{71}$$

The condition for the terms in this expression to increase in absolute value is that

$$\frac{2t(n-r-1)}{(r+1)(r+2)} > 1 \quad (r = 0, 1, \ldots, n-2),$$

which is certainly satisfied if $2t > n^2$. If this condition is satisfied, then, since the terms in (71) are of alternate signs and increase in absolute value, we have

$$|P_n(t)| < \frac{(2t)^n}{n!} \quad (2t > n^2). \tag{72}$$

Let $\epsilon > 0$ and $\phi > 0$ be assigned. Then

$$\lambda_n = \epsilon(1+\phi)^n > \tfrac{1}{2}n^2 \quad [n > n(\epsilon, \phi)]. \tag{73}$$

For $t \geqslant \lambda_n$, (72) holds, and so

$$\int_{\lambda_n}^{\infty} e^{-t} t^{\rho} |P_n|\, dt < \frac{2^n}{n!} \int_{\lambda_n}^{\infty} e^{-t} t^{\rho+n}\, dt.$$

We may suppose that ρ is a positive integer. Then, since

$$e^t > \frac{t^{n+\rho+s+1}}{(n+\rho+s+1)!},$$

where s is a positive integer, and since

$$\frac{(n+\rho+s+1)!}{n!} < K(\rho, s)\, n^{\rho+s+1},$$

where $K(\rho, s)$ is a constant which depends only on ρ and s, we have

$$\int_{\lambda}^{\infty} e^{-t} t^{\rho} |P_n|\, dt < 2^n K(\rho, s)\, n^{\rho+s+1} \int_{\lambda_n} \frac{dt}{t^{s+1}}.$$

The number ρ is fixed throughout the argument. We therefore write $K(\rho, s) = K(s)$, and the preceding expression is less than

$$2^n \frac{K(s)}{s} n^{\rho+s+1} \lambda_n^{-s} = \left[2\left(\frac{K(s)}{s}\right)^{1/n} (n^{1/n})^{\rho+s+1}\right]^n \Big/ [\epsilon^{s/n}(1+\phi)^s]^n. \tag{74}$$

We may suppose that $\epsilon < 1$. Since $\epsilon^{s/n}$ increases with n, there is a $\sigma = \sigma(\epsilon, \phi)$ such that

$$\epsilon^{\sigma/n}(1+\phi)^{\sigma} > 4 \quad [n \geqslant N(\sigma)].$$

We may take σ to be an integer. Let s have the fixed value σ. Then, for $n \geqslant n(\sigma)$, we have

$$\left(\frac{K(\sigma)}{\sigma}\right)^{1/n} < (\tfrac{3}{2})^{\frac{1}{2}}, \quad (n^{1/n})^{\rho+\sigma+1} < (\tfrac{3}{2})^{\frac{1}{2}},$$

and (74) is less than $(\frac{3}{4})^n$, which, for $n > n'(\epsilon, \phi)$, does not exceed $\frac{1}{2}\epsilon(1+\phi)^n$. We have thus proved that, given $\epsilon > 0$ and $\phi > 0$, there is a number depending only on ϵ and ϕ (and ρ), which we now denote by $n(\epsilon, \phi)$, such that

$$\int_{\epsilon(1+\phi)^n}^{\infty} e^{-t} t^{\rho} |P_n| \, dt < \tfrac{1}{2}\epsilon(1+\phi)^n \quad [n > n(\epsilon, \phi)]. \tag{75}$$

Further, by (66),

$$\int_0^{\epsilon(1+\phi)^n} |e^{-t} P_n(t)| \, t^{\rho} \, dt < \int_0^{\epsilon(1+\phi)^n} t^{\rho} \, dt$$

$$< \tfrac{1}{2}\epsilon^{\rho+1}[(1+\phi)^{\rho+1}]^n \quad (n = 1, 2, \ldots). \tag{76}$$

Now $\epsilon^{\rho+1} < \epsilon$, and $(1+\phi)^{\rho+1} > 1+\phi$, so that (75) and (76) give

$$\int_0^{\infty} e^{-t} t^{\rho} |P_n| \, dt < \epsilon[(1+\phi)^{\rho+1}]^n \quad [n > n(\epsilon, \phi)].$$

Since, however, $(1+\phi)^{\rho+1}$ can by suitable choice of ϕ be made equal to any number greater than 1, the preceding relation is the same as

$$\int_0^{\infty} e^{-t} t^{\rho} |P_n| \, dt < \epsilon(1+\phi)^n \quad [n > n'(\epsilon, \phi)]. \tag{77}$$

We may suppose that $|C(t)| < \frac{1}{2}t^{\rho}$ for $t > 0$. Then

$$\left| \int_0^{\infty} e^{-t} C(t) P_n(t)(1-e^{-yt}) \, dt \right| < \int_0^{\infty} e^{-t} |P_n| \, t^{\rho} \, dt$$

$$< \epsilon(1+\phi)^n \quad [n > n'(\epsilon, \phi)] \tag{78}$$

for all $y > 0$. Since

$$\lim_{y \to 0} \int_0^{\infty} (1-e^{-yt}) e^{-t} C(t) P_n(t) \, dt = 0,$$

there is a $y\{n'(\epsilon, \phi)\} = y(\epsilon, \phi)$ such that, for $n = 1, 2, \ldots n'$ and

$$0 < y \leqslant y(\epsilon, \phi),$$

the first member of (78) is less than $\epsilon(1+\phi)^n$. Hence (ii$a$) is satisfied.

11. The condition (i) of Theorem IX admits of an elegant transformation in virtue of the following theorem.

THEOREM XII. *Necessary and sufficient conditions for*

$$P(r, x) = \tfrac{1}{2}a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx) r^n$$

to satisfy

$$|P(r, x)| < 1 \quad (0 \leqslant r < 1) \tag{79}$$

are, with the notation

$$c_n = \tfrac{1}{2}(a_n - ib_n), \quad c_{-n} = \bar{c}_n,$$

$$D_n(x) = \begin{vmatrix} c_0 - x & c_1 & c_2 & \ldots & c_n \\ c_{-1} & c_0 - x & c_1 & \ldots & c_{n-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ c_{-n} & c_{-(n-1)} & & \ldots & c_0 - x \end{vmatrix},$$

that

$$D_n(-1) > 0, \quad (-1)^{n+1} D_n(+1) > 0 \quad (n = 0, 1, \ldots). \tag{80}$$

We first observe that (79) is equivalent to the condition that

$$\tfrac{1}{2}a_0 + \sum_1^\infty a_n \cos nx + b_n \sin nx \equiv \tfrac{1}{2}a_0 + \sum_1^\infty A_n(x) \tag{81}$$

shall be the Fourier series of a function f which satisfies $|f| \leqslant 1$, and which is not equivalent to 1 or to -1. We require the following lemma.

LEMMA 5. *A necessary and sufficient condition that* (79) *shall be satisfied is that, for* $n = 0, 1, \ldots$, *the values of the Laurentian form*

$$\sum_{j,\,k=0}^{n} c_{j-k} x_j \bar{x}_k \quad \left(\sum_0^n x_j \bar{x}_j = 1\right) \tag{82}$$

shall lie in the open interval $(-1, 1)$.

The condition is necessary. For let (79) hold. Let

$$g(x) = \sum_{j=0}^{n} \xi_j e^{-ixj},$$

where the ξ_j are any complex numbers which satisfy

$$\sum_0^n \xi_j \bar{\xi}_j = 1.$$

Then $$|g(x)|^2 = g(x)\bar{g}(x) = \tfrac{1}{2}\alpha_0 + \sum_1^n \alpha_m \cos mx + \beta_m \sin mx \quad (\tfrac{1}{2}\alpha_0 = 1).$$

Since $|f| \leqslant 1$, and $|f|$ is not equivalent to 1,

$$\left| \frac{1}{2\pi} \int_0^{2\pi} f(t)|g(t)|^2 dt \right| < \frac{1}{2\pi} \int_0^{2\pi} |g(t)|^2 dt$$

$$< 1.$$

But $$\frac{1}{2\pi} \int_0^{2\pi} f|g|^2 dt = \frac{1}{2\pi} \int_0^{2\pi} f \sum_{j,\,k=0}^n \xi_j \bar{\xi}_k e^{-i(j-k)t} dt$$

$$= \sum_{j,\,k=0}^n c_{j-k} \xi_j \bar{\xi}_k.$$

The condition is sufficient. For, if not, there would be an n and an x such that

$$\left| \tfrac{1}{2}a_0 + \sum_{m=1}^n \left(1 - \frac{m}{n+1}\right) A_m(x) \right| \geqslant 1. \tag{83}$$

But

$$\tfrac{1}{2}a_0 + \sum_{m=1}^n \left(1 - \frac{m}{n+1}\right) A_m(x) = \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{-n}^n c_m e^{imx} e^{imt} \right) \frac{1}{n+1} \left(\frac{\sin \frac{1}{2}nt}{\sin \frac{1}{2}t} \right)^2 dt,$$

and $$\frac{1}{n+1} \left(\frac{\sin \frac{1}{2}nt}{\sin \frac{1}{2}t} \right)^2 = \left(\sum_0^n \lambda_j e^{-ijt} \right) \left(\sum_0^n \bar{\lambda}_j e^{ijt} \right)$$

with $$\lambda_j = \bar{\lambda}_j = \frac{1}{\sqrt{(n+1)}}, \quad \Sigma \lambda_j \bar{\lambda}_j = 1.$$

Thus (83) gives

$$\left| \sum_{j,\,k=0}^n c_{j-k} \lambda_j \bar{\lambda}_k e^{i(j-k)x} \right| \geqslant 1. \tag{84}$$

Writing $x_j = \lambda_j e^{ijx}$, we have $\Sigma x_j \bar{x}_j = 1$, so that (84) contradicts the condition of the lemma.

We can now complete the proof of the theorem. If (79) holds, then the value of the Laurentian form (82) lies in the open interval $(-1, 1)$. Hence the roots of $D_n(x) = 0$ lie in that open interval. Hence $D_n(-1)$ is of the same

sign in the strict sense as $D_n(x)$ for large negative x, *i.e.* is positive; and $D_n(+1)$ is of the same sign in the strict sense as $D_n(x)$ for large positive x, *i.e.* is of the same sign as $(-1)^{n+1}$. Thus (80) holds.

Conversely, if (80) holds, then

$$\sum_{j,k=0}^{n} c_{j-k} x_j \bar{x}_k + \sum_{0}^{n} x_j \bar{x}_j$$

is positive definite; and

$$\sum_{j,k=0}^{n} c_{j-k} x_j \bar{x}_k - \sum_{0}^{n} x_j \bar{x}_j$$

is negative definite. Hence, for $\Sigma x_j \bar{x}_j > 0$,

$$\left| \sum_{j,k=0}^{n} c_{j-k} x_j \bar{x}_k \right| < \sum_{0}^{n} x_j \bar{x}_j,$$

so that the condition of Lemma 5 is satisfied. By that lemma, (79) holds.

12. We now wish to investigate under what conditions the harmonic function

$$J(\xi, \eta) = \int_0^\infty e^{-\eta t} \{A(t) \cos \xi t + B(t) \sin \xi t\} dt \tag{85}$$

satisfies not only

$$|J(\xi, \eta)| < 1 \tag{86}$$

but also

$$\lim_{\eta \to 0} J(\xi, \eta) = F(\xi) \tag{87}$$

with a continuous $F(\xi)$. Clearly, a necessary and sufficient condition is that the function $H(r, \theta)$ of Theorem IX shall satisfy the conditions (i) and (ii) [or (ii$a$)], and, further, that

$$\lim_{r \to 1} H(r, \theta) = f(\theta) \quad (\theta \neq \pi),$$

where f is continuous for $\theta \neq \pi$. What we have now to examine is the following problem. Given the trigonometric series

$$\tfrac{1}{2} a_0 + \sum_{1}^{\infty} A_n(x), \tag{88}$$

to find a necessary and sufficient condition for it to be the Fourier series of a bounded function which is continuous except at $x = \pi \pmod{2\pi}$. We consider first the case in which the exceptional point does not occur.

THEOREM XIII. *A necessary and sufficient condition for* (88) *to be the Fourier series of a continuous function is, with the notation*

$$c_n = \tfrac{1}{2}(a_n - ib_n), \quad c_{-n} = \bar{c}_n,$$

$$D(n, \nu, r) = \begin{vmatrix} 1 & c_1\nu(1-r) & c_2\nu(1-r^2) & c_n\nu(1-r^n) \\ c_{-1}\nu(1-r) & 1 & c_1\nu(1-r) & c_{n-1}\nu(1-r^{n-1}) \\ \cdots \quad \cdots & \cdots \quad \cdots \quad \cdots & \cdots \quad \cdots & \cdots \quad \cdots \\ c_{-n}\nu(1-r^n) & c_{-(n-1)}\nu(1-r^{n-1}) & \cdots & 1 \end{vmatrix},$$

that, given the positive integer ν, *there is an* $r = r(\nu)$ *satisfying* $0 < r < 1$ *such that*

$$D(n, \pm\nu, r) > 0 \quad (n = 0, 1, 2, \ldots). \tag{89}$$

The presence of the undetermined $r(\nu)$ is inevitable, and reflects the element of indetermination in the definition of a continuous function. The periodic function $f(x)$ is continuous if, given the positive integer ν, there is an $h(\nu) > 0$ such that $\nu|f(x)-f(y)| < 1$ for $|x-y| < h(\nu)$. In this definition, the way in which $h(\nu)$ depends on ν is unspecified.

If (88) is the Fourier series of a continuous function $f(x)$, then the functions

$$P(r, x) = \tfrac{1}{2}a_0 + \sum_1^\infty A_n(x)\, r^n \quad (0 < r < 1) \tag{90}$$

tend uniformly to $f(x)$ as $r \to 1$. Hence

$$\sum_1^\infty A_n(x)\,(1-r^n) \tag{91}$$

is the Fourier series of a function f_r which satisfies $|f_r| < 1/\nu$ for $r \geqslant r(\nu)$. If $r = r(\nu)$, then

$$\sum_1^\infty \nu A_n(x)(1-r^n) \tag{92}$$

is the Fourier series of a function ϕ_r which satisfies $|\phi_r| < 1$, and, by Theorem XII, the condition (89) is satisfied.

Conversely, if (89) holds, then, by Theorem XII, the series (92) with $r = r(\nu)$ is the Fourier series of a function νf_r with $|\nu f_r| \leqslant 1$, and (91) is the Fourier series of f_r. Hence a_n and b_n tend to zero as $n \to \infty$, and so (90) is a Fourier series for $r = r(\nu)$. By the addition of (90) and (91), we infer that (88) is a Fourier series. Let f be a function of which it is the Fourier series. Then $|f(x) - P(r_\nu, x)| \leqslant 1/\nu$ p.p. If E_ν is the set of measure zero at which

this inequality does not hold, then, for $x \subset C\Sigma E_\nu$,

$$|f(x)-P(r_\nu, x)| \leqslant 1/\nu \quad (\nu = 1, 2, \ldots).$$

Thus the sequence $P(r_\nu, x)$ of continuous functions tends uniformly to $f(x)$ on $C\Sigma E_\nu$. Hence $f(x)$ is equivalent to a continuous function, and so (88) is the Fourier series of a continuous function.

THEOREM XIV. *Let* (88) *be the Fourier series of a bounded function* $f(x)$. *A necessary and sufficient condition for* $f(x)$ *to be continuous except at* $x = \pi \pmod{2\pi}$ *is that*

$$\sum_{-\infty}^{\infty} (2c_n + c_{n-1} + c_{n+1}) e^{inx} \tag{93}$$

is the Fourier series of a continuous function.

The periodic function

$$g(x) = 2(1+\cos x) = 2 + e^{ix} + e^{-ix}$$

is continuous and vanishes only for $x = \pi \pmod{2\pi}$. The series (93) is the Fourier series of $f(x)g(x)$. If f is continuous for $x \neq \pi$, then, since f is bounded, fg is continuous for all x. Conversely, if fg is continuous for all x, then, since $g \neq 0$ for $x \neq \pi$, f is continuous for $x \neq \pi$.

13. We have now to examine the third condition mentioned at the beginning of § 10. We require the following lemma.

LEMMA 6. *Let* $\Phi(\xi)$ *be bounded in* $(-\infty, \infty)$, *and let* $\phi(\theta) = \Phi(\xi)$, *where*

$$\xi = \tan \tfrac{1}{2}\theta. \tag{94}$$

A necessary and sufficient condition that

$$\lim_{u\to\infty} \frac{1}{u} \int_0^u \Phi(\xi)\, d\xi = 0 \tag{95}$$

is that

$$\lim_{t\to 0} \frac{1}{t} \int_0^t \phi(\pi-\theta)\, d\theta = 0 \quad (t > 0). \tag{96}$$

We have

$$\frac{1}{u} \int_0^u \Phi\, d\xi = \frac{1}{2u} \int_0^{2\tan^{-1}u} \phi(\theta)(1+\tan^2 \tfrac{1}{2}\theta)\, d\theta = \frac{1}{2u} \int_0^{2\tan^{-1}u} \phi(\theta) \tan^2 \tfrac{1}{2}\theta + o(1).$$

Putting $\theta' = \pi - \theta$, we find that

$$\frac{1}{u}\int_0^{2\tan^{-1}u} \phi(\theta)\tan^2\tfrac{1}{2}\theta\,d\theta = -\frac{1}{u}\int_\pi^{2\cot^{-1}u} \phi(\pi-\theta)\cot^2\tfrac{1}{2}\theta\,d\theta$$

$$= \frac{4}{u}\int_\eta^\pi \frac{\phi(\pi-\theta)}{\theta^2}\,d\theta + o(1),$$

where $\eta = 2\cot^{-1}u$. If (96) holds, then, writing

$$\psi(t) = \int_0^t \phi(\pi-\theta)\,d\theta = o(t),$$

we have

$$\frac{1}{u}\int_\eta^\pi \frac{\phi(\pi-\theta)}{\theta^2}\,d\theta = \frac{1}{u}\int_\eta^\pi \frac{\psi'(\theta)}{\theta^2}\,d\theta = \frac{1}{u}\left[\frac{\psi(\theta)}{\theta^2}\right]_\eta^\pi + \frac{2}{u}\int_\eta^\pi \frac{\psi(\theta)}{\theta^3}\,d\theta$$

$$= o(1) + o\left(\frac{1}{u}\int_\eta^\pi \frac{d\theta}{\theta^2}\right) = o(1).$$

Suppose, conversely, that (95) holds. Then

$$\frac{1}{t}\int_0^t \phi(\pi-\theta)\,d\theta = \frac{1}{t}\int_{\pi-t}^\pi \phi(\theta)\,d\theta = \frac{1}{t}\int_{\cot\frac{1}{2}t}^\infty \Phi(\xi)\,\frac{2d\xi}{1+\xi^2}.$$

Writing
$$\Psi(u) = \int_0^u \Phi\,d\xi = o(u),$$
we have

$$\frac{1}{t}\int_{\cot\frac{1}{2}t}^\infty \Phi\,\frac{d\xi}{1+\xi^2} = \frac{1}{t}\left[\frac{\Psi(\xi)}{1+\xi^2}\right]_{\cot\frac{1}{2}t}^\infty + \frac{2}{t}\int_{\cot\frac{1}{2}t}^\infty \Psi(\xi)\,\frac{\xi\,d\xi}{(1+\xi^2)^2}$$

$$= o(1) + o\left(\frac{1}{t}\int_{\cot\frac{1}{2}t}^\infty \frac{d\xi}{\xi^2}\right) = o(1).$$

THEOREM XV. *If the function $J(\xi, \eta)$ of Theorem IX satisfies* (53), *then a necessary and sufficient condition for its limit function $F(\xi)$ to satisfy*

$$\lim_{u\to\infty}\frac{1}{u}\int_u^u |F(\xi)|\,d\xi = 0 \tag{97}$$

is, with the notation

$$A_n = \tfrac{1}{2}\sum_{p=-\infty}^{\infty} a_p a_{n-p} - \beta_p\beta_{n-p} \quad (n = 0, 1, \ldots),$$

that
$$\tfrac{1}{2}A_0 + \sum_1^\infty (-1)^n A_n$$
is summable $(C, 1)$ *to zero.*

In this enunciation, α_n, β_n have the meanings assigned in Theorem IX, $\alpha_{-n} = \alpha_n, \beta_{-n} = -\beta_n$. We observed at the end of § 9 that (97) is the same as

$$\lim_{u\to\infty} \frac{1}{u} \int_{-u}^{u} F^2(\xi)\, d\xi = 0. \tag{98}$$

If $H(r, \theta)$ has the meaning assigned in Theorem IX, then

$$\lim H(r, \theta) = f(\theta) \quad \text{p.p.},$$

where $f(\theta) = F(\xi)$, and $\xi = \tan \frac{1}{2}\theta$. By Lemma 6, (98) is the same as

$$\lim_{t\to 0} \frac{1}{t} \int_0^t \{f^2(\pi-\theta)+f^2(\pi+\theta)\}\, d\theta = 0.$$

This, however, is a necessary and sufficient condition† for the Fourier series of $f^2(\theta)$ to be summable $(C, 1)$ to zero at $\theta = \pi$. Since

$$f(\theta) \sim \sum_{-\infty}^{\infty} \gamma_n e^{in\theta},$$

we have

$$f^2(\theta) \sim \sum_{-\infty}^{\infty} C_n e^{in\theta} \quad \left(C_n = \sum_{p=-\infty}^{\infty} \gamma_p \gamma_{n-p}\right)$$

$$\sim \tfrac{1}{2}A_0 + \sum_{1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta),$$

where A_n has the value in the enunciation. For $\theta = \pi$, this gives the result.

14. If we examine the proof of Theorem X, we see that it amounts to a proof of the following theorem.

THEOREM XVI. *If $C(t)$ satisfies the condition* (ii a), *then*

$$J(\xi, \eta) = H(r, \theta),$$

where

$$re^{i\theta} = -(\zeta-i)/(\zeta+i).$$

We now have

THEOREM XVII. *If $C(t)$ satisfies the condition* (ii a), *and if $J(\xi, \eta) > 0$, for $\eta > 0$, then there is a bounded non-diminishing function $G(x)$ such that*

$$J(\xi, \eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\eta(1+x^2)}{\eta^2+(x-\xi)^2}\, dG(x). \tag{99}$$

† G. H. Hardy and J. E. Littlewood, *Journal London Math. Soc.*, 1 (1926), 134–138.

Further, (i) *if* $C(t)/(1+t^2)$ *is integrable in* $(0, \infty)$, *then, for almost all* $t>0$,

$$A(t) = \lim_{h\to 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-h|x|} \cos tx(1+x^2)\, dG,$$

$$B(t) = \lim_{h\to 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-h|x|} \sin tx(1+x^2)\, dG(x) \quad (h>0);$$

(ii) *if* $C(t) = o(t)$, *the preceding result holds*;

(iii) *if* $A(t)$ *is bounded in a neighbourhood of the origin, then* $A(t)$, $B(t)$ *are both equivalent to bounded continuous functions, the integral*

$$\int_{-\infty}^{\infty} (1+x^2)\, dG \tag{100}$$

exists, and, for almost all $t>0$,

$$A(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (1+x^2) \cos tx\, dG, \quad B(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (1+x^2) \sin tx\, dG.$$

By Theorem XVI,

$$H(r, \theta) = \tfrac{1}{2}a_0 + \sum_1^{\infty} (a_n \cos n\theta + \beta_n \sin n\theta) r^n > 0$$

for $0 \leqslant r < 1$. Hence

$$\tfrac{1}{2}a_0\theta + \sum_1^{\infty} \frac{a_n \sin n\theta - \beta_n \cos n\theta}{n} = g(\theta),$$

where $g(\theta)$ is non-diminishing, and so

$$\gamma_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta}\, dg(\theta) \quad (n = 0, 1, \ldots). \tag{101}$$

The function $g(\theta)$ is defined as the sum of an everywhere convergent series, which, except for the continuous term $\frac{1}{2}a_0\theta$, is a trigonometric series. Hence the discontinuities of $g(\theta)$ are regular. Thus

$$g(\pi) = \tfrac{1}{2}[g(\pi+0)+g(\pi-0)] = \tfrac{1}{2}[g(-\pi+0)+g(\pi-0)]+\tfrac{1}{2}a_0\pi,$$

$$g(\pi)-g(\pi-0) = \tfrac{1}{2}a_0\pi - \tfrac{1}{2}[g(\pi-0)-g(-\pi+0)],$$

and

$$g(-\pi) = \tfrac{1}{2}[g(-\pi+0)+g(-\pi-0)] = \tfrac{1}{2}[g(-\pi+0)+g(\pi-0)]-\tfrac{1}{2}a_0\pi,$$

$$g(-\pi+0)-g(-\pi) = \tfrac{1}{2}a_0\pi - \tfrac{1}{2}[g(\pi-0)-g(-\pi+0)]$$

$$= g(\pi)-g(\pi-0).$$

Since now the continuous function $e^{-in\theta}$ has the same values at $\theta = -\pi$, $\theta = \pi$, if we write

$$g'(\theta) = g(\theta) \quad (-\pi < \theta < \pi),$$

$$g'(\pi) = g(\pi - 0), \quad g'(-\pi) = g(-\pi + 0),$$

then g' will be a non-decreasing function, bounded in $-\pi \leqslant \theta \leqslant \pi$, and (101) remains true on replacing g by g'. Thus, since g' is continuous at the end points of the interval of integration, we can write

$$\gamma_n = \lim_{N\to\infty} \frac{1}{2\pi} \int_{-2\tan^{-1}N}^{2\tan^{-1}N} e^{-in\theta}\, dg'(\theta).$$

Let $x = \tan \frac{1}{2}\theta$, and let $G(x) = g'(2\tan^{-1}x)$. Then G is bounded and non-diminishing in $(-\infty, \infty)$, and, by the definition of a Stieltjes integral, we have

$$\int_{-2\tan^{-1}N}^{2\tan^{-1}N} e^{-in\theta}\, dg'(\theta) = (-1)^n \int_{-N}^{N} \left(\frac{x+i}{x-i}\right)^n dG(x).$$

Hence

$$\gamma_n = \frac{(-1)^n}{2\pi} \int_{-\infty}^{\infty} \left(\frac{x+i}{x-i}\right)^n dG(x).$$

Now

$$\begin{aligned} H(r,\theta) &= \gamma_0 + \sum_1^\infty \gamma_n r^n e^{in\theta} + \sum_1^\infty \gamma_{-n} r^n e^{-in\theta} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dG(x) \left[1 + \sum_1^\infty \left(\frac{\zeta - i}{\zeta + i}\right)^n \left(\frac{x+i}{x-i}\right)^n + \sum_1^\infty \left(\frac{\bar{\zeta}+i}{\bar{\zeta}-i}\right)^n \left(\frac{x-i}{x+i}\right)^n\right] \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\eta(1+x^2)}{\eta^2 + (x-\xi)^2}\, dG(x). \end{aligned}$$

This proves (99), *i.e.* the equation

$$\int_0^\infty e^{-\eta t} \{A(t)\cos \xi t + B(t) \sin \xi t\}\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\eta(1+x^2)}{\eta^2 + (x-\xi)^2}\, dG(x). \quad (102)$$

To prove (i), say the relation involving $A(t)$, we multiply both sides of (102) by $e^{-h|\xi|} \cos \tau\xi$ ($h > 0$, $\tau > 0$), and integrate over $(-\infty, \infty)$. We thus obtain

$$\begin{aligned} &\int_{-\infty}^{\infty} e^{-h|\xi|} \cos \tau\xi\, d\xi \int_0^\infty e^{-\eta t} A(t) \cos \xi t\, dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-h|\xi|} \cos \tau\xi\, d\xi \int_{-\infty}^{\infty} \frac{\eta(1+x^2)}{\eta^2 + (x-\xi)^2}\, dG(x) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} (1+x^2)\, dG(x) \int_{-\infty}^{\infty} e^{-h|\xi|} \cos \tau\xi\, \frac{\eta}{\eta^2 + (x-\xi)^2}\, d\xi. \quad (103) \end{aligned}$$

The first member of (103) is

$$\int_0^\infty e^{-\eta t} A(t)\, dt\, \tfrac{1}{2} \int_{-\infty}^{\infty} e^{-h|\xi|} \{\cos(\tau - t)\,\xi + \cos(\tau + t)\,\xi\}\, d\xi$$

$$= \int_0^\infty e^{-\eta t} A(t) \left[\frac{h}{h^2 + (\tau - t)^2} + \frac{h}{h^2 + (\tau + t)^2} \right] dt.$$

To transform the last member of (103), we observe that

$$\int_{-\infty}^{\infty} e^{-h|\xi|} \cos \tau\xi \, \frac{\eta}{\eta^2 + (x - \xi)^2}\, d\xi = \int_{-\infty}^{\infty} \frac{h}{h^2 + (\tau - y)^2}\, e^{-\eta|y|} \cos xy\, dy.$$

This relation is most easily verified by observing that the first expression is a bounded harmonic function of (τ, h) in $h > 0$, and that its boundary function $(h = 0)$ is

$$\int_{-\infty}^{\infty} \cos \tau\xi \, \frac{\eta}{\eta^2 + (x - \xi)^2}\, d\xi = \pi e^{-\eta|\tau|} \cos \tau x,$$

while the second expression is a bounded harmonic function of (τ, h) in $h > 0$ with the same boundary function. The two expressions are identical in virtue of an argument which we have already used. Thus (103) becomes

$$\int_0^\infty e^{-\eta t} A(t) \left[\frac{h}{h^2 + (\tau - t)^2} + \frac{h}{h^2 + (\tau + t)^2} \right] dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} (1 + x^2)\, dG \int_{-\infty}^{\infty} \frac{h}{h^2 + (\tau - y)^2}\, e^{-\eta|y|} \cos xy\, dy. \quad (104)$$

Since, by hypothesis, $A(t)/(1 + t^2)$ is integrable in $(0, \infty)$, it easily follows that the limit as $\eta \to 0$ of the first member in (104) is the expression obtained on putting $\eta = 0$. Thus

$$\int_0^\infty A(t) \left[\frac{h}{h^2 + (\tau - t)^2} + \frac{h}{h^2 + (\tau + t)^2} \right] dt$$

$$= \lim_{\eta \to 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} (1 + x^2)\, dG \int_{-\infty}^{\infty} \frac{h}{h^2 + (\tau - y)^2}\, e^{-\eta|y|} \cos xy\, dy. \quad (105)$$

Consider the function

$$I(x, \eta) = \int_{-\infty}^{\infty} \frac{h}{h^2 + (\tau - y)^2}\, e^{-\eta|y|} \cos xy\, dy$$

for fixed $h > 0$, $\tau > 0$. We have

$$I = \int_0^\infty + \int_{-\infty}^{0} = I_1 + I_2.$$

Now

$$I_1 = \int_0^\infty \frac{h}{h^2+(\tau-y)^2}\, e^{-\eta y}\, \frac{d}{dy}\left(\frac{\sin xy}{x}\right) dy$$

$$= -\frac{1}{x}\int_0^\infty \sin xy \left[-\frac{\eta h e^{-\eta y}}{h^2+(\tau-y)^2} + \frac{e^{-\eta y}\, 2h(\tau-y)}{[h^2+(\tau-y)^2]^2}\right] dy$$

$$= \frac{1}{x}\int_0^\infty \frac{d}{dy}\left(\frac{\cos xy}{x}\right)[\quad]\, dy$$

$$= -\frac{1}{x^2}\left[-\frac{\eta h}{h^2+\tau^2} + \frac{2h\tau}{(h^2+\tau^2)^2}\right]$$

$$-\frac{1}{x^2}\int_0^\infty \cos xy \left[\frac{\eta^2 h e^{-\eta y}}{h^2+(\tau-y)^2} - \frac{\eta h e^{-\eta y}\, 2(\tau-y)}{[h^2+(\tau-y)^2]^2}\right.$$

$$\left. - \frac{2he^{-\eta y}[\eta(\tau-y)+1]}{[h^2+(\tau-y)^2]^2} + \frac{8he^{-\eta y}(\tau-y)^2}{[h^2+(\tau-y)^2]^3}\right] dy.$$

Thus

$$|I_1| < \frac{1}{x^2}\left[\frac{\eta h}{h^2+\tau^2} + \frac{2h\tau}{(h^2+\tau^2)^2}\right]$$

$$+ \frac{1}{x^2}\int_0^\infty \left[\frac{\eta^2 h}{h^2+(\tau-y)^2} + \frac{4h\eta|\tau-y|}{[h^2+(\tau-y)^2]^2} + \frac{10h}{[h^2+(\tau-y)^2]^2}\right] dy$$

$$< \frac{K(\eta, h, \tau)}{x^2},$$

where K is a constant which depends only on η, h, τ, and which, for $0 < \eta \leqslant 1$ say, satisfies $K(\eta, h, \tau) < K(h, \tau)$. A similar relation holds for I_2. Thus

$$|I(x, \eta)| < \frac{K}{x^2} \quad (0 < \eta \leqslant 1),$$

where $K = K(h, \tau)$ is independent of x and relevant η.

Given $\epsilon > 0$, choose $N = N(\epsilon) > 1$ so that the bounded non-decreasing G satisfies

$$\left(\int_N^\infty + \int_{-\infty}^{-N}\right) dG < \frac{\epsilon}{2K}.$$

Then

$$\left|\left(\int_{-\infty}^\infty - \int_{-N}^{N}\right)(1+x^2)\, I(x, \eta)\, dG\right| < \left(\int_{-\infty}^{-N} + \int_N^\infty\right) 2x^2\, \frac{K}{x^2}\, dG$$

$$< \epsilon. \qquad (106)$$

Further,
$$|I| < \int_{-\infty}^{\infty} \frac{h}{h^2+(\tau-y)^2}\,dy = \pi,$$
so that
$$\lim_{\eta\to 0}\int_{-N}^{N}(1+x^2)\,I(x,\eta)\,dG = \int_{-N}^{N}(1+x^2)\lim_{\eta\to 0} I(x,\eta)\,dG$$
$$= \int_{-N}^{N}(1+x^2)\,\pi e^{-h|x|}\cos\tau x\,dG. \qquad (107)$$

In virtue of (106) and (107), we know the value of the second member of (105), and this equation now becomes

$$\int_0^{\infty} A(t)\left[\frac{h}{h^2+(\tau-t)^2}+\frac{h}{h^2+(\tau+t)^2}\right]dt = \tfrac{1}{2}\int_{-\infty}^{\infty}(1+x^2)\,e^{-h|x|}\cos\tau x\,dG. \quad (108)$$

The limit of the first member in (108), as $h\to 0$, is manifestly $\pi A(\tau)$ p.p. in $\tau > 0$. This is the required result.

To prove (ii), we again confine our attention to the first of the desired relations. We first observe that (104) holds. Next, the preceding proof shows that the second member of (104) has a limit as $\eta\to 0$, namely the second member of (108). Thus

$$\lim_{\eta\to 0}\int_0^{\infty} e^{-\eta t}A(t)\left[\frac{h}{h^2+(\tau-t)^2}+\frac{h}{h^2+(\tau+t)^2}\right]dt = \tfrac{1}{2}\int_{-\infty}^{\infty}(1+x^2)e^{-h|x|}\cos\tau x\,dG.$$

Since, by hypothesis, $A(t) = o(t)$, the function multiplying $e^{-\eta t}$ in the integrand on the left is $o(1/t)$, and so, by the analogue of Tauber's theorem, (108) holds, this equation implying that the first integral is convergent for all $h > 0$, $\tau > 0$. We now observe that, as $t\to\infty$, the differential coefficient of the rational function

$$\phi = \left[(1+t^2)\left\{\frac{1}{h^2+(t+\tau)^2}+\frac{1}{h^2+(t-\tau)^2}\right\}\right]^{-1}$$

is positive for $t > t(h,\tau)$, and so this function is a bounded monotone function for such t. If we write

$$\int_{t_2}^{t_2}\frac{A(t)}{1+t^2}\,dt = \int_{t_1}^{t_2}A(t)\left[\frac{1}{h^2+(\tau-t)^2}+\frac{1}{h^2+(\tau+t)^2}\right]\phi\,dt,$$

apply the mean value theorem, and use the convergence of the first member of (108), we see that $A(t)/(1+t^2)$ is integrable in $(0,\infty)$. The desired result thus follows from (108) as before.

To prove (iii), consider the function $e^{-\eta|t|}A(t)$, where $A(-t) = A(t)$. For a fixed η, this function is absolutely integrable in $(-\infty,\infty)$. Hence

its Fourier integral is summable $(C, 1)$ p.p. The equation (102) enables us to write down its Fourier integral. We have, p.p. in t,

$$e^{-\eta|t|}A(t) = \frac{1}{2\pi^2}\lim_{\omega\to\infty}\frac{1}{\omega}\int_0^\omega d\lambda\int_{-\lambda}^{\lambda}\cos t\xi\,d\xi\int_{-\infty}^{\infty}\frac{\eta(1+x^2)}{\eta^2+(x-\xi)^2}\,dG(x). \quad (109)$$

Further, since, by hypothesis, $e^{-\eta_1|t|}A(t)$ is bounded in a neighbourhood of $t=0$, the partial Cesàro sums of its Fourier integral are bounded for $t=0$; *i.e.*

$$\left|\frac{1}{\omega}\int_0^\omega d\lambda\int_{-\lambda}^{\lambda}d\xi\int_{-\infty}^{\infty}\frac{\eta(1+x^2)}{\eta^2+(x-\xi)^2}\,dG\right| < K.$$

Since, however, the integrand with respect to λ is an increasing function of λ, we have

$$\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}\frac{\eta(1+x^2)}{\eta^2+(x-\xi)^2}\,dG < K. \quad (110)$$

We may here invert the order of integration, and thus infer that the expression (100) is finite. In virtue of (110), we see that

$$\int_{-\infty}^{\infty}\cos t\xi\,d\xi\int_{-\infty}^{\infty}\frac{\eta(1+x^2)}{\eta^2+(x-\xi)^2}\,dG$$

exists for all t and is equal to

$$\int_{-\infty}^{\infty}(1+x^2)\,dG\int_{-\infty}^{\infty}\frac{\eta\cos t\xi}{\eta^2+(x-\xi)^2}\,d\xi = \pi\int_{-\infty}^{\infty}(1+x^2)\,e^{-\eta|t|}\cos tx\,dG.$$

Hence, for a t for which (109) holds, we have the first of the required relations; the second is proved similarly. Finally, in virtue of (100), the second members in these relations are bounded continuous functions.

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A NOTATION FOR THE CONTACT-PRIMES AND FAMILIES OF CONTACT-QUADRICS OF CANONICAL CURVES

(*Second Paper.*)

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Introductory statement. Our previous paper* upon this subject was written in the belief that, if ever the geometry of contact loci is to be profitably studied, some simpler notation than the unwieldy "marks" commonly used to denote the characteristics of theta-functions must be found. We suggested the extended use of "sets of $2p+2$ letters", already familiar for curves of genus 3 and 4. Each separate letter and each set of letters could represent a characteristic and take the place of a mark, a set of letters meaning the sum of the individual letters. Theorems proved in the treatises upon Abelian functions and theta-functions could be translated into the new notation, which was seen to be both simpler and more informative.

But it has become clear to us that the laborious proofs in these treatises are unnecessary also; the new notation leads more readily to the results which we require. To show this is the object of the present paper; it is complete in itself, but reference may be made to the previous paper. We propose to confine ourselves to contact loci of the first and second degrees in relation to canonical curves, but we consider also one method by which a canonical curve is transformed into one of lower genus. The first eight of the following sections explain the new notation.

Characteristics, marks, and sets of letters.

1. *Definitions.* In connection with algebraic curves of genus p, the term a *mark* is understood to mean a symbol formed of $2p$ integers

* See F. Bath and H. W. Richmond, *Journal London Math. Soc.*, 7 (1932), 183–192, where references are given.

α_n, β_n $(n = 1, 2, 3, \ldots, p)$ arranged in two rows thus:

$$J \equiv \begin{pmatrix} \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n, \ldots, \alpha_p \\ \beta_1, \beta_2, \beta_3, \ldots, \beta_n, \ldots, \beta_p \end{pmatrix}.$$

The integers α_n, β_n, known as the elements of the mark J, are distinguished only as even or odd; it may be assumed that each even integer is reduced to 0 and each odd integer to 1. We define two other numbers (J) and $(J \mid J')$ in relation to a single mark J and to a pair of marks J and J':

$$(J) \equiv \Sigma(\alpha_n \beta_n) \pmod 2; \qquad (J \mid J') \equiv \Sigma(\alpha_n \beta_n' + \beta_n \alpha_n') \pmod 2,$$

dashes distinguishing the elements of J'. The mark J is said to be *even* or *odd* according as (J) is even or odd. The two marks J and J' are said to be *evenly* or *oddly related* (or to be *syzygetic* or *asyzygetic*) according as $(J \mid J')$ is even or odd. We are again concerned only with the parity of these numbers and are at liberty to assume them reduced to 0 or 1. Since $J \mid J) = 2(J)$, an even number, any mark is evenly related to itself.

The sum of two or more marks is found by adding together the elements in each of the $2p$ places (and reducing each sum to 0 or 1) so as to form a new mark. The sum of two different marks is a third mark; but the result of adding any mark to itself is an exceptional mark, known as the zero-mark, in which all the elements are zero. To add the zero-mark to any other leaves the latter unaltered. In summing a number of marks, a mark that is repeated may be omitted.

The total number of marks, the zero-mark being excluded, is $2^{2p} - 1$. Clearly, if we take the $2p$ marks in which all the elements except one are 0, every mark can be expressed in one and only one way as a sum of certain of these $2p$ marks.

2. It is, however, better to express all the marks in terms of what is known as a "*fundamental system*" of $2p+1$ marks. From this point onward, a particular value, usually 8, will at times be given to p; general results will be as well understood from a complete table for $p = 8$ as from the conventional tables having sundry rows and columns filled with a series of dots. Consider the system of 17 $(= 2p+1)$ marks shown in Table I, and denoted for the present by $B, C, D, E, \ldots, R, S, T, U$. Each of these capital letters will be found in the table, and the mark which it represents can be read from the two rows of eight figures (all either 0 or 1), one slightly above and one slightly below the level of the capital letter. Each row of figures is the upper row of one mark and the lower row of another. It will be seen that each two of the marks $B, C, D, \ldots, T, U$

are oddly related, and that the sum of them all, but not of any smaller number, is the zero-mark. A family of $2p+1$ marks possessing these properties is defined as a *fundamental system* of marks.

3. The marks in which every element except one is zero are seen to be

$$U,\ T,\ U+T+S,\ U+T+R,\ U+T+S+R+Q,\ U+T+S+R+P,\ \ldots.$$

TABLE I.

	1	1	1	1	1	1	1	1
B								ab
	1	1	1	1	1	1	1	1
C								ac
	0	1	1	1	1	1	1	1
E								ae
	0	0	1	1	1	1	1	1
G								ag
	0	0	0	1	1	1	1	1
K								ak
	0	0	0	0	1	1	1	1
M								am
	0	0	0	0	0	1	1	1
P								ap
	0	0	0	0	0	0	1	1
R								ar
	0	0	0	0	0	0	0	1
T								at
	0	0	0	0	0	0	0	0
U								au
	0	0	0	0	0	0	0	1
S								as
	0	0	0	0	0	0	1	1
Q								aq
	0	0	0	0	0	1	1	1
N								an
	0	0	0	0	1	1	1	1
L								al
	0	0	0	1	1	1	1	1
H								ah
	0	0	1	1	1	1	1	1
F								af
	0	1	1	1	1	1	1	1
D								ad
	1	1	1	1	1	1	1	1
B								ab
	1	1	1	1	1	1	1	1

From this (or otherwise) we deduce that every mark can be expressed in one way and only one as a sum of certain of the marks $C, D, E, \ldots, T, U$; but this implies that the same mark is the sum of the remaining letters. Every mark except the zero-mark can therefore be expressed in just two ways as a sum of certain of the fundamental marks $B, C, D, \ldots, S, T, U$, the two sums being complementary, *i.e.*, containing between them all the

$2p+1$ fundamental marks. One sum consists of an even number and the other of an odd number of marks.

An easy rule determines whether two marks expressed in this way are evenly or oddly related. Suppose that two marks J and J' are respectively sums of m and m' fundamental marks, and that r is the number of fundamental marks appearing in both sums. $(J|J')$, being linear in the elements of both J and J', can be resolved into mm' expressions of the same kind, in each of which one of the m fundamental marks whose sum is J is associated with one of the m' whose sum is J'. Each of the mm' expressions is odd (because each two fundamental marks are oddly related) except r, in which the two marks are the same. Hence we have a rule and an important corollary:

"*J and J' are evenly or oddly related according as $mm'-r$ is even or odd*".

COROLLARY. *Every mark J is evenly related to an odd number of the marks in the fundamental system, and is oddly related to the remainder, an even number. Moreover, J is the sum of the marks in either group.*

4. Among the $2^{2p}-1$ marks there are many fundamental systems, as defined in § 2, any one of which would serve just as well as

$$B, C, D, \ldots, T, U, V.$$

We require a rule for finding these; or (to take a broader view) for deriving from any set of marks the equivalent sets which possess the same properties.

Consider the effect of adding a mark j to either or both of two marks J and J'. Under what conditions is the parity of the relation between J and J' unaffected? When j is added, (a) to J alone; (b) to J' alone; (c) to both J and J'; $(J|J')$ is increased by, (a) $(j|J')$; (b) $(J|j)$; (c) $(j|J')+(J|j)+(j|j)$; but $(j|j)$ is even and may be dropped. The parity of $(J|J')$ is unaltered if all the increments are even, which occurs if j is evenly related to both J and J'. But it occurs also under other conditions. For (a) is even if j is oddly related to J and not to J', (b) is even if j is oddly related to J' and not to J, (c) is even if j is oddly related to both J and J'. Thus we see that the parity of the relation $(J|J')$ is unaffected if j is added to whichever of the two, J and J', is oddly related to j. This result establishes a very general theorem:

Given any group of marks. Select at random a mark j, and derive a second group by adding j to those members of the first group which are oddly related to j, leaving the others untouched. The relation between two members

of the second group is of the same parity as that between the corresponding members of the first group.

The $2^{2p}-2$ marks (other than j and the zero-mark) fall into pairs whose sum is j, and the two members of a pair are both evenly related or both oddly related to j. In the latter case we interchange them; in the former we leave them as they were.

In particular, if the first group is a fundamental system, so is the second.

5. *A two-letter notation.* The importance of these results in their application to geometry lies in the fact that marks distinguish the $2^{2p}-1$ families of contact-quadrics possessed by the canonical curve. But marks have a second application to geometry which leads us to study another kind of property, the evenness or oddness of a single mark. Odd marks correspond to and are used to distinguish the contact-primes (finite in number) which touch the canonical curve wherever they meet it; and a correspondence between both odd and even marks and certain primes will be explained later. In this interpretation two primes, each separately represented by a mark, form in combination a contact-quadric which belongs to one of the former families; and the mark which represents the family is the sum of the two marks which represent the primes. For this and for other reasons it is convenient to express $B, C, D, \ldots, S, T, U$ as a sum of two marks. Let any one mark, chosen for the present completely at random, be called a; then there are definite marks $b, c, d, \ldots, s, t, u$ which, when added to a, give the marks $B, C, D, \ldots, S, T, U$. We may write either

$$b = a+B; \quad \text{or} \quad a+b = B; \quad \text{or} \quad a+b+B = 0 \quad \text{(the zero-mark)}.$$

Now let it be agreed that two or more letters written in sequence shall denote the sum of the marks represented by the single letters, so that

$$B = ab; \quad C = ac; \quad D = ad; \quad \ldots; \quad S = as; \quad T = at; \quad U = au;$$

a notation that has already been entered in Table I. We now have $2p+2$ letters $a, b, c, d, \ldots, s, t, u$, each representing a mark, the mark represented by a being arbitrary. The sum of them all is the zero-mark, *i.e.*

$$a+b+c+d+\ldots+s+t+u = abcd\ldots stu = 0 \quad \text{(the zero-mark)}.$$

Each mark formerly denoted by a sum of $B, C, D, \ldots, S, T, U$ in two complementary ways is now represented in two complementary ways by symbols denoting the sum of *an even number* of the letters $a, b, c, \ldots, s, t, u$.

Any two of these marks are seen to be evenly or oddly related according as the number of letters common to their representative symbols is even or odd. It is immaterial which of the two complementary symbols is used for either mark. The rule enunciated at the end of § 4 remains valid; but it must be understood that the selected mark j is to be one represented by an even number of letters $a, b, c, \ldots, s, t, u$. Should j be represented by two letters only, the rule proves that the two may be interchanged. The zero-mark may be regarded as corresponding to complementary sets consisting of all or none of the $2p+2$ letters. This notation for marks, as used to denote families of contact-quadrics, will always be used in what follows.

6. The even or odd relationship of two marks is plainly shown when the marks are expressed by an *even* number out of $2p+2$ letters. Equally simple rules, showing whether an individual mark is even or odd, exist for marks represented by an *odd* number of these letters. From the definition of (J) we derive a formula for the parity of the sum of two or more marks:

$$(J+J') = \Sigma\,[(\alpha_n+\alpha_n')(\beta_n+\beta_n')] \equiv (J)+(J')+(J\,|\,J') \pmod 2;$$

so

$$(J+J'+J'') \equiv (J)+(J')+(J'')+(J\,|\,J')+(J\,|\,J'')+(J'\,|\,J'') \pmod 2.$$

Hence
$$(b) \equiv (a+B) \equiv (a)+(B)+(a\,|\,B) \pmod 2,$$

or
$$(B)+(a\,|\,B) \equiv (a)+(b) \pmod 2.$$

From this result the rules follow: for, working always to modulus 2,

$$\begin{aligned}(a+B+C) &\equiv (a)+(B)+(C)+(a\,|\,B)+(a\,|\,C)+(B\,|\,C)\\ &\equiv (a)+(b)+(a)+(c)+(a)+(B\,|\,C).\end{aligned}$$

Now $a+B+C$ is abc; and $(B\,|\,C)$ is 1, because B and C are two marks of the fundamental system; hence we have, for the parity of abc,

$$(abc) \equiv (a)+(b)+(c)+1 \pmod 2.$$

Again,

$$\begin{aligned}&(a+B+C+D)\\ &\equiv (a)+(B)+(C)+(D)+(a\,|\,B)+(a\,|\,C)+(a\,|\,D)+(B\,|\,C)+(B\,|\,D)+(C\,|\,D)\\ &\equiv (a)+(b)+(a)+(c)+(a)+(d)+(a)+(B\,|\,C)+(B\,|\,D)+(C\,|\,D).\end{aligned}$$

Since the last three brackets represent odd numbers,

$$(bcd) \equiv (b)+(c)+(d)+1 \pmod 2.$$

In the same way, when we take the sum of a with 4, 5, 6, 7, ... of the marks $B, C, D, E, \ldots$, the number of brackets at the end of the line is 6, 10, 15, 21, ... Hence

$$(abcde) \equiv (a)+(b)+(c)+(d)+(e) \pmod 2,$$

$$(bcdef) \equiv (b)+(c)+(d)+(e)+(f) \pmod 2,$$

$$(abcdefg) \equiv (a)+(b)+(c)+(d)+(e)+(f)+(g)+1 \pmod 2,$$

$$(bcdefgh) \equiv (b)+(c)+(d)+(e)+(f)+(g)+(h)+1 \pmod 2.$$

The parity of a mark which is the sum of 5 *or* 9 *or* 13 ... *of the marks* $a, b, c, d, \ldots$ *is equal to the sum of the parities of the components; the parity of a mark which is the sum of* 3 *or* 7 *or* 11 ... *of them is opposite to the sum of the parities of the components.* There appears to be no such rule for the marks which are sums of an even number of the marks $a, b, c, \ldots$.

7. Unless something is known about the parities of $a, b, c, d, \ldots$, it is not easy to make practical use of this result. It will now be proved that there is one definite value of a which makes $b, c, d, e, \ldots$ have the same parity; if we give a this value (it will be remembered that up to now the value of a has been left wholly arbitrary), and derive the values of $b, c, d, \ldots$, marks which are sums of $b, c, d, \ldots$ in sets of 5 or 9 or 13 ... will be of the same parity as $b, c, d, \ldots$, while marks which are their sums in sets of 3 or 7 or 11, ... will be of the opposite parity. Each mark may equally well be represented as the sum of the rest of the $2p+2$ letters; this complementary sum will include a.

It remains to find this value of a. Suppose that a is the typical mark quoted in § 1; to find $b, c, d, \ldots$ we have to add a to $B, C, D, \ldots$ given in Table I. The values of b, c, d differ only in the first elements of each row, and the parities of b, c, d cannot be the same unless the product of the first two elements is the same for each of them. Hence

$$(\alpha_1+1)(\beta_1+1) \equiv \beta_1(\alpha_1+1) \equiv \alpha_1(\beta_1+1) \pmod 2,$$

i.e.

$$\alpha_1 \equiv \beta_1 \equiv 1 \pmod 2.$$

Replace α_1 and β_1 by 1 and consider the values of b, e, f, found by adding a to B, E, F of Table I. After the first two elements in each row the three

marks are identical, and the condition that e and f as well as c and d are of the same parity as b is that, to mod 2,

$$0\times 0+(\alpha_2+1)(\beta_2+1)\equiv 1\times 1+\beta_2(\alpha_2+1)\equiv 1\times 1+\alpha_2(\beta_2+1),$$

i.e. $$\alpha_2\equiv\beta_2\equiv 0 \pmod 2.$$

Replace α_2 and β_2 by 0 and continue the process. It is found that α_3 and β_3 are 1, α_4 and β_4 are 0, and so on alternately. Finally, we prove that for the value

$$a=\begin{pmatrix}1&0&1&0&1&0&1&0&\ldots\\1&0&1&0&1&0&1&0&\ldots\end{pmatrix} \quad (=\theta \text{ say}),$$

and for no other value of a, the marks b, c, d, ..., s, t, u are either all of them even or all of them odd. With this value of a we are in a position to draw up a table of the marks represented by the $2p+2$ single letters a, b, c, ..., s, t, u.

Occasionally we shall refer to this mark and its property without assigning to it the letter a. Every fundamental system of $2p+1$ marks has one definite mark associated with it, whose partners are all of the same parity. We shall denote the mark by θ, and describe it as the *leading mark* of the fundamental system.

8. Table II, like Table I, is a complete statement for a particular value of p, viz. $p=8$. Table II might supersede Table I, because the values of ab, ac, ... contained in Table I can be calculated at once from Table II. In fact, Table II shows every detail of the connection between the two notations, marks and sets of letters. To extend the table to values of p greater than 8 needs no explanation; for values less than 8, the table actually contains all the details. For, if the last column is erased, the four central rows become identical; t and u and the first and last of these rows can be dropped, as well as the last column. What is left is Table II for $p=7$. Again, if the last two columns, together with r, s, t, u, and four of the six central rows (which have now become identical) are erased, what is left is the table analogous to Table II for $p=6$. Similarly when p is 5 or 4 or 3 or 2. Table II is a compendium of results for all values of p up to and including $p=8$. A similar table of p columns and $2p+4$ rows may be constructed for any chosen value of p.

The structure of the table is plain and the facts which are embodied in it, *e.g.* that b, c, d, ..., s, t, u are of the same parity and that the addition of a to each of them gives Table I, are readily verified. We cannot assert

that the notation is always the best possible, but its applicability to all values of p is in its favour. To represent the families of contact-quadrics (or the period-characteristics of theta-functions) by complementary sets of letters, even in number, seems both simple and satisfactory; for this reason Table I is retained. But complementary sets, odd in number, some of which represent the contact-primes of the canonical curve and all

TABLE II.

	0	1	0	1	0	1	0	1
b								
	0	1	0	1	0	1	0	1
c								
	1	1	0	1	0	1	0	1
e								
	1	0	0	1	0	1	0	1
g								
	1	0	1	1	0	1	0	1
k								
	1	0	1	0	0	1	0	1
m								
	1	0	1	0	1	1	0	1
p								
	1	0	1	0	1	0	0	1
r								
	1	0	1	0	1	0	1	1
t								
	1	0	1	0	1	0	1	0
a								
	1	0	1	0	1	0	1	0
u								
	1	0	1	0	1	0	1	1
s								
	1	0	1	0	1	0	0	1
q								
	1	0	1	0	1	1	0	1
n								
	1	0	1	0	0	1	0	1
l								
	1	0	1	1	0	1	0	1
h								
	1	0	0	1	0	1	0	1
f								
	1	1	0	1	0	1	0	1
d								
	0	1	0	1	0	1	0	1
b								
	0	1	0	1	0	1	0	1

of which represent theta-characteristics, though simple when p has an even value, are less satisfactory when p is odd. It is a fact not to be ignored that for the contact-primes of the canonical curve when $p = 3$ (the bitangent lines of the plane quartic), a different notation (based it is true upon eight letters) is almost universally adopted. This needs consideration; but

the geometrical aspect of the subject of our investigation should first be clearly stated.

Canonical curves.

9. The canonical curve of genus p is of order $2(p-1)$ and lies in a space of $p-1$ dimensions. Through it pass $\frac{1}{2}(p-2)(p-3)$ linearly independent quadrics. Associated with the curve are $2^{2p}-1$ families of contact-quadrics, each family corresponding to one of the $2^{2p}-1$ period-characteristics, which touch the curve at $2(p-1)$ points; and in each family $2^{p-2}(2^{p-1}-1)$ quadrics factorize into two primes which touch the curve at $p-1$ points. The total number of contact-primes is $2^{p-1}(2^p-1)$, and each corresponds to one of the odd theta-characteristics in such a way that the characteristic of the family of quadrics to which the two primes in combination belong is the sum of the characteristics of the primes. This will be true also of the corresponding sets of letters which we have introduced in place of marks to represent characteristics. Geometry thus gives a satisfactory interpretation of the fact that any two odd theta-characteristics combine to give a period-characteristic.

But any two theta-characteristics, whether odd or even, combine in this way. We wish to offer a suggestion under which both odd and even theta-characteristics correspond to contact-primes of a modified canonical curve, and any two primes combine into a special contact-quadric. It explains at the same time one possible means by which a canonical curve, together with its concomitant contact-primes and quadrics, can be transformed into the canonical curve of the next lower genus. Further, the whole process, stage by stage, has a counterpart in simple modifications of the notation we have introduced in §§ 1–7.

For any curve of genus p (regarded preferably as an intersection of loci) to be modified into one of genus $p-1$, the method that first comes to mind is for the curve to acquire a double point. Projection from this point derives a second curve in space of one dimension fewer, of which the genus is less by one and the order by two, exactly what is desired for the canonical curve. But the matter is made clearer by a further modification. Suppose that the first canonical curve acquires a *cusp*. Certain of the contact-primes coalesce (not by twos, as in the case of a double point, but by threes) into primes which go through the cusp and touch the curve at $p-2$ other points; and which, when the curve is projected from the cusp, give rise to the contact-primes of the canonical curve of genus $p-1$. If the canonical curve of genus 4, the sextic curve of intersection of a cubic and a quadric surface in space of three dimensions, acquires a cusp, certain of the 120 tritangent planes move up to coincidence by threes, ultimately passing

through the cusp and touching the curve at two other points. After projection from the cusp, they provide the twenty-eight bitangents of the plane quartic and account for eighty-four of the 120 tritangent planes. There remain thirty-six planes which do not pass through the cusp, but touch the cusped curve thrice at ordinary points. The curve is of genus 3, and its contact-planes, some with three ordinary points of contact, some passing through the cusp and having two ordinary points of contact, interpret geometrically all the thirty-six even and the twenty-eight odd theta-characteristics of genus 3. Any two of the planes in combination form a quadric belonging to one of the families of contact-quadrics. Projection from the cusp as vertex gives us the plane quartic and its bitangents, together with an interpretation of the even theta-characteristics. In space we need use nothing more than planes and quadric loci. For any value of p both the odd and the even theta-characteristics can be interpreted by primes touching the canonical curve of genus $p+1$ (a curve of order $2p$ in space of p dimensions), *when that curve is modified so that it has a cusp, and its genus reduced by one.*

Consideration of Table II; even values of the genus.

10. Some properties of contact-primes of this cusped curve of genus p and order $2p$ will first be stated. Any two of them in combination form a contact-quadric, the set of letters distinguishing the family to which it belongs being the sum of the sets distinguishing the contact-primes. (The sum of two sets of letters, each odd in number, is a set of letters even in number.) If we select a family of contact-quadrics, all the 2^{2p} contact-primes fall into pairs which in combination form a quadric of the family; two such primes will be described as "partners" in the family of quadrics. Thus the result of § 7 is to prove that the fundamental system of families shown in Table I is associated with a definite prime (afterwards taken to be a) whose partners are all of the same parity. One prime is associated in this way with every fundamental system of families. The rule for changing the notation at the end of §§ 4 and 5 can be extended so as to apply to contact-primes. In its complete form the rule may be formulated thus:

"Select any set of letters, j, even in number. To change the notation, add j to those sets even in number (*i.e.* families of contact-quadrics) which are oddly related to j, and to those sets odd in number (*i.e.* contact-primes) in which the addition does not produce a change of parity. Leave the other sets as they are." Thus the loci whose symbols are changed fall into pairs whose symbols are interchanged. The chief property of two

quadrics of the same family, that their points of contact lie on a quadric, gives the result that four contact-primes represented by symbols whose sum is zero have their $4(p-1)$ points of contact on a quadric, over and above the quadrics which pass through the curve.

Table II contains a *possible* scheme of notation and enables us to verify these statements; we shall now examine that table, partly with the object of deciding whether a more convenient notation can be discovered. For an even value of the genus, little fault can be found with it. Not only is the parity of a the same as that of b, c, d, ..., but a takes a place with b, c, d, ... as members of a completely homogeneous system of $2p+2$ letters having zero sum. The separate letters denote contact-primes (or theta-characteristics) of the same parity as $\frac{1}{2}p$; sets of 5, 9, 13, ... of these letters denote the rest of the primes of that parity; sets of 3, 7, 11, ... of the letters denote the primes of the opposite parity. For an even value of p, the odd theta-characteristics (representing contact-primes of the canonical curve of genus p) correspond to sets of $p-1$, $p-5$, $p-9$, ... letters, or to the residual sets which consist of $p+3$, $p+7$, $p+11$, ... letters; the even theta-characteristics (represented as primes by help of a cusped curve of order $2p$) correspond to sets of $p+1$, $p-3$, $p-7$, ... letters, or to the residual sets which consist of $p+1$, $p+5$, $p+9$, ... letters. In the "mark" notation confusion occurs from the use of the same symbols for two different entities, primes and families of quadrics; particularly in the matter of the zero-mark, which is abnormal in the second sense and yet does not differ from other even marks in the first. In the new notation this confusion does not arise.

Odd values of the genus.

11. When p is odd, the parity of a is opposite to that of the other letters b, c, d, ..., t, u; and we have to distinguish between sets of letters odd in number which do or do not include a. The parity of b, c, d, ..., t, u is that of $\frac{1}{2}(p-1)$; other sets of the same parity are given by these letters in sets of 5, 9, 13, ..., or by the residual sets of 3, 7, 11, ... letters which include a. For an odd value of p, the odd theta-characteristics (representing contact-primes of the canonical curve of genus p) correspond to sets of $p\pm2$, $p\pm6$, ... letters excluding a, or to sets of p, $p\pm4$, $p\pm8$, ... letters including a; the residual of any member of the first group is to be found among the second. Similarly for the even theta-characteristics. Now this compares very unfavourably with the simple scheme in § 10. To abstain from using the letter a entirely would be possible, and to do so would undeniably make the *statement* of the rules simpler and clearer;

but when we discuss particular values of p in detail it is a great advantage to have a choice of two complementary sets of letters. It is convenient here to quote Tables I and II for the value $p = 3$, and Table II for $p = 5$,

TABLE I. ($p = 3$.)

	1	1	1
ab			
	1	1	1
ac			
	0	1	1
ae			
	0	0	1
ag			
	0	0	0
ah			
	0	0	1
af			
	0	1	1
ad			
	1	1	1
ab			
	1	1	1

TABLE II. ($p = 3$.)

	0	1	0		
b				or	θab
	0	1	0		
c				or	θac
	1	1	0		
e				or	θae
	1	0	0		
g				or	θag
	1	0	1		
a				or	θ
	1	0	1		
h				or	θah
	1	0	0		
f				or	θaf
	1	1	0		
d				or	θad
	0	1	0		
b				or	θab
	0	1	0		

TABLE II. ($p = 5$.)

	0	1	0	1	0		
b						or	θab
	0	1	0	1	0		
c						or	θac
	1	1	0	1	0		
e						or	θae
	1	0	0	1	0		
g						or	θag
	1	0	1	1	0		
k						or	θak
	1	0	1	0	0		
m						or	ϵam
	1	0	1	0	1		
a						or	θ
	1	0	1	0	1		
n						or	θan
	1	0	1	0	0		
l						or	θal
	1	0	1	1	0		
h						or	θah
	1	0	0	1	0		
f						or	θaf
	1	1	0	1	0		
d						or	θad
	0	1	0	1	0		
b						or	ϵab
	0	1	0	1	0		

with one modification, viz. to denote by θ the mark found in § 7 whose

partners are all of the same parity. To denote it by a and its partners by $b, c, d, \ldots$ serves excellently when p is even; when p is odd and the proper procedure is doubtful, it is better to leave ourselves as much freedom as possible; we may put a for θ later, if it is desirable. For the present we have gone back to the conditions of § 5; in which all that we know about $a, b, c, \ldots$ is in Table I; sums of an even number of these marks are known, but not the individual marks. The mark θ may be described as the *leading* mark of the fundamental system given in Table I. Its partners in that system are θab, θac, θad, ... as entered in Table II on the right.

With this notation we see that all theta-characteristics of the same parity as θ are denoted by θ with sets of 4, 8, 12, ... of the letters $a, b, c, \ldots$, and all those of the opposite parity are denoted by θ with sets of 2, 6, 10, ... of the letters. The complementary set of letters is included in the same group. The value of one of the letters $a, b, c, \ldots$ may be chosen at will. Thus we may take a to be the same as θ, and so recover the results stated at the beginning of this section. Or we may take a to be the mark all of whose elements are 0; if so, a is simply omitted from all the formulae.

For an odd value of p, the odd theta-characteristics are denoted by θ with sets of $p-1, p-5, p-9, \ldots$ of the letters $a, b, c, \ldots$, or with the residual sets of $p+3, p+7, \ldots$ letters. The even theta-characteristics are denoted by θ with sets of $p+1, p-3, p-7, \ldots$ letters, or with the residual sets of $p+1, p+5, p+9, \ldots$ letters. The presence of θ avoids the confusion between the notation for contact-primes and theta-characteristics and that for families of quadrics.

Hesse's notation for the double tangents of the plane quartic curve.

12. For genus 3 the canonical curve is the plane quartic. Its contact-primes, the twenty-eight double tangents, were studied by Hesse and others before Abelian functions had been applied to geometry, and many properties of these lines and of the sixty-three families of contact-conics were discovered. Hesse saw a connection between the double tangents and the lines which join each two of eight points common to three quadrics, and so was led to represent each double tangent by two out of a system of eight letters, a notation which is found to have far more than a superficial resemblance to that in § 11; for example, the sets of four double tangents whose eight points of contact lie on a conic, and the six pairs which in combination count as conics of the same family, are alike in the two notations. This is surprising, because Hesse could give no significance to the idea of adding symbols, while nothing in the present paper has suggested a connection between the single letters and points in space.

The notation of Hesse for the double tangents and that of § 11 are not identical, though very nearly. Where Hesse uses two out of the eight letters a, b, c, d, e, f, g, h, Table II for $p = 3$ gives $\theta ab, \theta ac, \ldots, \theta gh$; here the single letters stand for marks, θ being definite not only for $p = 3$ but for all values of p. But θ is definite only in the sense that it is determined without ambiguity when a fundamental system has been decided upon. As it happens, Table I shows a fundamental system (copied from a book upon Abelian functions with unimportant alterations), which we have accepted as the basis of all later results; but the initial choice of the particular fundamental system was almost accidental. A rule has already been given in § 10 by which we can derive other fundamental systems with the appropriate θ for each; it proves that the parity of θ must always be the same for the same value of p. If $p \equiv 3$ or 0 to mod 4, θ is even, if $p \equiv 1$ or 2, θ is odd. Once more we have a method of simplification applicable to certain values of p, but not to all. For if θ is even, certain fundamental systems have the mark whose elements are all 0 for their leading mark; in a sum such as θab the θ may be dropped. It amounts to this: that, when p is 3 or 7 or 11 ..., we can find systems of $2p+1$ marks which (regarded as period-characteristics) form a fundamental system, and (regarded as theta-characteristics) are all odd. (The same holds when p is 4 or 8 or 12 ..., if the last word is changed to even.) It does not seem necessary to follow up this notion except for the value 3 of the genus; but it is an important fact that Hesse's notation for the double tangents of the quartic fits in perfectly with one of the schemes of notation in terms of marks and sets of letters here developed.

Table II for $p = 3$ in the last section shows that *bef* is the mark whose elements are all 0; hence, if we apply the rule in § 10, taking "j" as *abef* or *cdgh*, we shall arrive at a fundamental system of the kind we require. We obtain seven marks which we denote by *ab*, *ac*, *ad*, *ae*, *af*, *ag*, *ah*; each two are oddly related, and the seven form a fundamental system.

$$\underset{ab}{\begin{pmatrix}1&1&1\\1&1&1\end{pmatrix}}\quad\underset{ad}{\begin{pmatrix}1&1&0\\0&1&0\end{pmatrix}}\quad\underset{ah}{\begin{pmatrix}1&0&1\\1&0&0\end{pmatrix}}\quad\underset{ae}{\begin{pmatrix}0&1&1\\0&0&1\end{pmatrix}}\quad\underset{ac}{\begin{pmatrix}0&1&0\\1&1&0\end{pmatrix}}\quad\underset{ag}{\begin{pmatrix}1&0&0\\1&0&1\end{pmatrix}}\quad\underset{af}{\begin{pmatrix}0&0&1\\0&1&1\end{pmatrix}}.$$

The sixty-three families of contact-conics can all be derived from these seven; each is represented by two or four of the letters a, b, c, d, e, f, g, h. The seven marks are all odd; obviously they are the partners of the mark whose elements are all 0 in these seven families of contact-conics. All the twenty-eight double tangents (contact-primes) are represented by a pair of the eight letters. There are also thirty-six even theta-characteristics

(often ignored in books on algebraic geometry), thirty-five of which correspond to sets of four letters, while the last corresponds to a set of all or none. (It should be stated that there are eight fundamental systems of this sort: *ca*, *cb*, *cd*, *ce*, *cf*, *cg*, *ch* is another such system.)

For $p = 7$, one such family of fifteen marks is made up of the following marks, taken from Tables I and II in §§ 3 and 8:

$$B,\ c,\ d,\ E,\ F,\ g,\ h,\ K,\ L,\ m,\ n,\ P,\ Q,\ r,\ s,$$

the tables being adapted to $p = 7$ by the omission of the last column. In a similar way for even values of p which are divisible by 4 we can make use of fundamental systems for which the leading mark θ is the zero mark. If p is 4, and the tables in §§ 3 and 8 are adapted to this value, $B, c, d, E, F, g, h, K, L$ form a fundamental system; they are also the partners of the zero mark in the families of conics they represent, and are all even, like the zero mark. No such simplification is possible if p is even but not divisible by 4.

Diminution of the genus to the next lower value.

13. The standard notation for the plane quartic and its extension to other values of the genus could not be passed over, but for the purpose of the present paper they must be regarded as side-tracks. They simplify the work for certain values of p; but our aim is to find methods applicable to all values of p and means of connecting different values of p. It was seen in § 8 that Tables I and II, drawn up for $p = 8$, could be modified by proper omissions into the analogous tables for $p = 7$. The last column of each table was removed, t and u became identical with a in Table II, and at and au in Table I became the zero mark. We omit the letters t and u, two of the central rows, and the last column in each table; this leaves us with the same two tables for $p = 7$. The same process repeated gives the two tables for $p = 6$, and so on.

In § 9 a possible geometrical conception of the modification of the canonical curve of genus p into that of genus $p-1$ was suggested. The curve of genus p acquires a cusp and so becomes a curve of genus $p-1$. of which the projection is the canonical curve of that genus. The contact-primes of the original curve separate themselves into two groups; some of them remain as primes which touch the curve at $p-1$ points, others become primes which pass through the cusp and touch the curve at $p-2$ points; but each prime in the latter group consists of three primes which have moved up to coincidence. Projection from the cusp gives the canonical curve of genus $p-1$; its contact-primes are derived from the second

group, and correspond to the odd theta-characteristics of genus $p-1$. The first group of primes corresponds to the even theta-characteristics, but loses the simple property that distinguishes them when projected. In order to interpret both odd and even theta-characteristics geometrically, it seems best to use the two types of contact-primes of the cusped curve of order $2p$ and genus p in space of p dimensions.

Now it will be found that the change of the notation described in the first paragraph of this section is in complete harmony with the modification of the canonical curve described in the second paragraph. The most satisfactory method of discussing this would be to trace out the effects which the vanishing of two of the periods must have upon the functions and the algebraic relations that connect them. But even if we had succeeded in doing this, it would be far removed from the rest of this paper. To justify the statement, we shall consider more fully what occurs when two of the letters become identical with a, the genus being thereby reduced by one. Tables I and II contain all the necessary facts, and we discuss the reduction of the genus from 8 to lower values.

When the last two letters (t and u when p is 8, but the results are always of much the same nature) become indistinguishable from a, the complementary sets of letters which form our notation fall into two categories: (α) all three letters a, t, u are to be found in one of the two complementary sets; (β) two of the three letters are to be found in one set and the third letter in the complementary set. In (α), acting upon the rule that a repeated letter may be omitted in any sum, we omit t and u and leave only a. In (β) we omit the two letters for the same reason and write the single letter as a. Thus the two initial complementary sets are replaced by complementary sets of letters whose number is lessened by two. If the number of letters in each of the initial sets is even, so is the number in the final sets; if odd, the number in the final sets is odd. In case (β), it is clear that three of the initial complementary sets will give the same final complementary sets, viz. three derived by permutation of the letters a, t, u; but in (α) there is no such ambiguity, there is a one-to-one correspondence between the initial and final sets.

We pass on to apply this, first to the value $p = 8$, and then to lower values.

14. *The canonical curve of genus* 8, *the* C_8^{14} *in* [7]. The notation employs eighteen letters, $a, b, c, d, e, f, g, h, k, l, m, n, p, q, r, s, t, u$. The $2^{16}-1$ families of contact-quadrics correspond to the sets of two, four, six, or eight letters, or to the residual sets of sixteen, fourteen, twelve, or ten letters. The separate letters denote eighteen of the even theta-characteristics;

and with the sets of five or nine letters denote the total number $2^7(2^8+1)$. Sets of three or seven letters, or the residual sets of fifteen or eleven letters, represent the odd theta-characteristics and the contact-primes, in number $2^7(2^8-1)$.

The cusped C_7^{14} *in* [7]. Consider the effect upon the contact-primes and families of contact-quadrics when t and u become identical with a. The former fall into two categories (α) and (β). Those in (α) correspond to complementary sets of letters in which one set consists of three, seven, eleven, or fifteen of the letters $b, c, d, \ldots, r, s$ and the other of a, t, u and twelve, eight, four, or none of these letters. After t and u have been dropped, as explained above, there remain complementary sets of the sixteen letters from a to s, consisting of three, seven, eleven, or fifteen letters without a, or sets of thirteen, nine, five, or one letters including a. No question of coincidence of these primes arises; they still are contact-primes touching the cusped curve at seven points. In category (β) the sets consist of two of the letters a, t, u with one, five, nine, or thirteen of $b, c, d, \ldots, r, s$, and the third letter with fourteen, ten, six, or two of $b, c, d, \ldots, r, s$. These (β) sets give primes which coincide in threes, a fact which we explain by one point of contact being now at the cusp and six at ordinary points of the curve. When t and u are dropped, these contact-primes correspond to complementary sets of the sixteen letters a to s consisting of one, five, nine, or thirteen letters without a, or sets of fifteen, eleven, seven, or three letters including a.

The effect upon the families of contact-quadrics is different. Three of them degenerate into the abnormal pair of complementary sets which consist of all or none of the sixteen letters, viz. the three in which one of the two complementary sets of letters is at or au or tu. At first sight it would appear that the others become identical by fours; for four such as those in which one set is either $****$, or $at****$, or $au****$, or $tu****$ (the asterisks denoting an even number of the letters $b, c, d, \ldots, r, s$) are indistinguishable when a, t, and u are identical. But in reality the first differs fundamentally from the other three. For a pair of contact-primes that are partners in the first family have the property that when the curve acquires a cusp both of them or neither of them will pass through the cusp. In the last three families, of two contact-primes which are partners, both may but one must pass through the cusp. Or, in the first family, the general quadric does not pass through the cusp; in the last three families (which do actually become indistinguishable) the general quadric passes through the cusp; but in all of them certain quadrics have a double point at the cusp, and are cones with the cusp as vertex. It is these cones that are important for the purpose of projection into lower space. A

finite number of them factorize into two primes through the cusp. When we confine our attention to these cones, ignoring all the other contact-quadrics when the curve acquires a cusp, it is a fact that the families of contact-quadrics will have become identical by fours, and the number of families reduced from $2^{2p}-1$ to $2^{2(p-1)}-1$.

The canonical curve of genus 7; *the* C_7^{12} *in* [6]. This curve is the projection of the curve just discussed, the cusp being the centre of projection, and those primes which pass through the cusp and touch the C_7^{14} at six other points become contact-primes of the C_7^{12} in [6]. From Table II it is seen that, when the eighth column is omitted, $b, c, d, \ldots, r, s$ become odd theta-characteristics, while a remains even; the primes in category (β) correspond to the odd theta-characteristics of genus 7, and those in (α) to the even ones. But, as regards (α), projection of a prime that does not pass through the centre of projection is not practicable; and no prime will touch a curve of order 12 in seven points. We can define the seven points as points of contact of certain quadrics related in special ways to the point of the C_7^{12} which is the projection of a point indefinitely near to the cusp. But we have to abandon the convenient method of representing all theta-characteristics, odd and even, by primes, so that any two in combination form a contact-quadric. For this reason it is advisable to link together two consecutive canonical curves by means of the cusped curve when the subject is viewed geometrically.

A contact-quadric cannot well be projected into lower space unless it has a double point at the centre of projection, *i.e.* is a cone with the cusp as vertex. A family of contact-quadrics contains certain of these cones, and some of them factorize into a pair of primes passing through the cusp. We infer that in the projected curve some of the contact-quadrics break up into two contact-primes. The processes by which we have passed from the canonical curve of genus 8 to that of genus 7 may be repeated; and the genus reduced successively to 6, 5, 4, 3. Tables I and II give an explanation of any difficulties or irregularities that arise. These depend for the most part upon the changes of parity of the single letters $a, b, c, \ldots$, which are shown to be as follows:

$p=8$ a even, $b, c, d, \ldots, t, u$ even,

$p=7$ a even, $b, c, d, \ldots, r, s$ odd,

$p=6$ a odd, $b, c, d, \ldots, p, q$ odd,

$p=5$ a odd, $b, c, d, \ldots, m, n$ even,

$p=4$ a even, $b, c, d, \ldots, k, l$ even,

$p=3$ a even, $b, c, d, \ldots, g, h$ odd.

15. Many further questions might be investigated in the general theory of contact curves and particular values of p might be considered in fuller detail; but we shall conclude our paper by a short discussion of the cusped C_2^4.

The case of genus $p = 2$.

The sequence of canonical curves of genus p and order $2(p-1)$ in a space of $p-1$ dimensions comes to an end when p has been diminished to 3. For $p = 2$, it is necessary to choose a standard curve, and a plane quartic with a node is the most obvious choice. Clebsch used the curve with a biflecnode

$$z^2 u_2(x, y) = u_4(x, y).$$

Present-day familiarity with geometry of many dimensions suggests the curve in [4] of order 6 in which the coordinates are proportional to

$$t^3,\ t^2,\ t,\ 1,\ T,$$

where T^2 is a sextic function of the parameter t; from this, any nodal quartic may be derived by projection. But the present paper indicates that the cusped plane quartic† brings the value $p = 2$ better into line with higher values of p than either of these. Six lines through the cusp touch the curve at a second point; they correspond in some way to the six odd theta-characteristics. Plücker's formulae show that the curve has ten double tangents; they correspond to the even ones. Furthermore, in the notation that we are proposing the odd theta-characteristics are represented by six letters a, b, c, d, e, f whose sum vanishes; and the even theta-characteristics by either of two complementary sums of these letters in sets of three.

The equation of a plane cusped quartic curve may be exhibited in a standard form which is in accord with the notation. The first polar of the cusp is a cusped cubic curve, the equation of which can be reduced to

$$zx^2 + y^3 = 0;$$

$x = y = 0$ is the cusp, and $x = 0$ the tangent there for the quartic as well as for the cubic. The equation of the quartic in the same coordinates is

$$z^2 x^2 + 2y^3 z = u_4(x, y),$$

the right-hand member being a homogeneous function of x and y of

† *Cf.* H. W. Richmond, *Quart. J. of Math.*, 26 (1892), 5–26, where the curve is studied from what may be described as Salmon's point of view.

order 4; if a factor x^2 is introduced, it may be written

$$(zx^2+y^3)^2 = y^6+x^2u_4(x, y)$$
$$= (y+Ax)(y+Bx)(y+Cx)(y+Dx)(y+Ex)(y+Fx),$$

or, shortly, $$U^2 = \alpha\times\beta\times\gamma\times\delta\times\epsilon\times\zeta,$$

where $$A+B+C+D+E+F = 0.$$

The lines $\alpha = 0$, $\beta = 0$, ... are the six tangents from the cusp; and ten equations (from each of which a superfluous factor x^2 may be removed), similar to

$$\alpha\beta\gamma+\delta\epsilon\zeta = 2U,$$

give the ten double tangents. The correspondence of the six tangents with the separate letters a, b, c, d, e, f, and of the ten double tangents with the pairs of complementary sets of three of these letters, is obvious. We may represent each of the six by the single letter to which it corresponds and each of the ten by either of the sets of three letters; and then certain of the general rules already obtained can be applied. For example, we find that, besides the sets of four double tangents such as *abc* or *def*, *acd* or *bef*, *ade* or *bcf*, *aeb* or *cdf*, whose points of contact lie on a conic, there are sets of two double tangents and two tangents from the cusp, such as *acd* or *bef*, *bcd* or *aef*, *a* and *b*, whose points of contact lie on a conic through the cusp; and sets of one double tangent and three tangents from the cusp, such as *abc* or *def*, *a*, *b*, and *c*, whose points of contact lie on a conic which touches $x = 0$ at the cusp. In each of these groups of four the representative sets of letters have zero sum; but the converse theorem, that when the representative sets of letters have sum zero the points of contact lie on a conic, is sometimes false. For example, *a*, *abc*, *acd*, *adb* have sum zero, but it is impossible that their eight points of contact should lie on a conic because one and only one lies at the cusp; no conic can meet the quartic at the cusp and at seven other points. To elucidate the matter, consider the families of contact-conics (deriving them, if need be, from those of the non-singular quartic), as in § 14.

The equation of the curve $\alpha\beta\gamma\delta\epsilon\zeta = U^2$ (containing an unnecessary factor x^2) can be written as

$$\alpha\beta\gamma(\alpha\beta\gamma+\delta\epsilon\zeta-2U) = (\alpha\beta\gamma-U)^2.$$

This shows that the points of contact of *a*, *b*, *c*, and *abc* lie on the conic

$$\alpha\beta\gamma = U$$

which passes through the cusp and touches $x = 0$ there. (A factor x can be rejected.) But we may also obtain from this the equation of a family of contact-conics,

$$k^2x^2\alpha\beta + 2kx(\alpha\beta\gamma - U) + (\alpha\beta\gamma + \delta\epsilon\zeta - 2U)\gamma = 0,$$

or
$$\alpha\beta(\gamma + kx)^2 - 2U(\gamma + kx) + \gamma\delta\epsilon\zeta = 0,$$

or, better,
$$\alpha\beta(y + \lambda x)^2 - 2U(y + \lambda x) + \gamma\delta\epsilon\zeta = 0.$$

This equation contains a factor x^2 which is to be rejected, and a family of conics remains. The conics all go through the cusp; they factorize into a tangent from the cusp and a double tangent when $\lambda = C$, D, E, or F, and into two tangents α, β from the cusp when λ is infinite (this being a double root of the discriminant).

A second type of a family of contact-conics may be derived from two of the double tangents, for example *acd* or *bef* and *bcd* or *aef*. Their equations are

$$\alpha\gamma\delta + \beta\epsilon\zeta = 2U; \quad \beta\gamma\delta + \alpha\eta\zeta = 2U;$$

Their four points of contact, $U = \alpha\gamma\delta = \beta\epsilon\zeta$ and $U = \beta\gamma\delta = \alpha\epsilon\zeta$, lie on the conic

$$(\alpha + \beta)U = \alpha\beta(\gamma\delta + \epsilon\zeta)$$

which passes also through the cusp. The identity

$$\alpha\beta(\alpha\gamma\delta + \beta\epsilon\zeta - 2U)(\beta\gamma\delta + \alpha\epsilon\zeta - 2U) - [(\alpha + \beta)U - \alpha\beta(\gamma\delta + \epsilon\zeta)]^2$$
$$= (\alpha - \beta)^2(\alpha\beta\gamma\delta\epsilon\zeta - U^2)$$

proves that the conic passes through the points of contact of a and b, two tangents from the cusp; all the eight intersections of the conic with the cusped quartic are thus accounted for. From the identity we derive the equation of a second type of family of contact-conics, viz.:

$$0 = (2U - \alpha\gamma\delta - \beta\epsilon\zeta)(2U - \beta\gamma\delta - \alpha\epsilon\zeta) - 2kx^2[(\alpha + \beta)U - \alpha\beta(\gamma\delta + \epsilon\zeta)] + \alpha\beta k^2x^4,$$

k being the variable parameter of the family and powers of x being introduced to make the expression homogeneous. A factor x^4 can be removed from the whole expression, but to do so is obviously undesirable. We may write it

$$4U^2 - 2U(\alpha + \beta)(\gamma\delta + \epsilon\zeta + kx^2) + \alpha\beta(\gamma\delta + \epsilon\zeta + kx^2)^2 + (\alpha - \beta)^2\gamma\delta\epsilon\zeta = 0.$$

When k is 0, this gives the two double tangents *acd* and *bcd*; when k is infinite, it gives the two tangents from the cusp a and b. The discriminant

has three infinite roots and three finite; one of the latter is known, and the other two are easily found because there are two values of k which make

$$\gamma\delta+\epsilon\zeta+kx^2=\gamma\epsilon+\delta\zeta \quad \text{and} \quad \gamma\delta+\epsilon\zeta+kx^2=\gamma\zeta+\delta\epsilon$$

respectively. It is better to write the equation of the family of conics as

$$4U^2-2U(\alpha+\beta)\phi+\alpha\beta\phi^2+(\alpha-\beta)^2\gamma\delta\epsilon\zeta=0,$$

where

$$\phi=2y^2+(C+D+E+F)xy+\lambda x^2.$$

The conics factorize into two lines, *acd* and *bcd*, *ace* and *bce*, *acf* and *bcf*, for the three values of λ, $CD+EF$, $CE+DF$, $CF+DE$; and into a and b when λ is infinite (this being a triple root of the discriminant).

All this is in agreement with the statements of § 14. Also it will be found that the properties of the cusped quartic can be derived from those of the non-singular quartic either in the notation suggested here or in the more familiar Hessian notation, by supposing that three of the eight letters employed, a, g, and h, become indistinguishable.

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HARMONIC FUNCTIONALS IN A RIEMANNIAN SPACE

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In this paper I have attempted to generalise some of the results of my paper entitled "A Dirichlet problem for harmonic functionals, with applications to analytic varieties"†. In any Riemannian space in which the parameters are $(x^1, \ldots, x^n)$, and in which the positive definite metric is given by

$$g_{ij}\,dx^i\,dx^j,$$

a skew-symmetric tensor $B_{i_1 \ldots i_p}$ $(1 \leqslant p \leqslant n)$ is said to be harmonic when it satisfies the equations

$$\sum_{r=1}^{p+1} (-1)^{r-1} B_{i_1 \ldots i_{r-1} i_{r+1} \ldots i_p, i_r} = 0$$

and

$$g^{rs} B_{i_1 \ldots i_{p-1} r, s} = 0$$

for all sets of suffixes, the suffixes after the comma implying covariant differentiation. In § 1 I have proposed a boundary value problem for harmonic tensors, and have shown that this can be solved under simple conditions. From this it is possible to investigate at length the properties of these tensors, but I have confined myself, in § 2, to those theorems which I have to use in § 3. This third section is devoted to Riemannian spaces which are also absolute manifolds in the topological sense, and I have shown that, if the p-th Betti number of the manifold is R_p, there are exactly R_p independent tensors

$$B^{(i)}_{i_1 \ldots i_p} \quad (i = 1, \ldots, R_p)$$

† W. V. D. Hodge, *Proc. London Math. Soc.* (2), 36 (1934), 257–303, referred to as I.

on it which are everywhere harmonic, and that the integrals

$$\int B^{(i)}_{i_1 \dots i_p} dx^{i_1} \dots dx^{i_p}$$

have independent periods on the p-cycles of the manifold.

§ 1. *Harmonic functionals and the boundary value problem.*

1. Let us consider a portion of a Riemannian space, in which the parameters are $(x^1, \dots, x^n)$, and let the distance element in this space be given by the fundamental definite quadratic form

$$g_{ij} dx^i dx^j.$$

In this, as in all that follows, we adopt the summation convention of tensor calculus. The functions g_{ij} will be assumed to be analytic functions of $(x^1, \dots, x^n)$ in the region considered. The region of the Riemannian space in which we are interested is obtained by regarding $(x^1, \dots, x^n)$ as the coordinates of a point in a Euclidean n-dimensional space, and allowing this point to vary in an n-cell. It will be convenient to suppose that the coordinates in the Riemannian space are the normal coordinates† associated with some point of our region, and we shall confine ourselves to a small (but *finite*) neighbourhood of this point. The portion of the Riemannian space is then mapped on a Euclidean n-cell, and in the present section we shall think of this mapping rather than of the Riemannian space itself, though the final result is to be translated back into the Riemannian space.

Let $B_{i_1 \dots i_p}$ be a one-valued anti-symmetric covariant tensor, whose components have continuous derivatives of the second order, and which satisfies the equations

$$\sum_{r=1}^{p+1} (-1)^{r-1} \frac{\partial}{\partial x^{i_r}} B_{i_1 \dots i_{r-1} i_{r+1} \dots i_{p+1}} = 0 \quad (\text{all } i_1, \dots, i_{p+1}) \qquad (1)$$

at every point of our n-cell. This can be written in the form

$$\sum_{r=1}^{p+1} (-1)^{r-1} B_{i_1 \dots i_{r-1} i_{r+1} \dots i_{p+1}, i_r} = 0, \qquad (1')$$

where ", r_j" denotes covariant differentiation with respect to x_j. Then the integral

$$\int_{\delta_p} B_{i_1 \dots i_p} dx^{i_1} \dots dx^{i_p}$$

† *Cf.* O. Veblen, "Invariants of quadratic differential forms" (*Cambridge Tract*, No. 24), Chap. 6. We also suppose that at the origin $g_{ij} = \delta_{ij}$ (Kronecker δ).

over a p-dimensional field of integration δ_p depends only on the boundary j_{p-1} of this field, and is said to be a functional of this boundary. Now let us write

$$g^{i_1 \dots i_p, j_1 \dots j_p} = \begin{vmatrix} g^{i_1 j_1} \dots g^{i_1 j_p} \\ \vdots \\ g^{i_p j_1} \dots g^{i_p j_p} \end{vmatrix},$$

where g^{ij} is defined by the equations

$$g^{ij} g_{jk} = \delta_k^i,$$

and write $(g)^{-1} = g^{1 \dots n, 1 \dots n}$. If the integral

$$\int_{\delta_{n-p}} \sum_i \pm \sqrt{g}\, g^{i_1 \dots i_p, j_1 \dots j_p} B_{j_1 \dots j_p} dx^{i_{p+1}} \dots dx^{i_n},$$

where $(i_1, \dots, i_n)$ is a derangement of the first n integers, and where we take the positive or negative sign according as the derangement is even or odd, and sum over all derangements, defines a functional of the boundary j_{n-p-1} of δ_{n-p}, we have

$$\sum_k \frac{\partial}{\partial x^k} \{\sqrt{g}\, g^{i_1 \dots i_{p-1} k, j_1 \dots j_p} B_{j_1 \dots j_p}\} = 0.$$

Now these equations can be written†

$$g^{rs} B_{i_1 \dots i_{p-1} r, s} = 0 \quad (\text{all } i_1, \dots, i_{p-1}). \tag{2}$$

In this case we shall say that the functionals of the cycles j_{p-1}, j_{n-p-1} are harmonic, the relation being symmetrical. The forms of the conditions (1′) and (2) show that the conditions for a harmonic functional are invariant under a change of the coordinate system.

From equations (2) we deduce that

$$g^{rs} \sum_{l=1}^{p} (-1)^{l-1} B_{j_1 \dots j_{l-1} j_{l+1} \dots j_p r, s, j_l} = 0,$$

and hence, by the rule for the interchange of the order of covariant differentiation,

$$g^{rs} \sum_{l=1}^{p} (-1)^{l-1} B_{j_1 \dots j_{l-1} j_{l+1} \dots j_p r, j_l, s}$$

† T. Levi-Civita, *The absolute differential calculus* (English translation), 155.

is equal to a linear expression in the components of the tensor $B_{i_1 \dots i_p}$ with analytic coefficients. By using (1) we see that this expression is equal to

$$(-1)^{p-1} g^{rs} B_{j_1 \dots j_p, r, s},$$

and hence we see that

$$\sum_{r=1}^{n} \sum_{s=1}^{n} g^{rs} \frac{\partial^2}{\partial x^r \partial x^s} B_{j_1 \dots j_p}$$

is equal to an expression which is linear in the components of the tensor $B_{i_1 \dots i_p}$ and their first derivatives, the coefficients being analytic functions of $(x^1, \dots, x^n)$.

2. In our n-cell let us consider a locus Γ enclosing a sub-cell Δ of the n-cell. We shall suppose that Γ is regular (*i.e.* that at every point of Γ the principal curvatures are defined, and that they vary continuously on Γ). Now let $b_{i_1 \dots i_p}$ be a one-valued anti-symmetric tensor having continuous first partial derivatives which satisfy (1) in our n-cell. This defines a functional of the bounding $(p-1)$-cycles of Δ. We now propose the boundary value problem—can we find a functional of the bounding $(p-1)$-cycles of Δ which is harmonic in Δ and takes the same values as the given functional on the $(p-1)$-cycles of Γ? We shall suppose that $p < n$, for if $p = n$ we must have $B_{1 \dots n} = k\sqrt{g}$, and the boundary requirement merely leads to an equation to determine the constant k.

The boundary requirement can be expressed in another way. Let Γ be given, in the neighbourhood of some point P, by the equations

$$x^i = f^i(u^1, \dots, u^{n-1}) \quad (i = 1, \dots, n).$$

If the harmonic functional is defined by the tensor $B_{i_1 \dots i_p}$, as we approach P in any direction, the tensor is to tend to a definite finite limit, and at P

$$B_{i_1 \dots i_p} \frac{\partial(x^{i_1}, \dots, x^{i_p})}{\partial(u^{j_1}, \dots, u^{j_p})} = b_{i_1 \dots i_p} \frac{\partial(x^{i_1}, \dots, x^{i_p})}{\partial(u^{j_1}, \dots, u^{j_p})}$$

for all $(j_1, \dots, j_p)$.

The solution of this problem is achieved by a process similar to that described by Courant, Friedrichs, and Lewy†, but before proceeding to this we must give some simple lemmas on combinatorial complexes.

† *Math. Annalen*, 100 (1928), 32.

3. Suppose that an n-cell, together with its boundary, is subdivided into a complex K_n, and that its boundary is thereby subdivided into a complex K_{n-1}. Let there be a_r r-cells of K_n, and a_r' r-cells of K_{n-1}, and let $a_r = a_r' + a_r''$. Then

$$a_n - a_{n-1} + a_{n-2} - \ldots + (-1)^n a_0 = (-1)^n,$$

$$a_{n-1}' - a_{n-2}' + \ldots + (-1)^{n-1} a_0' = 1 + (-1)^{n-1},$$

$$a_n'' - a_{n-1}'' + a_{n-2}'' - \ldots + (-1)^n a_0'' = 1.$$

Let the r-cells of K_n be denoted by E_r^i $(i = 1, \ldots, a_r)$ and let the incidence relations be given by

$$E_r^i \to {}_r\eta_j^i E_{r-1}^j.$$

Let us associate with each $(p-1)$-cell a number, say A_{p-1}^i with E_{p-1}^i. Then with each p-cell we associate a number A_p^i given by the equations

$${}_p\eta_j^i A_{p-1}^j = A_p^i. \tag{3}$$

If now

$$a_i E_p^i \to 0, \tag{4}$$

then

$$a_i A_p^i = 0. \tag{5}$$

Conversely, suppose that we are given numbers associated with the p-cells, such that (4) implies (5). Then the equations (3) are solvable for the numbers A_{p-1}^j. Any relation connecting the left-hand sides of (3) can be supposed to be an integral combination of them, since ${}_p\eta_j^i$ is an integer. Then any such relation implies a set of equations

$$a_{i\,.\,p}\eta_j^i = 0 \quad (j = 1, \ldots, a_{p-1}),$$

where the a's are integers, and hence

$$a_i E_p^i \to 0,$$

and, therefore, the right-hand sides of (3) are connected by the same relation. The equations (3) are therefore consistent, and are hence solvable. Let us consider the solution in the case $A_p^i = 0$ (all i). First let us determine the number of independent equations (3). There is a relation between these for each p-cycle of K_n. But these cycles can all be expressed in terms of the bounding relations of the $(p+1)$-cells. In this way there are a_{p+1} relations, but there is a relation between them corresponding to each $(p+1)$-cycle. We can find a base for these new relations, and proceed in

this way. We thus see that there are, altogether,

$$\alpha_p - \alpha_{p+1} + \alpha_{p+2} - \ldots + (-1)^{n-p}\alpha_n = \alpha_{p-1} - \alpha_{p-2} + \ldots + (-1)^{p-1}\alpha_0 + (-1)^p$$

independent equations. We can construct the solutions as follows. With the $(p-2)$-cells E^j_{p-2} we associate numbers A^j_{p-2}, and let

$$A^i_{p-1} = {}_{p-1}\eta_j{}^i A^j_{p-2}. \tag{3'}$$

The numbers A^i_{p-1} satisfy the required equations. But not every set of numbers yields different numbers A^i_{p-1}. We must subtract the number of solutions of (3') with $A^i_{p-1} = 0$. Proceeding, we see that our process leads to

$$\alpha_{p-2} - \alpha_{p-3} + \ldots + (-1)^p \alpha_0 + (-1)^{p-1}$$

independent solutions of the homogeneous equations derived from (3), as we expect.

We shall have to solve equations (3) with $A_p{}^i = 0$, with the additional restriction that $A^i_{p-1} = 0$ on the cells of K_{n-1}. There is an equation (3) for each p-cell of $K_n - K_{n-1}$, and a relation for each $(p+1)$-cell of $K_n - K_{n-1}$, and so on. We thus see that there are

$$\alpha''_p - \alpha''_{p+1} + \ldots + (-1)^{n-p}\alpha''_n + (-1)^{n-p+1}$$

independent equations, the last term being due to the fact that K_n is a cycle (mod K_{n-1}). The solutions can be constructed in the same way as before.

4. Now return to the solution of our boundary value problem. Cover the n-cell in which Δ lies by a network of points $x^i = 0, \pm h, \pm 2h, \ldots$ $(i = 1, \ldots, n)$, and let K_n be the complex formed by the n-cubes formed from this network which lie entirely within Δ. The most convenient way of orienting the cells of K_n is as follows. An r-cell is formed by points $(x^1, \ldots, x^n)$ which satisfy the relations

$$m^{i_s} h < x^{i_s} < (m^{i_s}+1)h \quad (s = 1, \ldots, r),$$

$$x^{i_s} = m^{i_s} h \qquad (s = r+1, \ldots, n).$$

The point $P, x^i = m^i h$ $(i = 1, \ldots, n)$ is called the vertex associated with the cell. Let PA_i be a line through the vertex parallel to the x^i-axis. If $i_1 < i_2 < \ldots < i_r$, we orient the cell by the indicatrix $(PA_{i_1} \ldots A_{i_r})$. Denote a cell of this type by $E_{i_1 \ldots i_r}$ (using an affix, if necessary, to distinguish cells of this type, and denote the number to be associated with it by $h^r A_{i_1 \ldots i_r}$.

At present we shall suppose the numbers $h^{p-1}A_{i_1\dots i_{p-1}}$ associated with the $(p-1)$-cells of $K_n - K_{n-1}$ to be undetermined. For the numbers associated with the $(p-1)$-cells of K_{n-1} we proceed as follows. By Stokes's theorem we can find an anti-symmetric tensor $a_{i_1\dots i_{p-1}}$ such that

$$\sum_{r=1}^{p} (-1)^{r-1} \frac{\partial a_{i_1\dots i_{r-1}\, i_{r+1}\dots i_p}}{\partial x^{i_r}} = b_{i_1\dots i_p} \quad (\text{all } i_1, \dots, i_p)$$

at all points of Δ and its boundary. If $E_{i_1\dots i_{p-1}}$ is a cell of K_{n-1} we associate with it the number

$$h^{p-1}A_{i_1\dots i_{p-1}} = \int_{E_{i_1\dots i_{p-1}}} a_{i_1\dots i_{p-1}}\, dx^{i_1} \dots dx^{i_{p-1}}.$$

We shall further denote by $h^p B_{i_1\dots i_p}$ the derived number associated with the p-cells of K_n. We can now associate numbers $B_{i_1\dots i_p}$ with each vertex P of K_n. Through an internal vertex P there is one p-cell $E_{i_1\dots i_p}$ (for each set of suffixes $i_1 \dots i_p$) having P as its associated vertex. The number $B_{i_1\dots i_p}$ associated with this cell is then associated with P. If P is on K_{n-1} we follow the same rule, except that, when there is no cell $E_{i_1\dots i_p}$ of K_n having P as associated vertex, we put $B_{i_1\dots i_p} = 0$. At each vertex of K_n we form the sum

$$h^n \sqrt{g}\, g^{i_1\dots i_p, j_1\dots j_p} B_{i_1\dots i_p} B_{j_1\dots j_p} = B_{i_1\dots i_p} B^{i_1\dots i_p} \quad (\text{say}),$$

and let D be the sum of these expressions over the vertices of K_n. Then D is positive unless all the B's are zero. Now, if ϕ is any function of the points of the network, we define ϕ_k, $\phi_{\bar{k}}$ at the vertex point $(x^1, \dots, x^n)$ by the equations

$$\phi_k = \frac{1}{h}\{\phi(x^1, \dots, x^{k-1}, x^k + h, x^{k+1}, \dots, x^n) - \phi(x^1, \dots, x^n)\},$$

$$\phi_{\bar{k}} = \frac{1}{h}\{\phi(x^1, \dots, x^n) - \phi(x^1, \dots, x^{k-1}, x^k - h, x^{k+1}, \dots, x^n)\}.$$

By simple algebra we find that

$$D = -h^n \sum_{K_n - K_{n-1}} A_{i_1\dots i_{p-1}} \left\{ \sum_k (B^{i_1\dots i_{p-1}k})_{\bar{k}} \right\} + h^{n-1} \sum_{K_{n-1}} A_{i_1\dots i_{p-1}} C^{i_1\dots i_{p-1}}, \quad (6)$$

where the first summation on the right-hand side is over the vertices associated with the a'_{p-1} $(p-1)$-cells of $K_n - K_{n-1}$, and the second over the vertices associated with the $(p-1)$-cells of K_{n-1}. $C^{i_1\dots i_{p-1}}$ is a linear

function of the B's which can be calculated if necessary. If we impose the condition that D is to have its minimum value consistent with the boundary conditions for the A's, we find that the A's must satisfy the equations

$$\sum_k (B^{i_1 \dots i_{p-1} k})_{\bar{k}} = 0 \tag{7}$$

at all the points which appear in the first summation on the right-hand side of (6). There are, in fact, as many equations as unknown A's (the number is α''_{p-1}). Let us first show that these equations are solvable. We construct the dual complex $K_n{}^*$ of K_n by taking as the vertices of the necessary subdivision the mid-points of the cells. If $E^*_{i_{p+1} \dots i_n}$ is the dual of $E_{i_1 \dots i_p}$, we associate with $E^*_{i_{p+1} \dots i_n}$ the number

$$h^{n-p} B^{i_1 \dots i_p} = \pm h^{n-p} \sqrt{g}\, g^{i_1 \dots i_p, j_1 \dots j_p} B_{j_1 \dots j_p}$$

calculated at the vertex associated with $E_{i_1 \dots i_p}$. Equations (7) imply that the sum of the numbers $h^{n-p} B^{i_1 \dots i_p}$ over the cells of any $(n-p)$-cycle of $K_n{}^*$ is zero. We can then calculate, as in No. 3, the number of independent equations (7). They are in number

$$\alpha''_{p-1} - \alpha''_{p-2} + \dots + (-1)^{p-1} \alpha_0{}''.$$

We have to show that the equations to determine the unknowns $A_{i_1 \dots i_{p-1}}$ on the cells of $K_n - K_{n-1}$ are consistent. It is sufficient to show that there are exactly

$$\alpha''_{p-2} - \alpha''_{p-3} + \dots + (-1)^p \alpha_0{}''$$

independent solutions when $a_{i_1 \dots i_{p-1}} = 0$ everywhere. For if this is the case, any relation between the equations (7) which holds when the constant members are replaced by zero will hold for general values of the constant members. To prove that this is true we observe that if (7) is true, and $A_{i_1 \dots i_{p-1}} = 0$ on K_{n-1}, it follows from (6) that $D = 0$, or $B_{i_1 \dots i_p} = 0$ everywhere. But we have seen in No. 3 that the number of solutions in this case is exactly

$$\alpha''_{p-2} - \alpha''_{p-3} + \dots + (-1)^p \alpha_0{}'',$$

so that equations (7) are solvable. Also, two different solutions of (7) yield the same value of $B_{i_1 \dots i_p}$, hence equations (7) are uniquely solvable for the B's, subject to the condition that the sum of the expressions $h^p B_{i_1 \dots i_p}$ over the cells of a p-cycle of K_n is zero.

Since (7) is the condition that D should be a minimum,

$$D \leqslant h^n \sum_{K_n} \sqrt{g}\, g^{i_1 \ldots i_p,\, j_1 \ldots j_p} b_{i_1 \ldots i_p} b_{j_1 \ldots j_p},$$

where on the cell $E_{i_1 \ldots i_p}$ we give $b_{i_1 \ldots i_p}$ its mean value. Since the right-hand side of this inequality tends to

$$\int_\Delta \sqrt{g}\, g^{i_1 \ldots i_p,\, j_1 \ldots j_p}\, b_{i_1 \ldots i_p} b_{j_1 \ldots j_p} dx^1 \ldots dx^n$$

as h tends to zero, this integral gives an upper bound for D as h tends to zero.

We can also prove, by the algebra that leads to (6), that

$$h^n \sum_{K_n} \sqrt{g}\, g^{i_1 \ldots i_p,\, j_1 \ldots j_p} B_{i_1 \ldots i_p} (b_{j_1 \ldots j_p} - B_{j_1 \ldots j_p}) = 0. \tag{8}$$

From the manner in which the B's are constructed, we see that, at the vertex associated with $(p+1)$-cell of $K_n - K_{n-1}$,

$$\sum_1^{p+1} (-1)^{r-1} (B_{i_1 \ldots i_{r-1} i_{r+1} \ldots i_{p+1}})_{i_r} = 0. \tag{9}$$

Equation (9) is now the approximate form of (1). If we repeat the analysis of No. 1, using difference quotients instead of derivatives, and taking a suitable definition of covariant difference quotients, we can replace (7) by

$$g^{rs} (B_{i_1 \ldots i_{p-1} r})_{\bar{s}} = 0, \tag{10}$$

which is the approximate form of (2). Still following the reasoning of No. 1, we can show that

$$g^{rs} (B_{i_1 \ldots i_p})_{r\bar{s}}$$

is equal to a linear expression in the B's and their first difference quotients, at an internal point.

If it is desired to find the A's explicitly, we must give a rule for fixing a unique solution. For certain reasons the following is the most convenient manner of giving this rule. With the dual of a $(p-1)$-cell $E_{i_1 \ldots i_{p-1}}$, we associate the number

$$h^{n-p+1} \sqrt{g}\, g^{i_1 \ldots i_{p-1},\, j_1 \ldots j_{p-1}} A_{j_1 \ldots j_{p-1}} = h^{n-p+1} A^{i_1 \ldots i_{p-1}},$$

just as in the case of the B's. We then impose the conditions that the sum of the numbers over the cells of any $(n-p+1)$-cycle of $K_n{}^*$ is zero. We

easily verify that these give the right number of conditions, and yield the equations

$$\sum_k (A^{i_1 \cdots i_{p-2} k})_{\bar{k}} = 0,$$

or

$$g^{rs}(A_{i_1 \cdots i_{p-2} r})_{\bar{s}} = 0,$$

at each vertex. The equations for the A's are then uniquely solvable.

The situation is now similar to that described by Courant, Friedrichs, and Lewy (*loc. cit.*). Using the methods described there, we can show that as h tends to zero the B's tend to limits from which we are able to define a tensor $B_{i_1 \cdots i_p}$ which is continuous in Δ, and indeed possesses continuous derivatives of any order, and which satisfies equations (1) and (2)†.

The form of the solution in the limit will be

$$B_{i_1 \cdots i_p} = \int_\Gamma K^{j_1 \cdots j_p}_{i_1 \cdots i_p} b_{j_1 \cdots j_p} \, d\Gamma,$$

where $K^{j_1 \cdots j_p}_{i_1 \cdots i_p}$ is a continuous function of the points of Δ, and of the points of Γ, which is independent of the boundary values‡. A similar result will hold if we find the A's instead of the B's.

5. The theorem that the solution which we have obtained satisfies the boundary conditions is proved by an adaptation of the method given by Courant, Friedrichs, and Lewy in the case of Laplace's equation in a plane. Though the method is quite general, the proof will be clearer if we give it in the particular case $p = 2$, $n = 3$.

The functions A_1, A_2, A_3 are defined for each 1-cell of K_3. It is easy to define A_1 everywhere in K_3 so that it is a continuous function of x^2, x^3, and similarly for A_2, A_3. Consider, for instance, the 3-cell $0 \leqslant x^i \leqslant h$. We know A_1 at (0, 0, 0), (0, 1, 0), (0, 0, 1), (0, 1, 1), since these are vertices associated with 1-cells E_1 of the boundary of the 3-cell. We shall denote the values of A_1 and its difference quotients at the origin by the index "0". For any point in the 3-cell we define A_1 by the equation

$$A_1 = A_1{}^0 + (A_1{}^0)_{x^2} x^2 + (A_1{}^0)_{x^3} x^3 + (A_1{}^0)_{x^2 x^3} x^2 x^3.$$

† I have verified that the slight extension of this result which I use here can be made without any essential difficulty if we use normal coordinates and remain sufficiently near the origin. The calculations, which are somewhat lengthy, are not of sufficient interest to merit inclusion in detail in this paper.

‡ Although this is clearly the case, I have not been able to obtain a simple proof of this. *Cf.* Bouligand, *Ann. de la Soc. Polon. de Math.*, 4 (1926), 59, in particular Lemma III.

We proceed similarly for A_2, A_3, and in any other cell of K_3. Then we find that

$$B_{23} = B^0_{23} + (B^0_{23})_{x^1} x^1,$$

etc., and B_{23} is hence a continuous function of x^1, and takes the prescribed values on the 2-cells on which it was originally defined. From the given tensors a_i, b_{ij} we can define a_i, b_{ij} on each 1-cell or 2-cell of K_3 as its mean value on the cell, and define them everywhere in K_3 by the foregoing rule. We then write

$$\beta_{ij} = B_{ij} - b_{ij}$$

in K_3, and zero elsewhere. The conditions satisfied by β_{ij} are sufficient to ensure that the integral

$$\iint \beta_{ij}\, dx^i\, dx^j$$

over any closed cycle is zero.

Now consider any point $a(a^1, a^2, a^3)$ of Γ, and let $(\lambda^1, \lambda^2, \lambda^3)$ be the direction cosines of the normal to Γ at this point. One λ, say λ^3, is at least $1/\sqrt{3}$ in absolute magnitude. Let Γ_1 be a portion (cell) of Γ about a, bounded by the section by the planes $x^1 = a^1 \pm \alpha^1$, $x^2 = a^2 \pm \alpha^2$, in which $|\lambda^3| \geqslant \frac{1}{2}$. The lines through points of Γ_1 of length d, drawn into Δ parallel to the x^3-axis, define a region Δ_1 of Δ. Let $\xi(\xi^1, \xi^2, \xi^3)$ be any point of Γ_1. We then associate with each point x (x^1, x^2, x^3) of Γ_1 a 2-cell, bounded by sections by planes parallel to the coordinate x^1-, x^2-planes passing through ξ and x. We can then define a 2-cell, denoted by E_x, associated with each point x (x^1, x^2, x^3) of Δ_1. The points of this cell are at a constant distance measured parallel to the x^3-axis from Γ_1, and the cell projects on to Γ_1 into the cell associated with the projection of (x^1, x^2, x^3).

$$f(x^1, x^2, x^3) = \int_{E_x} \beta_{ij}\, dx^i\, dx^j$$

is then a function of x. We have to show that $f(x^1, x^2, x^3)$ tends to zero as x tends to a point of Γ_1, independently of h and ξ.

Now $f(x^1, x^2, x^3)$ can be expressed as an integral over the locus of projection of the boundary of E_x, and by simple calculation we have

$$f^2 \leqslant d\mu \Big[\Big(\iint \beta^2_{13}\, dx^1\, dx^3 \Big)_{x^2 = \xi^2} + \Big| \Big| \beta^2_{13}\, dx^2\, dx^3 + \Big(\Big| \Big| \beta^2_{23}\, dx^2\, dx^3 \Big)_{x^1 = \xi^1} + \iint \beta^2_{23} dx^2\, dx^3 \Big], \quad (11)$$

where μ depends only on $\alpha^1 \alpha^2$, and where the first and second integrals

are evaluated over sections of Δ_1 by planes parallel to the x^2-plane, and the third and fourth over sections by planes parallel to the x^1-plane. Integrate over the line through x parallel to the x^3-axis, and then with respect to x^2 and x^3 over the range

$$a^i - \alpha^i \leqslant x^i \leqslant a^i + \alpha^i.$$

We thus have

$$\int_{\Delta_1} f^2\, dx^1\, dx^2\, dx^3 \leqslant d^2\mu \Big[4\alpha^1\alpha^2 \Big(\iint \beta_{13}^2\, dx^1\, dx^3 \Big)_{x^2=\xi^2} + 2\alpha^1 \int_{\Delta_1} \beta_{13}^2\, dx^1\, dx^2\, dx^3$$
$$+ 4\alpha^1\alpha^2 \Big(\iint \beta_{23}^2\, dx^1\, dx^3 \Big)_{x^2=\xi^2} + 2\alpha^2 \int_{\Delta_1} \beta_{23}^2\, dx^1\, dx^2\, dx^3 \Big]. \quad (12)$$

Each side of this inequality is still a function of ξ^1, ξ^2. We integrate with respect to ξ_1 and ξ_2 over the range

$$a_i - \alpha_i \leqslant \xi^i \leqslant a_i + \alpha_i,$$

and we have

$$\int_{a^1-\alpha^1}^{a^1+\alpha^1} \int_{a^2-\alpha^2}^{a^2+\alpha^2} \Big[\int_{\Delta_1} f^2\, dx^1\, dx^2\, dx^3 \Big] d\xi^1\, d\xi^2 \leqslant d^2 k,$$

where k is an absolute constant (independent of h), since

$$\int_{\Delta_1} \beta_{ij}^2\, dx^1\, dx^2\, dx^3$$

is bounded. This inequality remains true as h tends to zero. We deduce from this that, as x tends to a point of Γ_1, $f(x^1, x^2, x^3)$ tends to zero almost everywhere, whatever be the point ξ. Hence we deduce that our boundary conditions are satisfied almost everywhere.

If $p = 2$, $n > 3$, we must integrate equation (12) with respect to the remaining variables, and the result is to be obtained for each set x^i, x^j, as has been done for x^1, x^2. The method is otherwise similar for general values of p and n. We can further prove in the same way that the A's (determined by the rule of No. 4) tend to the assigned boundary values in the same sense. We define 1-cells e_x associated with each point x of A_1, and so get functions

$$\phi(x^1, x^2, x^3) = \int_{e_x} \alpha_i\, dx^i \quad (\alpha_i = A_i - a_i).$$

$\phi(x^1, x^2, x^3)$ is then represented as an integral over the locus of projection of the boundary of e_x, together with an integral over the locus of projection

of the cell itself. This adds a term to (11). The presence of this last term does not lead to any serious complication in the subsequent proof.

6. With the restrictions put on the boundary Γ and on the boundary values, we can see that the second form of the boundary conditions given in No. 2 is satisfied at every point of Γ. The result just proved says that these conditions are satisfied almost everywhere. Let P be any point of the boundary and choose it as the origin of normal coordinates. Consider the form of the solution given at the end of No. 4. Then, as we approach P, the K's must approach the forms which they have for the metric $g_{ij} = \delta_{ij}$, since the values of $B_{i_1 \dots i_p}$ near P depend only on the boundary values in the neighbourhood of P. But we know (from my paper I) that in this case, if

$$B_{i_1 \dots i_p} \frac{\partial(x^{i_1}, \dots, x^{i_p})}{\partial(u^{j_1}, \dots, u^{j_p})}$$

approaches a continuous and differentiable value almost everywhere in the neighbourhood of P, it does so at P. Hence in our present case it follows that the boundary conditions are satisfied everywhere.

Similar remarks may be made with regard to the A's. Thus we can say that there is a tensor $A_{i_1 \dots i_{p-1}}$ such that†

$$g^{rs} A_{i_1 \dots i_{p-2} r, s} = 0,$$

$$\Sigma (-1)^{r-1} A_{i_1 \dots i_{r-1} i_{r+1} \dots i_p, i_r} = B_{i_1 \dots i_p},$$

and such that

$$(A_{i_1 \dots i_{p-1}} - a_{i_1 \dots i_{p-1}}) \frac{\partial(x^{i_1}, \dots, x^{i_{p-1}})}{\partial(u^{j_1}, \dots, u^{j_{p-1}})}$$

tends to zero for all values of $j_1, \dots, j_{p-1}$ as we approach a point of the boundary.

7. A remark must be made here with regard to the boundary conditions. In order to solve the boundary value problem, we only need to know the a's, which must satisfy the relations

$$\Sigma (-1)^{r-1} \frac{\partial a_{i_1 \dots i_{r-1} i_{r+1} \dots i_p}}{\partial x^{i_r}} = b_{i_1 \dots i_p},$$

† Whenever we deal with the tensor $A_{i_1 \dots i_{p-1}}$, we shall assume that this condition is satisfied.

and on Γ the a's and b's must satisfy the conditions necessary for the solution in the case $g_{ij} = \delta_{ij}$. These will be satisfied if (α) the above relation is satisfied right up to the boundary, and (β) the b's satisfy uniform Hölder conditions on Γ, *i.e.* if

$$b'_{j_1 \dots j_\rho} = b_{i_1 \dots i_\rho} \frac{\partial(x^{i_1}, \dots, x^{i_\rho})}{\partial(u_{j_1}, \dots, u_{j_\rho})},$$

$$|b'_{j_1 \dots j_\rho}(P) - b'_{j_1 \dots j_\rho}(Q)| < M |PQ|^\alpha,$$

whenever $|PQ| < \delta$, which is independent of the points P, Q on the boundary, where M and α are positive constants on Γ.

In the applications which we are going to consider, the situation in which we shall most often find ourselves is as follows. We shall be given the b's as functions which are continuous everywhere in a region which extends beyond Δ, and which have continuous derivatives [satisfying (1)] except at a finite number of analytic loci. The derivatives remain finite as we approach these loci, but the derivatives normal to these loci undergo finite discontinuities there. We easily verify that the b's satisfy uniform Hölder conditions on Γ, and the a's can be defined by the following rule. Let k be the greatest integer less than $i_1, \dots, i_{\rho-1}$, and let $b^l_{i_1 \dots i_\rho}$ denote the value of $b_{i_1 \dots i_\rho}$ when $x^1, \dots, x^l$ are put equal to $a^1, \dots, a^l$, where $(a^1, \dots, a^n)$ is some point of our domain. We then write

$$a_{i_1 \dots i_{\rho-1}} = \int_{a^1}^{x^1} b_{1 i_1 \dots i_{\rho-1}} dx^1 + \int_{a^2}^{x^2} b^1_{2 i_1 \dots i_{\rho-1}} dx^2 + \dots + \int_{a^k}^{x^k} b^{k-1}_{k i_1 \dots i_{\rho-1}} dx^k,$$

it being assumed that the b's are defined in an n-cube containing Δ.

If the b's satisfy the foregoing conditions in the region considered, we shall say that they satisfy conditions A. In this case, the boundary value problem can be solved, and, by considering the case $g_{ij} = \delta_{ij}$ we find that *all* the B's remain finite as we approach the boundary.

Further, we shall always assume in the sequel that the regions to which we apply our boundary value solution are sufficiently small (but none the less *finite*) for the validity of the argument of this section.

§ 2. *Some theorems on harmonic functionals.*

8. Having solved the boundary value problem, it is clear that a number of the results which I obtained in paper I in the case of a flat space can be extended to the case of a more general metric. It is not my intention in the present paper to do this, and in this section I confine myself to some theorems which I shall need for the applications to analytic varieties.

It will be more convenient now to revert to the language and notations of I. The expression

$$\phi = A_{i_1 \dots i_{p-1}} dx^{i_1} \dots dx^{i_{p-1}}$$

will be called a $(p-1)$-form, and we denote by ϕ_x, ϕ^x the derived forms

$$\phi_x = B_{i_1 \dots i_p} dx^{i_1} \dots dx^{i_p},$$

$$\phi^x = \Sigma \pm \sqrt{g}\, g^{i_1 \dots i_p, j_1 \dots j_p} B_{j_1 \dots j_p} dx^{i_{p+1}} \dots dx^{i_n}.$$

If $(\phi^x)_x = 0$, we say that ϕ_x is a harmonic form. We write

$$D_\Delta(\phi) = \int_\Delta \sqrt{g}\, g^{i_1 \dots i_p, j_1 \dots j_p} B_{i_1 \dots i_p} B_{j_1 \dots j_p} dx^1 \dots dx^n,$$

$$D_\Delta(\phi, \phi') = \int_\Delta \sqrt{g}\, g^{i_1 \dots i_p, j_1 \dots j_{p'}} B_{i_1 \dots i_p} B'_{j_1 \dots j_p} dx^1 \dots dx^n.$$

From equation (8) we see that, if ϕ_x is the harmonic form derived from boundary conditions given by ϕ_x',

$$D_\Delta(\phi' - \phi, \phi) = 0.$$

Now
$$\begin{aligned} D_\Delta(\phi') &= D_\Delta(\phi + \overline{\phi' - \phi}) \\ &= D_\Delta(\phi) + 2D_\Delta(\phi' - \phi, \phi) + D_\Delta(\phi' - \phi) \\ &= D_\Delta(\phi) + D_\Delta(\phi' - \phi). \end{aligned}$$

Hence
$$D_\Delta(\phi) < D_\Delta(\phi'),$$

except when
$$D_\Delta(\phi' - \phi) = 0,$$

or
$$B_{i_1 \dots i_p} = B'_{i_1 \dots i_p}$$

everywhere. In this case ϕ_x' is harmonic.

In our applications we shall have to consider a form

$$\phi_x = b_{i_1 \dots i_p} dx^{i_1} \dots dx^{i_p}$$

defined throughout a region R, and satisfying conditions A in R. Let Δ be a cell in R, bounded by a geodesic sphere Γ, whose centre is P. If ϕ_x is not harmonic in Δ, we can replace ϕ_x by $\bar{\phi}_x$ in Δ, where $\bar{\phi}_x$ is the harmonic form which defines the same functional on Γ as ϕ_x. The new form is not, however, continuous in R, there being discontinuities on Γ. We shall now

show that we can find a form $\tilde{\phi}_x$, which satisfies conditions A in R, such that

$$D_R(\tilde{\phi}) < D_R(\phi), \text{ and } \tilde{\phi}_x = \phi_x \text{ outside } \Delta.$$

Since ϕ_x satisfies conditions A, we can determine a form

$$\phi = a_{i_1 \dots i_{p-1}} dx^{i_1} \dots dx^{i_{p-1}}$$

from which ϕ_x is derived. The argument is similar to that given in No. 24 of I. We use normal coordinates with P as origin, and let Γ' be a sphere of radius b ($b < a$, the radius of Γ) concentric with Γ.

Let

$$\tilde{\phi} = \bar{\phi} \qquad (r \leqslant b),$$

$$\tilde{\phi} = \bar{\phi} + \left(\frac{r-b}{a-b}\right)^2 (\phi - \bar{\phi}) \qquad (b \leqslant r \leqslant a),$$

$$\tilde{\phi} = \phi \qquad (r \geqslant a),$$

where $r^2 = (x^1)^2 + \dots + (x^n)^2 = g_{ij} x^i x^j$.

We deduce at once that $\tilde{\phi}_x$ satisfies conditions A in R. We have then to show that

$$D_\Delta(\tilde{\phi}) < D_\Delta(\phi) + \epsilon,$$

where ϵ is arbitrarily small, provided that $a-b$ is small enough. The proof is the same as in I (*q.v.*). The only difference is that in place of

$$\int_{\Delta - \Delta'} \Sigma\, C^2_{i_1 \dots i_p} dx' \dots dx^n$$

we have now

$$\int_{\Delta - \Delta'} \sqrt{g}\, g^{i_1 \dots i_p, j_1 \dots j_p} C_{i_1 \dots i_p} C_{j_1 \dots j_p} dx^1 \dots dx^n.$$

But since this expression is less than

$$k \int_{\Delta - \Delta'} \Sigma\, C^2_{i_1 \dots i_p}\, dx^1 \dots dx^n,$$

where k is a constant independent of the C's, the result is obtained as before.

9. Though we have proved the uniqueness of the solution of the equations of finite differences used in solving the boundary value problem, the uniqueness of the solution of the boundary value problem itself does not

follow directly from this. I shall now prove this uniqueness in a special case. This is the case in which the form ϕ_x', used to give the boundary values, is itself harmonic in a region containing Δ. We have

$$D_\Delta(\phi) = D_\Delta(\phi' + \overline{\phi - \phi'})$$
$$= D_\Delta(\phi') + 2D_\Delta(\phi', \phi - \phi') + D_\Delta(\phi - \phi').$$

Now $$D_\Delta(\phi', \phi - \phi') = \int_\Gamma \phi - \phi' \,.\, \phi'^x + (-1)^p \int_\Delta \phi - \phi' \,.\, (\phi'^x)_x$$
$$= 0.$$

Hence $$D_\Delta(\phi) \geqslant D_\Delta(\phi').$$

But we know that $$D_\Delta(\phi) \leqslant D_\Delta(\phi').$$

Hence we must have $$D_\Delta(\phi - \phi') = 0,$$

that is $$\phi_x = \phi'_x.$$

If we now suppose that

$$g^{rs} A_{i_1 \ldots i_{p-2} r, s} = 0,$$

$$g^{rs} A'_{i_1 \ldots i_{p-2} r, s} = 0,$$

and let $$\psi_x = \phi - \phi',$$

then $$(\psi^x)_x = 0,$$

and $\psi_x = 0$ on Γ. Hence $$\psi_x = 0$$

everywhere, since

$$D_\Delta(\psi) = \int_\Gamma \psi \,.\, \psi^x + (-1)^{p-1} \int_\Delta \psi \,.\, (\psi^x)_x = 0;$$

that is $$\phi = \phi'$$

everywhere.

Now suppose that $$\phi_x = B_{i_1 \ldots i_p} dx^{i_1} \ldots dx^{i_p}$$

is a harmonic form in a certain region. Take a point P of this as origin of normal coordinates, and let Γ_r be a geodesic sphere, with centre at the origin, and of radius r. Then, within this sphere,

$$B_{i_1 \ldots i_p} = \int_{\Gamma_r} K^{j_1 \ldots j_p}_{i_1 \ldots i_p} B_{j_1 \ldots j_p} d\Gamma.$$

Use this formula to evaluate $B_{i_1 \dots i_\rho}$ at the origin. As r tends to zero,

$$|\Gamma_r| K^{j_1 \dots j_\rho}_{i_1 \dots i_\rho} \quad \text{tends to} \quad (p!)^{-1} \delta^{j_1 \dots j_\rho}_{i_1 \dots i_\rho},$$

where $|\Gamma_r|$ is the measure of Γ_r. Then

$$B_{i_1 \dots i_\rho} \int_0^r |\Gamma_r| \, dr = \int_{\Delta_r} k^{j_1 \dots j_\rho}_{i_1 \dots i_\rho} B_{j_1 \dots j_\rho} dx^1 \dots dx^n,$$

where $k^{j_1 \dots j_\rho}_{i_1 \dots i_\rho}$ is a finite and continuous function everywhere in Δ_r. Thus, if Δ is any cell with a spherical boundary, the value of $B_{i_1 \dots i_\rho}$ at the centre is given by the formula

$$B_{i_1 \dots i_\rho} = \frac{1}{\Delta} \int_\Delta k^{j_1 \dots j_\rho}_{i_1 \dots i_\rho} B_{j_1 \dots j_\rho} dx^1 \dots dx^n.$$

10. Levi's inequality for the Dirichlet integrals of a system of harmonic functions† can be extended to our integrals $D_\Delta(\phi)$. If

$$\phi_x{}^1, \ \phi_x{}^2, \ \dots$$

is a system of forms satisfying certain conditions, which are unspecified, except that they are such that, if $\phi_x{}^i$, $\phi_x{}^j$ satisfy the conditions, so does

$$\frac{\lambda \phi_x{}^i + \mu \phi_x{}^j}{\lambda + \mu} \quad (\lambda + \mu \neq 0);$$

then, if $$D_\Delta(\phi^n) \geqslant d \quad (\text{all } n),$$

$$D_\Delta(\phi^i - \phi^j) \leqslant [\sqrt{\{D_\Delta(\phi^i) - d\}} + \sqrt{\{D_\Delta(\phi^j) - d\}}]^2.$$

Weyl's proof can be applied to our case, and gives the result immediately.

§ 3. *Application to analytic varieties.*

11. We are now in a position to make our applications to analytic varieties. We shall say that an analytic variety of n dimensions is a Riemannian manifold, denoted by M, if it can be "triangulated" to form an absolute orientable manifold in the topological sense, and if at the same time it is the seat of a positive definite Riemannian metric. In the region in which the parameters $(x^1, \dots, x^n)$ are valid, the metric will be given by

† Weyl, *Die Idee der Riemannschen Fläche*, 101.

the quadratic differential form

$$g_{ij}\,dx^i\,dx^j.$$

The g's will be supposed to be analytic functions of the x's.

12. Let the Betti numbers of M be $R_0, \ldots, R_n$. Then it has been shown by de Rham† that there is a one-valued p-form

$$\phi_x = b_{i_1 \ldots i_p}\,dx^{i_1} \ldots dx^{i_p},$$

satisfying conditions A everywhere on M, and such that

$$\int \phi_x$$

has assigned periods in the R_p independent p-cycles of M. In actual fact de Rham's integrals satisfy more stringent conditions, but conditions A are all that we require.

We can find a finite number r of geodesic spheres on M such that every point of M is internal to at least one of these spheres. Let the spheres be

$$\Gamma_1, \Gamma_2, \ldots, \Gamma_r,$$

and let them bound cells

$$\Delta_1, \Delta_2, \ldots, \Delta_r$$

respectively. Suppose that ϕ_x is not harmonic in Δ_1. Then we can replace ϕ_x by $\tilde{\phi}_x$, satisfying conditions A, and having the same periods, such that

$$D(\tilde{\phi}) = D_M(\tilde{\phi}) < D(\phi).$$

The process of No. 8 is used. If $\tilde{\phi}_x$ is not harmonic in Δ_2 we proceed in the same way. Going through the sequence $\Delta_1, \Delta_2, \ldots, \Delta_r$ an infinite number of times, we construct a sequence of forms $\{\phi_x{}^m\}$ such that

$$D(\phi^m) < D(\phi^{m'}) \quad \text{if} \quad m' < m.$$

The sequence is only finite if we have a number μ such that $\phi_x{}^\mu$ is harmonic everywhere on M. If this is the case our problem is really solved, so we shall suppose that no such μ exists.

Since $\{D(\phi^m)\}$ is a decreasing sequence, and

$$D(\phi^m) > 0,$$

† de Rham, *Journal de Math.* (9), 10 (1931), 115–200.

the sequence has a lower bound which we denote by d. We have to show that, as $D(\phi^m)$ tends to d, $\{\phi_x{}^m\}$ converges to a limit ϕ_x, which satisfies the equation

$$(\phi^x)_x = 0$$

everywhere on M.

13. Let us consider a point P of M, and take this as origin of normal coordinates. Let Δ be the cell bounded by a geodesic sphere Γ of radius a, whose centre is P. Let $\bar{\phi}_x{}^m$ be the harmonic form in Δ which defines the same functional as $\phi_x{}^m$ on Γ. Write

$$\phi_x{}^m = B^m_{i_1 \dots i_p} dx^{i_1} \dots dx^{i_p},$$

$$\bar{\phi}_x{}^m = \bar{B}^m_{i_1 \dots i_p} dx^{i_1} \dots dx^{i_p}.$$

Consider

$$\psi^m_{j_1 \dots j_p} = \frac{1}{|\Delta|} \int_\Delta k^{i_1 \dots i_p}_{j_1 \dots j_p} B^m_{i_1 \dots i_p} dx^1 \dots dx^n,$$

$$|\psi^{m_1} - \psi^{m_2}| \leqslant \frac{1}{|\Delta|^2} \int_\Delta \sum_i (k^{i_1 \dots i_p}_{j_1 \dots j_p})^2 dx^1 \dots dx^n \int_\Delta \sum_i (B^{m_1}_{i_1 \dots i_p} - B^{m_2}_{i_1 \dots i_p})^2 dx^1 \dots dx^n$$

$$\leqslant k \int_\Delta \sum_i (B^{m_1}_{i_1 \dots i_p} - B^{m_2}_{i_1 \dots i_p})^2 dx^1 \dots dx^n,$$

where k is a constant not dependent on the B's. Hence we have

$$|\psi^{m_1} - \psi^{m_2}| < k' D(\phi^{m_1} - \phi^{m_2}) \leqslant k' [\sqrt{\{D(\phi^{m_1}) - d\}} + \sqrt{\{D(\phi^{m_2}) - d\}}]^2,$$

by Levi's inequality. Hence $\{\psi^m\}$ converges. Similarly, if

$$\bar{\psi}^m_{j_1 \dots j_p} = \frac{1}{|\Delta|} \int_\Delta k^{i_1 \dots i_p}_{j_1 \dots j_p} \bar{B}^m_{i_1 \dots i_p} dx^1 \dots dx^n,$$

$$|\bar{\psi}^{m_1} - \bar{\psi}^{m_2}| \leqslant k' D_\Delta(\bar{\phi}^{m_1} - \bar{\phi}^{m_2})$$

$$\leqslant k' D_\Delta(\phi^{m_1} - \phi^{m_2})$$

$$\leqslant k' [\sqrt{\{D(\phi^{m_1}) - d\}} + \sqrt{\{D(\phi^{m_2}) - d\}}]^2.$$

Hence $\{\bar{\psi}^m\}$ also converges. A similar argument will hold if, in place of Γ, we use any sphere Γ' lying in Δ. Now

$$\bar{\psi}^m_{j_1 \dots j_p} = \bar{B}^m_{j_1 \dots j_p}$$

at the centre of the sphere. Hence it follows that $\{\bar{\phi}_x{}^m\}$ converges at all points of Δ to a limit ϕ_x. Also

$$B^m_{i_1 \dots i_\rho} = \int_{\Gamma'} K^{j_1 \dots j_\rho}_{i_1 \dots i_\rho} \bar{B}_{j_1 \dots j_\rho} d\Gamma'$$

at all points in Δ' (but not necessary on Γ'). Hence, proceeding to the limit, we see that ϕ_x defines a harmonic functional in Δ.

Now consider $\psi^m - \bar{\psi}^m$. Using Δ' and Γ' as above, in place of Δ, Γ,

$$|\psi^m - \bar{\psi}^m| < k' D_{\Delta'}(\phi^m - \bar{\phi}^m) \leqslant k' D_\Delta(\phi^m - \bar{\phi}^m)$$
$$= k' \{D_\Delta(\phi^m) - D_\Delta(\bar{\phi}^m)\} \leqslant k' \{D(\phi^m) - d\}.$$

We thus see that the sequence $\{\psi^m_{i_1 \dots i_\rho}\}$ converges, whatever sphere Δ' is used in Δ, to the value of $B_{i_1 \dots i_\rho}$ at the centre of the sphere, where

$$\phi_x = B_{i_1 \dots i_\rho} dx^{i_1} \dots dx^{i_\rho}.$$

14. In place of Δ, take the regions $\Delta_1, \Delta_2, \dots, \Delta_r$ in turn. For each of these we derive from the sequences $\{\psi^m\}, \{\bar{\psi}^m\}$ harmonic forms which we shall write as $(\phi_x)_1, \dots, (\phi_x)_r$. Taking as our Δ' of No. 13 a spherical cell lying in $(\Delta_i . \Delta_j)$, it follows that the sequences $\{\psi^m\}_i, \{\psi^m\}_j$ have the same limit. Thus, at the centre of Δ',

$$(\phi_x)_i = (\phi_x)_j,$$

and thus these forms coincide wherever they are both defined. Thus we deduce a single form ϕ_x which is harmonic everywhere on M.

Since the sequences $\{\psi^m\}, \{\bar{\psi}^m\}$ have the same limit for all points P of M, this limit being independent of the radius a, it follows that $\{\phi_x{}^m\}$ converges to ϕ_x almost everywhere on M. Let δ_p be any p-cell of integration lying entirely in Δ_i, say. It is obtained by putting

$$x^i = f^i(u^1, \dots, u^\nu),$$

where the u's vary in a certain p-cell of the Euclidean p-space $(u^1, \dots, u^\nu)$, and the f's have continuous partial derivatives of the first order. Let

$$\delta_p \to j_{p-1}.$$

Then we can find a family of cells δ_p, depending on $n-p$ parameters $(u^{p+1}, \dots, u^n)$, which vary in an $(n-p)$-cell E_{n-p}, all the cells of the family having the same boundary j_{p-1}. No two cells of the family are to have any points in common except on j_{p-1}, and, as the parameters vary, the p-cells

are to describe an n-cell in Δ_i. Now

$$\int_{E_{n-p}} \left[\int_{\delta_p} \phi_x{}^m\right] du^{p+1} \dots du^n \quad \text{tends to} \quad \int_{E_{n-p}} \left[\int_{\delta_p} \phi_x\right] du^{p+1} \dots du^n$$

as m tends to infinity, or

$$\int_{\delta_p} \phi_x{}^m \quad \text{tends to} \quad \int_{\delta_p} \phi_x$$

for almost every cell δ_p. But these integrals depend only on j_{p-1} and not on the particular δ_p. Hence

$$\int_{\delta_p} \phi_x{}^m \quad \text{tends to} \quad \int_{\delta_p} \phi_x$$

for every δ_p. From this we deduce that, if Γ_p is any p-cycle of M, on which the integrals

$$\int \phi_x{}^m$$

have period π,

$$\int_{\Gamma_p} \phi_x = \pi.$$

Hence $\int \phi_x$ is a harmonic functional on M which has the assigned periods with respect to the p-cycles of M.

We can regard ϕ_x as the limit of the minimising sequence $\{\phi_x{}^m\}$. It is certainly the limit of this sequence, except perhaps at a set of measure zero. If there are exceptional points, we can regard these as due to a faulty choice in our minimising sequence. ϕ_x is indeed a form which minimises the integral

$$\int_M \sqrt{g}\, g^{i_1 \dots i_p, j_1 \dots j_p} B_{i_1 \dots i_p} B_{j_1 \dots j_p} dx^1 \dots dx^n$$

for the system of de Rham integrals with the assigned periods.

15. We can thus obtain a harmonic integral $\int \phi_x$ having assigned periods on the p-cycles of M. To show that it is unique, we merely need to show that there is no harmonic integral having zero periods. Suppose that $\int \psi_x$ is a harmonic integral with zero periods. Then we shall show that $D(\psi) = 0$, or $\psi_x = 0$ everywhere.

Let M' be any image of M on which we shall denote any entity by the symbol for the corresponding entity on M, with a dash attached. If

ϕ_x is any harmonic form, let $\int \phi_x$, $\int \phi^x$ have periods

$$\alpha_1, \ldots, \alpha_{R_p} \quad \text{on the } p\text{-cycles} \quad \Gamma_p^1, \ldots, \Gamma_p^{R_p},$$

and $$\beta_1, \ldots, \beta_{R_{n-p}} \quad \text{on the } (n-p)\text{-cycles} \quad \Gamma_{n-p}^1, \ldots, \Gamma_{n-p}^{R_{n-p}},$$

of M_1 respectively. The integral $\int \phi_x . \phi'^{x'}$ defines a functional on the product manifold $M \times M'$, and this integral has zero periods on the cycles $\Gamma_r^i \times \Gamma_{n-r}'^j$ if $r \neq p$, and has the period $\alpha_i \beta_j$ if $r = p$. Hence, if $\phi_x = \psi_x$,

$$\int \psi_x . \psi'^{x'}$$

has no periods on $M \times M'$. Consider, in particular, the cycle Γ of $M \times M'$ given by $x^i = x'^i$ (the diagonal cycle). Then

$$\int_\Gamma \psi_x . \psi'^{x'} = 0.$$

But clearly $$\int_\Gamma \psi_x . \psi'^{x'} = D(\psi).$$

Hence $\psi_x = 0$, everywhere.

In the general case,

$$D(\phi) = \int_\Gamma \phi^x . \phi'^{x'} = a^{ij} \alpha_i \beta_j,$$

where (a^{ij}) is the transpose of the inverse of the matrix $(\Gamma_p^{\ i} \Gamma_{n\ p}^{j})$ of the intersection of the p-cycles and the $(n-p)$-cycles†.

16. We can now sum up our results in the theorem:

Attached to M there are R_p p-fold harmonic integrals $I = \int \phi_x$, say

$$I_1, \ I_2, \ \ldots, \ I_{R_p},$$

such that, if

$$\Gamma_p^{\ 1}, \ \ldots, \ \Gamma_p^{R_p}$$

are the R_p independent p-cycles of M, and $I_i(\Gamma_p^{\ j})$ denotes the period of I_i with respect to $\Gamma_p^{\ j}$, then

$$I_i(\Gamma_p^{\ j}) = \delta_i^{\ j}.$$

Further, any harmonic p-fold integral attached to M is a linear combination of these, with constant coefficients.

† Hodge, *Proc. Amer. Nat. Acad. of Sci.*, 17 (1931), 643.

This result holds for $p = 1$ to $p = n-1$. We can complete the series in this way. In the case $p = 0$, take

$$\phi_x = 1,$$

defining an integral of order zero by the rule that, if

$$\Gamma_0 = a_i P^i$$

is any zero chain, $$\int_{\Gamma_0} 1 = \sum_i a_i.$$

In the case $p = n$, we take

$$\phi_x = \frac{\sqrt{g}\, dx^1 \dots dx^n}{|M|},$$

where $|M|$ denotes the volume of M.

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THE LATTICE-POINTS IN A CIRCLE

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1. Let $R(x)$ denote the number of lattice-points inside or on the circle $u^2+v^2=x$; *i.e.* the number of solutions in integers m, n of the inequality $m^2+n^2 \leqslant x$. It is easily proved that, as $x \to \infty$, $R(x) \sim \pi x$, and, in fact, that

$$R(x) = \pi x + O(x^\alpha), \tag{1}$$

for some values of α less than 1. But the problem of the order of $R(x)-\pi x$ remains unsolved*.

It is a question of finding the lower bound, ϑ say, of numbers α for which (1) is true. It was proved by Hardy and by Landau† that $\vartheta \geqslant \frac{1}{4}$. The best result hitherto obtained on the other side is that $\vartheta \leqslant \frac{27}{82}$. This was proved by Nieland, by van der Corput's method‡.

Here I prove that $\vartheta \leqslant \frac{15}{46}$, a slightly better result. The method is an extension to two variables of van der Corput's method, and has already been used in the problem of the order of Epstein's zeta-function§. The present paper is on the same general lines as the previous one, but most of the details turn out to be more complicated than in the Epstein zeta-function case.

The one-dimensional van der Corput method depends on the repeated use of a lemma due to Weyl‖. It will be seen that we extend this to two variables in two different ways (Lemmas β and β'). In the application we use Lemma β' twice and then Lemma β once. But Lemma β' is really

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* A detailed discussion is given by Landau, *Vorlesungen über Zahlentheorie*, 2, Achter Teil.

† Landau, *ibid.*, Kap. 6.

‡ See E. C. Titchmarsh, *Quart. J. of Math.* (Oxford Series), 2 (1931), 161–73, for a simplified proof and references.

§ E. C. Titchmarsh, *Proc. London Math. Soc.* (2), 36 (1934), 485–500. We shall refer to this paper as E.

‖ Lemma 3 of my *Quart. J. of Math.* paper.

only a makeshift. If we could carry out a similar argument, using Lemma β each time, we should, apparently, obtain a still better result, in fact, that $\vartheta \leqslant \frac{1}{3}\frac{2}{7}$. Unfortunately, this method would introduce still further complications. The proof as it stands depends on the fact that the function § 11(4) is everywhere positive. The corresponding function obtained by using Lemma β each time vanishes along certain curves, and special arguments are required to deal with the situation in the neighbourhood of these curves. It seems likely that this peculiarity would only make a difference of a logarithmic factor in the result, but I have not succeeded in carrying out the necessary analysis.

We begin by proving a series of lemmas*.

2. LEMMA α. *Let $a_{\mu,\nu}$ be any numbers, real or complex, such that, if*

$$s_{m,n} = \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} a_{\mu,\nu},$$

then

$$|s_{m,n}| \leqslant G \quad (1 \leqslant m \leqslant M,\ 1 \leqslant n \leqslant N).$$

Let $b_{m,n}$ denote real numbers, $0 \leqslant b_{m,n} \leqslant H$, and let each of the expressions

$$b_{m,n}-b_{m,n+1}, \quad b_{m,n}-b_{m+1,n}, \quad b_{m,n}-b_{m+1,n}-b_{m,n+1}+b_{m+1,n+1}$$

be of constant sign for the values of m and n in question. Then

$$\left| \sum_{m=1}^{M} \sum_{n=1}^{N} a_{m,n} b_{m,n} \right| \leqslant 5GH.$$

This is Lemma α of E.

3. LEMMA β. *Let $f(x, y)$ be a real function of x and y, and*

$$S = \Sigma\Sigma\, e^{2\pi i f(m,n)},$$

the sum being taken over the lattice-points of a region D included in the

* For the meaning of A and O, see E, § 1; but it should be added that $O(1)$, for example, is also used in the sense of "bounded for all functions considered", without reference to any limiting process. The first line of p. 495 of E should read "consider any square, at some point of which". If $|x-y\sqrt{a}| < A_2 L$ at some point of the square, of side $l \leqslant L$, then $|x-y\sqrt{a}| < A_2 L + Al$ throughout the square. The conditions of Lemma ζ are satisfied throughout such a square if, as can plainly be arranged, l/L is small enough. In the remaining squares, referred to in line 8, $|x-y\sqrt{a}| \geqslant A_2 L$ throughout the square. This appears to meet the main objection raised by Walfisz, *Zentralblatt für Math.*, 8 (1934), 301.

rectangle $a \leqslant x \leqslant b$, $\alpha \leqslant y \leqslant \beta$. *Let*

$$S' = \Sigma\Sigma e^{2\pi i\{f(m+\mu,\, n+\nu)-f(m,\, n)\}}, \quad S'' = \Sigma\Sigma e^{2\pi i\{f(m+\mu,\, n-\nu)-f(m,\, n)\}},$$

where μ *and* ν *are integers, and* S' *is taken over values of* m *and* n *such that both* (m, n) *and* $(m+\mu, n+\nu)$ *belong to* D; *and similarly for* S''. *Let* ρ *be a positive integer not exceeding* $b-a$, *and let* ρ' *be a positive integer not exceeding* $\beta-\alpha$. *Then*

$$S = O\left\{\frac{(b-a)(\beta-\alpha)}{(\rho\rho')^{\frac{1}{2}}}\right\} + O\left[\left\{\frac{(b-a)(\beta-\alpha)}{\rho\rho'}\sum_{\mu=1}^{\rho-1}\sum_{\nu=0}^{\rho'-1}|S'|\right\}^{\frac{1}{2}}\right]$$

$$+ O\left[\left\{\frac{(b-a)(\beta-\alpha)}{\rho\rho'}\sum_{\mu=0}^{\rho-1}\sum_{\nu=1}^{\rho'-1}|S''|\right\}^{\frac{1}{2}}\right].$$

The proof is practically the same as that of Lemma β of E, in which D is the whole rectangle $(a, b;\ \alpha, \beta)$. The only difference is that we replace certain terms, of modulus 1, by 0.

We may, in particular, take $\rho' = 1$. Then the sum involving S'' disappears, and we obtain

LEMMA β'. *If* $0 < \rho \leqslant b-a$,

$$S = O\left\{\frac{(b-a)(\beta-\alpha)}{\rho^{\frac{1}{2}}}\right\} + O\left[\left\{\frac{(b-a)(\beta-\alpha)}{\rho}\sum_{\mu=1}^{\rho-1}|S'''|\right\}^{\frac{1}{2}}\right],$$

where

$$S''' = \Sigma\Sigma e^{2\pi i\{f(m+\mu,\, n)-f(m,\, n)\}}.$$

4. LEMMA γ. *Let* $f(x, y)$ *be a real differentiable function of* x *and* y. *Let* $f_x(x, y)$ *be a monotonic function of* x *for each value of* y *considered, and* $f_y(x, y)$ *a monotonic function of* y *for each value of* x *considered. Let* $|f_x| \leqslant \frac{3}{4}$, $|f_y| \leqslant \frac{3}{4}$, *for* $a \leqslant x \leqslant b$, $\alpha \leqslant y \leqslant \beta$, *where* $b-a \leqslant l$, $\beta-\alpha \leqslant l$ $(l \geqslant 1)$. *Let* D *be the rectangle* $(a, b;\ \alpha, \beta)$, *or part of the rectangle cut off by a continuous monotonic curve. Then*

$$\Sigma\Sigma_D e^{2\pi i f(m,\, n)} = \iint_D e^{2\pi i f(x,\, y)}\, dx\, dy + O(l).$$

This is equivalent to Lemma γ of E.

5. In Lemma δ of E, an inequality was obtained for the integral

$$\iint e^{2\pi i f(x,\, y)}\, dx\, dy$$

under the conditions

$$|f_{xx}f_{yy}-f_{xy}^2| \geqslant r^2, \quad r \leqslant |f_{xx}| < Ar,$$

etc. It is now necessary, or at any rate very convenient, to have a similar result when $|f_{xx}f_{yy}-f_{xy}^2|$ may be small compared with any of $|f_{xx}|^2$, $|f_{xy}|^2$, and $|f_{yy}|^2$.

We suppose that $f(x, y)$ is a real function of x and y with continuous partial derivatives of as many orders as may be required; and also that any curve defined by equating to zero a polynomial of given degree in these derivatives has $O(1)$ intersections with any other such curve, or with any straight line. In the case considered all such curves are algebraic, so that this condition is satisfied.

LEMMA δ. *Let $f(x, y)$ satisfy the above conditions in the rectangle $(a, b;\ \alpha, \beta)$. Let*

$$R \leqslant |f_{xx}| < AR, \quad R \leqslant |f_{yy}| < AR, \quad |f_{xy}| < AR, \tag{1}$$

and

$$|f_{xx}f_{yy}-f_{xy}^2| \geqslant r^2, \tag{2}$$

where $0 < r \leqslant R$, throughout the rectangle.

Let $|f_x| \leqslant r_1$, $|f_y| \leqslant r_1$, $|f_{xxx}| \leqslant r_3$, $|f_{xxy}| \leqslant r_3$, $|f_{xyy}| \leqslant r_3$, $|f_{yyy}| \leqslant r_3$, and let

$$r_1 r_3 < Kr^2, \tag{3}$$

where K is a sufficiently small constant (depending on the other constants). Then

$$\int_a^b \int_\alpha^\beta e^{2\pi i f(x, y)}\, dx\, dy = O\left(\frac{1}{r}\right).$$

If $f(x, y)$ was a quadratic, the third-order derivatives would vanish, and condition (3) would not be required. In general, K has to be so small that the third-order derivatives do not affect the argument.

Divide up the rectangle by the network of curves

$$f_x = m\sqrt{R}, \quad f_y - f_x f_{xy}/f_{xx} = nr/\sqrt{R},$$

where m and n are integers. On the former curves,

$$f_{xx} + f_{xy}\frac{dy}{dx} = 0, \quad \left|\frac{dy}{dx}\right| = \left|\frac{f_{xx}}{f_{xy}}\right| > A.$$

On the latter set,

$$f_{xy}-f_{xx}\frac{f_{xy}}{f_{xx}}+O\left(\frac{r_1r_3}{R}\right)+\left\{f_{yy}-f_{xy}\frac{f_{xy}}{f_{xx}}+O\left(\frac{r_1r_3}{R}\right)\right\}\frac{dy}{dx}=0,$$

so that, if K is small enough,

$$\frac{dy}{dx}=\frac{O(r_1r_3)}{f_{xx}f_{yy}-f_{xy}^2+O(r_1r_3)}=O\left(\frac{r_1r_3}{r^2}\right).$$

Hence there is a positive lower bound to the angles between curves of the two sets.

Consider first the region $f_x \geqslant \sqrt{R}$, say D. We have

$$(1)\quad \iint_D e^{2\pi i f(x,y)}\,dx\,dy=\int\left[\frac{e^{2\pi i f(x,y)}}{2\pi i f_x}\right]_{\chi(y)}^{\omega(y)}dy+\frac{1}{2\pi i}\iint_D\frac{f_{xx}}{f_x^2}e^{2\pi i f(x,y)}\,dx\,dy,$$

where $x=\chi(y)$, $x=\omega(y)$ are the boundaries of D. Let D_1 be the strip between the curves

$$f_y-f_x\frac{f_{xy}}{f_{xx}}=\pm\frac{r}{\sqrt{R}},$$

and let D_2 be the remainder of D. Let $x=\phi(u,y)$ be the solution of $f_x=u$. Then

$$\frac{\partial}{\partial y}f\{\phi(u,y),y\}=f_y-f_x\frac{f_{xy}}{f_{xx}}.$$

By the general conditions on f, this is monotonic in $O(1)$ y-intervals, for a fixed u, and in D_2 its absolute value is not less than $r/\sqrt{R}$. Hence

$$(2)\qquad \int e^{2\pi i f\{\phi(u,y),y\}}\,dy=O\left(\frac{\sqrt{R}}{r}\right),$$

and (limits not specified depending on the figure)

$$(3)\quad \iint_{D_2}\frac{f_{xx}}{f_x^2}e^{2\pi i f(x,y)}\,dx\,dy=\int dy\int\frac{f_{xx}}{f_x^2}e^{2\pi i f(x,y)}\,dx=\int dy\int_{\sqrt{R}}e^{2\pi i f\{\phi(u,y)\}}\frac{du}{u^2}$$

$$=\int_{\sqrt{R}}\frac{du}{u^2}\int e^{2\pi i f\{\phi(u,y),y\}}\,dy=\int_{\sqrt{R}}O\left(\frac{\sqrt{R}}{r}\right)\frac{du}{u^2}$$

$$=O\left(\frac{1}{r}\right).$$

Also $$\left|\iint_{D_1}\frac{f_{xx}}{f_x^2}e^{2\pi i f(x,y)}\,dx\,dy\right|<AR\iint_{D_1}\frac{dx\,dy}{f_x^2}.$$

Here we change to variables u, v, defined by

$$u = f_x, \quad v = f_y - f_x f_{xy}/f_{xx}.$$

Then

$$\frac{\partial(u, v)}{\partial(x, y)} = f_{xx}\left\{f_{yy} - f_{xy}\frac{f_{xy}}{f_{xx}} + O\left(\frac{r_1 r_3}{R}\right)\right\} - f_{xy}\left\{f_{xy} - f_{xx}\frac{f_{xy}}{f_{xx}} + O\left(\frac{r_1 r_3}{R}\right)\right\}$$

$$= f_{xx} f_{yy} - f_{xy}^2 + O(r_1 r_3),$$

$$\left|\frac{\partial(u, v)}{\partial(x, y)}\right| > Ar^2.$$

Hence

$$\iint_{D_1} \frac{dx\,dy}{f_x^{\,2}} \leqslant \int_{\sqrt{R}}^{r/\sqrt{R}} \frac{1}{u^2}\left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv < \frac{A}{r^2}\int_{\sqrt{R}}^{\infty}\frac{du}{u^2}\int_0^{r/\sqrt{R}} dv = \frac{A}{rR},$$

and the required result follows for this part.

Next consider the integrated terms in (1). Here there are two cases; the function $\chi(y)$ is either the solution of $f_x = \sqrt{R}$, or it is a constant. In the former case, the integrated term at the lower limit is

$$-\frac{1}{2\pi i\sqrt{R}}\int e^{2\pi i f\{\chi(y), y\}}\,dy,$$

the integral being over $O(1)$ intervals, by the general conditions on $f(x, y)$. As before, the part of the integral corresponding to D_2 is $O(\sqrt{R}/r)$, and the required result follows for this part. The remainder is

$$O\left(\frac{1}{\sqrt{R}}\int_{|v|\leqslant r/\sqrt{R}} dy\right) = O\left(\frac{1}{\sqrt{R}}\int_0^{r/\sqrt{R}}\left|\frac{dy}{dv}\right| dv\right),$$

v being now taken on $u = \sqrt{R}$. Here

$$\frac{dv}{dy} = \left\{f_{xy} - f_{xx}\frac{f_{xy}}{f_{xx}} + O\left(\frac{r_1 r_3}{R}\right)\right\}\left(-\frac{f_{xy}}{f_{xx}}\right) + f_{yy} - f_{xy}\frac{f_{xy}}{f_{xx}} + O\left(\frac{r_1 r_3}{R}\right)$$

$$= \frac{f_{xx} f_{yy} - f_{xy}^2}{f_{xx}} + O\left(\frac{r_1 r_3}{R}\right),$$

$$\left|\frac{dv}{dy}\right| > A\,\frac{r^2}{R}.$$

Hence we obtain

$$O\left(\frac{1}{\sqrt{R}}\int_0^{r/\sqrt{R}}\frac{R}{r^2}\,dv\right) = O\left(\frac{1}{r}\right).$$

Secondly, if $\chi(y)$ is a constant, a say, the term considered is

$$-\frac{1}{2\pi i}\int \frac{e^{2\pi i f(a,y)}}{f_x(a,y)}\,dy.$$

Since $|f_{yy}| \geqslant R$,

$$\int e^{2\pi i f(a,y)}\,dy = O\left(\frac{1}{\sqrt{R}}\right),$$

and, since here $|f_x(a,y)| \geqslant \sqrt{R}$, and $f_x(a,y)$ is monotonic over $O(1)$ intervals, the whole is

$$O(1/R) = O(1/r).$$

There remains the original integral over the strip $0 \leqslant f_x \leqslant \sqrt{R}$, say D'. We divide this into D_1' and D_2' as before. Putting $f_x = u$, we have

$$\iint_{D_2'} e^{2\pi i f(x,y)}\,dx\,dy = \int dy \int_0^{\sqrt{R}} \frac{e^{2\pi i f(x,y)}}{f_{xx}}\,du = \int_0^{\sqrt{R}} du \int \frac{e^{2\pi i f(x,y)}}{f_{xx}}\,dy.$$

As before, for a fixed u we have (2), and hence

$$\int \frac{e^{2\pi i f(x,y)}}{f_{xx}}\,dy = O\left(\frac{1}{r\sqrt{R}}\right),$$

and the required result follows for this part. Finally,

$$\left|\iint_{D_1'} e^{2\pi i f(x,y)}\,dx\,dy\right| \leqslant \iint_{D_1'} dx\,dy = \int_0^{\sqrt{R}}\int_0^{r/\sqrt{R}} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \int_0^{\sqrt{R}}\int_0^{r/\sqrt{R}} O\left(\frac{1}{r^2}\right) du\,dv = O\left(\frac{1}{r}\right).$$

This proves the result stated for the region $f_x \geqslant 0$, and a similar proof applies to the region $f_x < 0$.

6. LEMMA ϵ. *Let $f(x)$ satisfy the same conditions as in Lemma δ, but in a rectangle with sides inclined at a fixed angle to the axes, and of length $O(l)$. Then*

$$\iint e^{2\pi i f(x,y)}\,dx\,dy = O\left(\frac{1+|\log l|+|\log R|}{r}\right).$$

The proof goes as before, except that, in the integrated terms arising from D, we may now have $\chi(y) = \lambda y + \mu$, where λ and μ are constants. We have then to consider

$$\int \frac{e^{2\pi i f(\lambda y+\mu,\, y)}}{f_x(\lambda y+\mu,\, y)}\,dy.$$

Here $$\left|\frac{d^2}{dy^2} f(\lambda y+\mu, y)\right| = |\lambda^2 f_{xx}+2\lambda f_{xy}+f_{yy}| > AR,$$

and the result follows as before, if $\lambda < \lambda_0$, where λ_0 depends on the various constants of the lemma.

If $\lambda \geqslant \lambda_0$, we have

$$|f_x(\lambda y+\mu, y)| = \left|\int_{x_0}^{\lambda y+\mu} f_{xx}(x, y)\,dx\right| \geqslant (\lambda y+\mu-x_0)\,R,$$

and also (in D) $|f_x| \geqslant \sqrt{R}$. Hence

$$\int \frac{dy}{|f_x(\lambda y+\mu, y)|} \leqslant \frac{1}{\sqrt{R}} \int_{|\lambda y+\mu-x_0|\leqslant 1/\sqrt{R}} dy + \frac{1}{R} \int_{|\lambda y+\mu-x_0|>1/\sqrt{R}} \frac{dy}{|\lambda y+\mu-x_0|}$$

$$= O\left(\frac{1}{\lambda R}\right) + O\left\{\frac{1}{\lambda R}\left(\log R + \log|\lambda y_1+\mu-x_0|\right)\right\},$$

where $1/\sqrt{R} \leqslant |\lambda y_1+\mu-x_0| < O(\lambda l)$; and the result again follows.

7. LEMMA ζ. *Let $f(x, y)$ satisfy the conditions of Lemma δ, with the exception of the lower bounds for $|f_{xx}|$ and $|f_{yy}|$ in (1); and let $b-a \leqslant l$, $\beta-\alpha \leqslant l$, and*

$$lr_3 < Kr,$$

where K is a sufficiently small constant. Then

$$\int_a^b \int_\alpha^\beta e^{2\pi i f(x, y)}\,dx\,dy = O\left(\frac{1+|\log l|+|\log R|}{r}\right).$$

The variation of f_{xx}, f_{xy}, or f_{yy} in the rectangle is at most $2lr_3 = c$, say. Hence there are numbers λ, μ, ν such that

$$\lambda c \leqslant |f_{xx}| \leqslant (\lambda+1)\,c, \quad \mu c \leqslant |f_{xy}| \leqslant (\mu+1)\,c, \quad \nu c \leqslant |f_{yy}| \leqslant (\nu+1)\,c.$$

We may suppose, without loss of generality, that $\nu \leqslant \lambda$. We then consider the following cases.

(i) Let $\lambda \leqslant 10\nu$, $\mu \leqslant 10\lambda$, $\nu \geqslant 1$. Then

$$(\mu+1)\,c \leqslant (100\nu+1)\,c \leqslant 101\nu c,$$

and the result follows from Lemma δ.

(ii) Let $\lambda \leqslant 10\nu$, $\mu > 10\lambda$, $\mu \geqslant 2$. Let

$$\sqrt{2}x = \xi+\eta, \quad \sqrt{2}y = -\xi+\eta.$$

Then the integral becomes

$$\iint e^{2\pi i\,\phi(\xi,\,\eta)}\,d\xi\,d\eta,$$

where $\phi(\xi, \eta) = f(x, y)$. Hence

$$\phi_{\xi\xi} = \tfrac{1}{2}(f_{xx}+f_{yy})-f_{xy}, \quad \phi_{\eta\eta} = \tfrac{1}{2}(f_{xx}+f_{yy})+f_{xy},$$

$$\phi_{\xi\eta} = \tfrac{1}{2}(f_{xx}-f_{yy}).$$

Hence

$$\min(|\phi_{\xi\xi}|,\ |\phi_{\eta\eta}|) \geqslant \{\mu-\tfrac{1}{2}(\lambda+\nu+2)\}c \geqslant \{\mu-(\tfrac{1}{10}\mu+1)\}c = (\tfrac{9}{10}\mu-1)c,$$

and

$$\max(|\phi_{\xi\xi}|,\ |\phi_{\xi\eta}|,\ |\phi_{\eta\eta}|) \leqslant (\lambda+\mu+2)c \leqslant (\tfrac{11}{10}\mu+2)c.$$

The result now follows from Lemma ϵ.

(iii) Let $\lambda > 10\nu$, $\mu \leqslant \frac{2}{5}\lambda$. Transforming as in (ii), we have

$$\min(|\phi_{\xi\xi}|, |\phi_{\eta\eta}|) \geqslant \{\tfrac{1}{2}(\lambda-\nu-1)-\mu-1\}c \geqslant \{\tfrac{1}{2}(\tfrac{9}{10}\lambda-1)-\tfrac{2}{5}\lambda-1\}c$$

$$= (\tfrac{1}{20}\lambda-\tfrac{3}{2})c,$$

and

$$\max(|\phi_{\xi\xi}|, |\phi_{\xi\eta}|, |\phi_{\eta\eta}|) \leqslant (\lambda+\mu+2)c \leqslant (\tfrac{7}{5}\lambda+2)c.$$

The result follows from Lemma ϵ if $\lambda \geqslant 40$.

(iv) Let $\lambda > 10\nu$, $\mu \geqslant \frac{3}{5}\lambda$. Transforming as in (ii),

$$\min(|\phi_{\xi\xi}|,\ |\phi_{\eta\eta}|) \geqslant \{\mu-\tfrac{1}{2}(\tfrac{5}{3}\mu+1+\tfrac{1}{6}\mu+1)\}c = (\tfrac{1}{12}\mu-1)c,$$

and

$$\max(|\phi_{\xi\xi}|,\ |\phi_{\xi\eta}|,\ |\phi_{\eta\eta}|) \leqslant (\tfrac{8}{3}\mu+2)c.$$

The result again follows from Lemma ϵ if $\mu \geqslant 24$.

(v) The above argument fails if $\mu-\frac{1}{2}\lambda$ is too small, but then we can use a different transformation. Let $\lambda > 10\nu$, $\frac{2}{5}\lambda \leqslant \mu \leqslant \frac{3}{5}\lambda$, and let

$$\sqrt{10}x = 3\xi+\eta, \quad \sqrt{10}y = -\xi+3\eta.$$

Then

$$10\phi_{\xi\xi} = 9f_{xx}+f_{yy}-6f_{xy}, \quad 10\phi_{\eta\eta} = f_{xx}+9f_{yy}+6f_{xy},$$

$$10\phi_{\xi\eta} = 3(f_{xx}-f_{yy})+8f_{xy}.$$

Hence

$$10|\phi_{\xi\xi}| \geqslant \{9\lambda-(\tfrac{1}{10}\lambda+1)-6(\tfrac{3}{5}\lambda+1)\}c = (\tfrac{53}{10}\lambda-7)c,$$

$$10|\phi_{\eta\eta}| \geqslant \{\tfrac{12}{5}\lambda-(\lambda+1)-9(\tfrac{1}{10}\lambda+1)\}c = (\tfrac{1}{2}\lambda-10)c,$$

$$\max(|\phi_{\xi\xi}|,\ |\phi_{\xi\eta}|,\ |\phi_{\eta\eta}) \leqslant 16(\lambda+1)c,$$

and the result again follows if $\lambda \geqslant 40$.

In all the remaining cases we have $\lambda < A$, $\mu < A$, $\nu < A$, so that

$$r^2 \leqslant |f_{xx}f_{yy}-f_{xy}^2| < Ac^2 = Al^2r_3^2.$$

This is excluded if the K of the lemma is small enough. Hence the result.

8. *The lattice-point problem.* It is known* that, if

$$R_1(x) = \int_0^x R(y)\,dy,$$

then
$$R_1(x) = \tfrac{1}{2}\pi x^2 + \frac{x}{\pi}\sum_{\nu=1}^{\infty}\frac{r(\nu)}{\nu}J_2\{2\pi\sqrt{(\nu x)}\}.$$

Hence, if $0 < \alpha < 1$,

$$\begin{aligned}\int_x^{x+x^\alpha}\{R(y)-\pi y\}\,dy &= \sum_{\nu=1}^{\lambda}\frac{r(\nu)}{\sqrt{\nu}}\int_x^{x+x^\alpha}\sqrt{y}\,J_1\{2\pi\sqrt{(\nu y)}\}\,dy \\ &\quad + \frac{x+x^\alpha}{\pi}\sum_{\nu=\lambda+1}^{\infty}\frac{r(\nu)}{\nu}J_2\{2\pi\sqrt{(\nu x+\nu x^\alpha)}\} \\ &\quad - \frac{x}{\pi}\sum_{\nu=\lambda+1}^{\infty}\frac{r(\nu)}{\nu}J_2\{2\pi\sqrt{(\nu x)}\} \\ &= \Sigma_1+\Sigma_2+\Sigma_3,\end{aligned}$$

say. Now

$$J_1\{2\pi\sqrt{(\nu y)}\} = \frac{\sin\{2\pi\sqrt{(\nu y)}-\frac{1}{4}\pi\}}{\pi(\nu y)^{\frac{1}{4}}} + O\left\{\frac{1}{(\nu y)^{\frac{3}{4}}}\right\}.$$

Hence
$$\Sigma_1 = O\left\{\int_x^{x+x^\alpha}|\phi(y)|\,y^{\frac{1}{4}}\,dy\right\} + O(x^{\alpha-\frac{1}{4}}),$$

where
$$\phi(y) = \sum_{\nu=1}^{\lambda}\frac{r(\nu)}{\nu^{\frac{3}{4}}}e^{2\pi i\sqrt{(\nu y)}} = \sum_{m^2+n^2\leqslant\lambda}\sum\frac{e^{2\pi i\sqrt{\{(m^2+n^2)\,y\}}}}{(m^2+n^2)^{\frac{3}{4}}}.$$

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* Landau, *ibid.*, 204, Satz 508.

Using the corresponding formula for J_2, we obtain similarly

$$\Sigma_3 = O\{x^{\frac{3}{4}}|\psi(x)|\}+O(x^{\frac{1}{4}}),$$

where

$$\psi(x) = \sum_{\nu=\lambda+1}^{\infty} \frac{r(\nu)}{\nu^{\frac{5}{4}}} e^{2\pi i\sqrt{(\nu x)}} = \sum\sum_{m^2+n^2>\lambda} \frac{e^{2\pi i\sqrt{\{(m^2+n^2)x\}}}}{(m^2+n^2)^{\frac{5}{4}}}.$$

There is also a similar result for Σ_2, with $x+x^a$ instead of x.

9. Certain parts of $\phi(y)$ and $\psi(x)$ can be dealt with directly. We have

$$\sum_{n=1}^{\infty} \frac{1}{(m^2+n^2)^{\frac{3}{4}}} \leqslant \sum_{n=1}^{m} \frac{1}{m^{\frac{3}{2}}} + \sum_{n=m+1}^{\infty} \frac{1}{n^{\frac{3}{2}}} = O\left(\frac{1}{m^{\frac{1}{2}}}\right).$$

Hence

$$\sum_{m=1}^{x^\beta} \sum_{n=1}^{\lambda-m^2} \frac{e^{2\pi i\sqrt{\{(m^2+n^2)y\}}}}{(m^2+n^2)^{\frac{3}{4}}} = \sum_{m=1}^{x^\beta} O\left(\frac{1}{m^{\frac{1}{2}}}\right) = O(x^{\frac{1}{2}\beta}),$$

and similarly for the sum with $n \leqslant x^\beta$. This will be found to be of the required order if $\beta = \frac{7}{46}$. Hence

$$\phi(y) = \phi_1(y)+O(x^{\frac{7}{92}}), \tag{1}$$

where $\phi_1(y)$ is the sum of those terms of $\phi(y)$ for which $m > x^{\frac{7}{46}}$ and $n > x^{\frac{7}{46}}$.

Again,

$$\sum_{n=1}^{\infty} \frac{1}{(m^2+n^2)^{\frac{5}{4}}} \leqslant \sum_{n=1}^{m} \frac{1}{m^{\frac{5}{2}}} + \sum_{n=m+1}^{\infty} \frac{1}{n^{\frac{5}{2}}} = O\left(\frac{1}{m^{\frac{3}{2}}}\right).$$

Hence

$$\sum_{m=x^\gamma}^{\infty} \sum \frac{e^{2\pi i\sqrt{\{(m^2+n^2)x\}}}}{(m^2-n^2)^{\frac{5}{4}}} = \sum_{m=x^\gamma}^{\infty} O\left(\frac{1}{m^3}\right) = O(x^{-\frac{1}{2}\gamma}),$$

and, if $x^{2\delta} < \frac{3}{4}\lambda$,

$$\sum_{m=1}^{x^\delta} \sum_{n=\sqrt{(\lambda-m^2)}}^{\infty} \frac{e^{2\pi i\sqrt{\{(m^2+n^2)x\}}}}{(m^2+n^2)^{\frac{5}{4}}} = \sum_{m=1}^{x^\delta} \sum_{m=\frac{1}{2}\sqrt{\lambda}}^{\infty} O\left(\frac{1}{n^{\frac{5}{2}}}\right) = O\left(\frac{x^\delta}{\lambda^{\frac{1}{4}}}\right).$$

These will be found to be of the required order if $\gamma = \frac{9}{46}$ and $\delta = \frac{1}{9}\frac{5}{2}$. Thus

$$\psi(x) = \psi_1(x)+O(x^{-\frac{\delta}{2}})+O(x^{\frac{1}{3}\delta}\lambda^{-\frac{1}{4}}), \tag{2}$$

where $\psi_1(x)$ is the sum of those terms of $\psi(x)$ for which m and n both lie between $x^{\frac{1}{3}\delta}$ and $x^{\frac{9}{46}}$.

10. In view of Lemma α, the problem now turns on the behaviour of sums of the form

$$\Sigma\Sigma e^{2\pi i f(m,n)}, \tag{1}$$

where $f(m, n) = \sqrt{\{(m^2+n^2)x\}}$. Let S denote a sum of this form, where the region of summation D is either a rectangle $M \leqslant m \leqslant M', N \leqslant n \leqslant N'$, or part of such a rectangle bounded by $m^2+n^2 = \lambda$, or by $m = n$, or both.

Let $L = \max(M, N) \geqslant x^{\frac{7}{16}}$. If $M'-M < x^{\frac{3}{23}}$, we have

$$S = O(Lx^{\frac{3}{23}}),$$

$$\Sigma\Sigma_D \frac{e^{2\pi i f(m,n)}}{(m^2+n^2)^{\frac{3}{4}}} = O\left(\frac{x^{\frac{3}{23}}}{L^{\frac{1}{2}}}\right) = O(x^{\frac{5}{92}}), \tag{2}$$

and

$$\Sigma\Sigma_D \frac{e^{2\pi i f(m,n)}}{(m^2+n^2)^{\frac{5}{4}}} = O\left(\frac{x^{\frac{3}{23}}}{L^{\frac{3}{2}}}\right) = O(x^{-\frac{9}{92}}).$$

A similar result holds if $N'-N < x^{\frac{3}{23}}$. We may therefore suppose that $M'-M \geqslant x^{\frac{3}{23}}$, $N'-N \geqslant x^{\frac{3}{23}}$.

We first apply Lemma β', and obtain

$$S = O\left(\frac{L^2}{\rho^{\frac{1}{2}}}\right) + O\left\{\frac{L}{\rho^{\frac{1}{2}}}\left(\sum_{m_1=1}^{\rho-1}|S_1|\right)^{\frac{1}{2}}\right\}, \tag{3}$$

where

$$S_1 = \Sigma\Sigma e^{2\pi i\{f(m+m_1, n)-f(m, n)\}},$$

and both $(m+m_1, n)$ and (m, n) belong to D. The number ρ satisfies the conditions of Lemma β' if

$$1 \leqslant \rho \leqslant \tfrac{1}{2}x^{\frac{3}{23}}. \tag{4}$$

Next, applying Lemma β', with σ instead of ρ, to S_1, we obtain

$$S_1 = O\left(\frac{L^2}{\sigma^{\frac{1}{2}}}\right) + O\left\{\frac{L}{\sigma^{\frac{1}{2}}}\left(\sum_{m_2=1}^{\sigma-1}|S_2|\right)^{\frac{1}{2}}\right\}, \tag{5}$$

where

$$S_2 = \Sigma\Sigma e^{2\pi i\{f(m+m_1+m_2, n)-f(m+m_1, n)-f(m+m_2, n)+f(m, n)\}},$$

the points $(m+m_1+m_2, n)$, etc., all belonging to D, and

$$1 \leqslant \sigma \leqslant \tfrac{1}{4}x^{\frac{3}{23}}. \tag{6}$$

By (5), the second term on the right of (3) is of the form

$$O\left(\frac{L^2}{\sigma^{\frac{1}{2}}}\right) + O\left[\frac{L^{\frac{3}{2}}}{\rho^{\frac{1}{2}}\sigma^{\frac{1}{2}}}\left\{\sum_{m_1=1}^{\rho-1}\left(\sum_{m_2=1}^{\sigma-1}|S_2|\right)^{\frac{1}{2}}\right\}^{\frac{1}{2}}\right].$$

The first term here is of the same form as the first term on the right of (3) if $\sigma = \rho^2$. With this value of σ we obtain

$$S = O\left(\frac{L^2}{\rho^{\frac{1}{2}}}\right) + O\left[\frac{L^{\frac{3}{2}}}{\rho}\left\{\sum_{m_1=1}^{\rho-1}\left(\sum_{m_2=1}^{\rho^2-1}|S_2|\right)^{\frac{1}{2}}\right\}^{\frac{1}{2}}\right], \tag{7}$$

and (4) and (6) are true if

$$1 \leqslant \rho \leqslant \tfrac{1}{2}x^{\frac{3}{4}}. \tag{8}$$

Finally we apply Lemma β, with ρ and ρ' each replaced by τ, to S_2. We obtain

$$S_2 = O\left(\frac{L^2}{\tau}\right) + O\left\{\frac{L}{\tau}\left(\sum_{m_3=1}^{\tau-1}\sum_{n_3=0}^{\tau-1}|S_3|\right)^{\frac{1}{2}}\right\} + O\left\{\frac{L}{\tau}\left(\sum_{m_3=0}^{\tau-1}\sum_{n_3=1}^{\tau-1}|S_3'|\right)^{\frac{1}{2}}\right\}, \tag{9}$$

where

$$S_3 = \Sigma\Sigma\, e^{2\pi i g(m,\,n)},$$

$$\begin{aligned} g(m, n) = {} & f(m+m_1+m_2+m_3,\ n+n_3) - f(m+m_1+m_3,\ n+n_3) \\ & - f(m+m_2+m_3,\ n+n_3) - f(m+m_1+m_2,\ n) + f(m+m_1,\ n) \\ & + f(m+m_2,\ n) + f(m+m_3,\ n+n_3) - f(m,\ n), \end{aligned}$$

and S_3' is the corresponding sum with $-n_3$ for n_3. The sums are taken over values of m and n such that all the points concerned belong to D.

Inserting the first terms on the right of (9) in (7), we find as before that we must take $\tau = \rho^2$. We thus obtain

$$S = O\left(\frac{L^2}{\rho^{\frac{1}{2}}}\right) + O\left[\frac{L^{\frac{1}{2}}}{\rho^{\frac{3}{2}}}\left[\sum_{m_1=1}^{\rho-1}\left\{\sum_{m_2=1}^{\rho^2-1}\left(\sum_{m_3=1}^{\rho^2-1}\sum_{n_3=0}^{\rho^2-1}|S_3|\right)^{\frac{1}{2}}\right\}^{\frac{1}{2}}\right]^{\frac{1}{2}}\right] \tag{10}$$

$$+ \text{a similar term involving } S_3'.$$

Since $\tau = \sigma = \rho^2$, the final restriction on ρ is (8).

11. Let

$$\begin{aligned} G(u, v) = {} & \sqrt{\{(u+m_1+m_2+m_3)^2 + (v+n_1+n_2+n_3)^2\}} \\ & - \Sigma\sqrt{\{(u+m_1+m_2)^2 + (v+n_1+n_2)^2\}} \\ & + \Sigma\sqrt{\{(u+m_1)^2 + (v+n_1)^2\}} - \sqrt{(u^2+v^2)}. \end{aligned}$$

Then $g(m, n) = \sqrt{x}\, G(m, n)$ if $n_1 = n_2 = 0$. It is, however, convenient to keep the general form of G.

If $|u_1| \leqslant \eta$, $|u_2| \leqslant \eta$, $\max(u, v) \geqslant L$,

$$\frac{\partial^2}{\partial u^2}\sqrt{\{(u+u_1)^2+(v+v_1)^2\}} = \frac{(v+v_1)^2}{\{(u+u_1)^2+(v+v_1)^2\}^{\frac{3}{2}}}$$

$$= \frac{v^2}{(u^2+v^2)^{\frac{3}{2}}}\left(1+\frac{2v_1}{v}+\frac{v_1^2}{v^2}\right)\left(1+2\,\frac{uu_1+vv_1}{u^2+v^2}+\frac{u_1^2+v_1^2}{u^2+v^2}\right)^{-\frac{3}{2}}$$

$$= \frac{v^2}{(u^2+v^2)^{\frac{3}{2}}}\left(1+\frac{2v_1}{v}+\frac{v_1^2}{v^2}\right)\left\{1-3\,\frac{uu_1+vv_1}{u^2+v^2}-\frac{3}{2}\,\frac{u_1^2+v_1^2}{u^2+v^2}\right.$$

$$\left.+\frac{15}{2}\left(\frac{uu_1+vv_1}{u^2+v^2}\right)^2+\frac{15}{2}\,\frac{(uu_1+vv_1)(u_1^2+v_1^2)}{(u^2+v^2)^2}-\frac{35}{2}\left(\frac{uu_1+vv_1}{u^2+v^2}\right)^3+O\left(\frac{\eta^4}{L^4}\right)\right\}$$

$$= \frac{v^2}{(u^2+v^2)^{\frac{3}{2}}}\left\{1+\phi_1+\phi_2+\frac{15}{2}\,\frac{(uu_1+vv_1)(u_1^2+v_1^2)}{(u^2+v^2)^2}-\frac{35}{2}\left(\frac{uu_1+vv_1}{u^2+v^2}\right)^3\right.$$

$$\left.-3\,\frac{v_1}{v}\,\frac{u_1^2+v_1^2}{u^2+v^2}+15\,\frac{v_1}{v}\left(\frac{uu_1+vv_1}{u^2+v^2}\right)^2-3\,\frac{v_1^2}{v^2}\,\frac{uu_1+vv_1}{u^2+v^2}+O\left(\frac{\eta^4}{L^4}\right)\right\},$$

ϕ_1 and ϕ_2 representing the linear and quadratic terms. When we substitute this expression in G_{uu}, with appropriate values of u_1 and v_1, the constant, linear, and quadratic terms vanish; and, writing

$$X = 6m_1m_2m_3, \quad Y = 2\Sigma m_1m_2n_3, \quad Z = 2\Sigma m_1n_2n_3, \quad W = 6n_1n_2n_3,$$

we obtain

$$G_{u,u} = \frac{v^2}{(u^2+v^2)^{\frac{5}{2}}}\left\{\frac{15}{2}\,\frac{(X+Z)u+(Y+W)v}{u^2+v^2}-\frac{35}{2}\,\frac{Xu^3+3Yu^2v+3Zuv^2+Wv^3}{(u^2+v^2)^2}\right.$$

$$\left.-3\,\frac{Y+W}{v}+15\,\frac{Yu^2+2Zuv+Wv^2}{v(u^2+v^2)}-3\,\frac{Zu+Wv}{v^2}\right\}+O\left(\frac{\eta^4}{L^5}\right),$$

where now $\eta = \max(m_1+m_2+m_3,\ n_1+n_2+n_3)$.

If $n_1 = n_2 = 0$, $Z = W = 0$, and hence

$$(1)\quad g_{u,u} = \frac{v^2\sqrt{x}}{(u^2+v^2)^{\frac{5}{2}}}\left\{\frac{15}{2}\,\frac{Xu+Yv}{u^2+v^2}-\frac{35}{2}\,\frac{Xu^3+3Yu^2v}{(u^2+v^2)^2}\right.$$

$$\left.-3\,\frac{Y}{v}+15\,\frac{Yu^2}{v(u^2+v^2)}\right\}+O\left\{\frac{\eta^4\sqrt{x}}{L^5}\right\}.$$

Interchanging u and v, X and W, and Y and Z, and then putting $Y = Z = 0$, we obtain

$$(2)\quad g_{v,v} = \frac{u^2\sqrt{x}}{(u^2+v^2)^{\frac{5}{2}}}\left\{\frac{15}{2}\,\frac{Xu+Yv}{u^2+v^2}-\frac{35}{2}\,\frac{Xu^3+3Yu^2v}{(u^2+v^2)^2}-3\,\frac{X}{u}\right.$$

$$\left.+15\,\frac{Xu+2Yv}{u^2+v^2}-3\,\frac{Xu+Yv}{u^2}\right\}+O\left(\frac{\eta^4\sqrt{x}}{L^5}\right).$$

Again,

$$\frac{\partial^2}{\partial u\,\partial v}\sqrt{\{(u+u_1)^2+(v+v_1)^2\}} = -\frac{(u+u_1)(v+v_1)}{\{(u+u_1)^2+(v+v_1)^2\}^{\frac{3}{2}}}$$

$$= -\frac{uv}{(u^2+v^2)^{\frac{3}{2}}}\left\{1+\psi_1+\psi_2+\tfrac{15}{2}\frac{(uu_1+vv_1)(u_1^2+v_1^2)}{(u^2+v^2)^2}-\tfrac{35}{2}\left(\frac{uu_1+vv_1}{u^2+v^2}\right)^3\right.$$

$$-\tfrac{3}{2}\left(\frac{u_1}{u}+\frac{v_1}{v}\right)\frac{u_1^2+v_1^2}{u^2+v^2}+\tfrac{15}{2}\left(\frac{u_1}{u}+\frac{v_1}{v}\right)\left(\frac{uu_1+vv_1}{u^2+v^2}\right)^2$$

$$\left.-3\,\frac{u_1v_1}{uv}\,\frac{uu_1+vv_1}{u^2+v^2}\right\}+O\left(\frac{\eta^4}{L^5}\right),$$

where ψ_1 and ψ_2 represent the linear and quadratic terms; and hence, as before,

$$(3)\quad g_{u,v} = -\frac{uv\sqrt{x}}{(u^2+v^2)^{\frac{5}{2}}}\left\{\tfrac{15}{2}\frac{Xu+Yv}{u^2+v^2}-\tfrac{35}{2}\frac{Xu^3+3Yu^2v}{(u^2+v^2)^2}-\tfrac{3}{2}\left(\frac{X}{u}+\frac{Y}{v}\right)\right.$$

$$\left.+\tfrac{15}{2}\frac{Xu+2Yv+Yu^2/v}{u^2+v^2}-3\,\frac{Y}{v}\right\}+O\left(\frac{x^{\frac{1}{2}}\eta^4}{L^5}\right).$$

Hence

$$g_{u,u}\,g_{v,v}-g_{u,v}^2$$

$$= \frac{u^2v^2x}{(u^2+v^2)^5}\left[\left\{\tfrac{15}{2}\frac{Xu+Yv}{u^2+v^2}-\tfrac{35}{2}\frac{Xu^3+3Yu^2v}{(u^2+v^2)^2}\right\}\left\{-3\,\frac{Xu+Yv}{u^2}+6\,\frac{Y}{v}\right\}\right.$$

$$+\left\{3\,\frac{X}{u}-15\,\frac{Xu+2Yv}{u^2+v^2}+3\,\frac{Xu+Yv}{u^2}\right\}\left\{3\,\frac{Y}{v}-15\,\frac{Yu^2}{v(u^2+v^2)}\right\}$$

$$\left.-\left\{\tfrac{3}{2}\left(\frac{X}{u}+\frac{Y}{v}\right)-\tfrac{15}{2}\frac{Xu+2Yv+Yu^2/v}{u^2+v^2}+3\,\frac{Y}{v}\right\}^2\right]+O\left\{\frac{x(X+Y)\eta^4}{L^9}\right\}$$

$$= -\frac{3x}{4(u^2+v^2)^7}\{(8u^4+6u^2v^2+3v^4)v^2X^2-2(8u^4-4u^2v^2+3v^4)uvXY$$

$$+3(4u^6+4u^4v^2+21u^2v^4+6v^6)Y^2\}+O\left\{\frac{x(X+Y)\eta^4}{L^9}\right\}$$

$$= -\frac{9x\,m_1^2m_2^2}{(u^2+v^2)^7}\{3(8u^4+6u^2v^2+3v^4)v^2m_3^2-2(8u^4-4u^2v^2+3v^4)uvm_3n_3$$

$$+(4u^6+4u^4v^2+21u^2v^4+6v^6)n_3^2\}+O\left\{\frac{xm_1m_2(m_3+n_3)\eta^4}{L^9}\right\}.$$

Now

$$24u^4v^2m_3^2-16u^5vm_3n_3+4u^6n_3^2 = 4u^4\{2v^2m_3^2+(2vm_3-un_3)^2\}$$

$$\geqslant 8u^4v^2m_3^2,$$

and

$$18u^2v^4m_3^2-6uv^5m_3n_3+6v^6n_3^2=\tfrac{1}{2}v^4(6um_3-vn_3)^2+\tfrac{11}{2}v^6n_3^2\geqslant\tfrac{11}{2}v^6n_3^2.$$

Hence the above expression in brackets is not less than

$$(8u^4v^2+9v^6)m_3^2+(4u^4v^2+21u^2v^4+\tfrac{11}{2}v^6)n_3^2\geqslant v^2(u^2+v^2)^2(m_3^2+n_3^2).$$

Hence

$$|g_{u,u}g_{v,v}-g_{u,v}^2|\geqslant\frac{9xv^2m_1^2m_2^2(m_3^2+n_3^2)}{(u^2+v^2)^5}-O\left\{\frac{xm_1m_2(m_3+n_3)\eta^4}{L^9}\right\}.$$

Hence, if $v\geqslant u$,

$$|g_{u,u}g_{v,v}-g_{u,v}^2|\geqslant\tfrac{9}{32}\frac{xm_1^2m_2^2(m_3^2+n_3^2)}{v^8}-O\left\{\frac{xm_1m_2(m_3+n_3)\eta^4}{L^9}\right\}.$$

The O term is negligible if

$$(m_3+n_3)(m_1^4+m_2^4+m_3^4+n_3^4)<Km_1m_2(m_3^2+n_3^2)L$$

with a sufficiently small K. For $m_3\geqslant n_3$ this is true if

$$m_1^4+m_2^4+2m_3^4<\tfrac{1}{2}Km_1m_2m_3L,$$

i.e.

$$\frac{m_1^3}{m_2m_3}+\frac{m_2^3}{m_1m_3}+\frac{2m_3^3}{m_1m_2}<\tfrac{1}{2}KL.$$

Since $m_1\geqslant1$, etc., this is true if

$$m_1^3+m_2^3+2m_3^3<KL,$$

and this is true if ρ^6/L is sufficiently small. A similar argument holds if $m_3<n_3$. Hence

$$|g_{u,u}g_{v,v}-g_{u,v}^2|>Axm_1^2m_2^2(m_3^2+n_3^2)v^{-8} \tag{4}$$

if ρ^6/L is small enough.

12. We now apply these results to the sum S_3. Since $X=O(\rho^5)$, $Y=O(\rho^5)$, $\eta=O(\rho^2)$, we have

$$g_{u,u}=O\left(\frac{x^{\frac{1}{2}}\rho^5}{L^4}\right)+O\left(\frac{x^{\frac{1}{2}}\rho^8}{L^5}\right)=O\left(\frac{x^{\frac{1}{2}}\rho^5}{L^4}\right),$$

provided that $\rho^3=O(L)$. A similar result holds for $g_{u,v}$ and $g_{v,v}$. Hence, if

$$l=aL^4x^{-\frac{1}{2}}\rho^{-5}$$

with a sufficiently small a, the variation of g_u and g_v in a square of side l is less than $\frac{1}{2}$. Suppose the region of summation of S_3 divided up into such squares, or parts of such squares. Then to every square correspond integers μ, ν such that, if

$$h(u, v) = g(u, v) - \mu u - \nu v,$$

then $|h_u| \leqslant \frac{3}{4}$, $|h_v| \leqslant \frac{3}{4}$. Hence, by Lemma γ, for each square,

$$\Sigma\Sigma\, e^{2\pi i g(m, n)} = \iint e^{2\pi i h(u, v)}\, du\, dv + O(l). \tag{1}$$

If the square is not crossed by the circular boundary, we can apply Lemma ζ to the integral. By § 11 we can take

$$r = Ax^{\frac{1}{2}} m_1 m_2 (m_3^2 + n_3^2)^{\frac{1}{2}} L^{-4}.$$

Also
$$r_1 = O(1), \quad r_3 = O(x^{\frac{1}{2}} \rho^5 L^{-5}),$$

so that condition (3) of Lemma δ is satisfied if

$$L^3 \rho^5 < Kx^{\frac{1}{2}} m_1^2 m_2^2 (m_3^2 + n_3^2). \tag{2}$$

Lemma ζ also requires that $lr_3 < Kr$, *i.e.*

$$L^3 < Kx^{\frac{1}{2}} m_1 m_2 (m_3^2 + n_3^2)^{\frac{1}{2}}. \tag{3}$$

Since $m_1 m_2 (m_3^2 + n_3^2)^{\frac{1}{2}} = O(\rho^5)$, (3) is satisfied if (2) is satisfied. Hence, if (2) holds, Lemma ζ gives

$$\iint e^{2\pi i h(u, v)}\, du\, dv = O\left\{\frac{L^4 \log x}{x^{\frac{1}{2}} m_1 m_2 (m_3^2 + n_3^2)^{\frac{1}{2}}}\right\}, \tag{4}$$

assuming that $L = O(x^A)$.

The number of such terms is $O(L^2/l^2)$, provided that $l \leqslant L$. Hence their total contribution to S_3 is

$$O\left\{\frac{L^6 \log x}{l^2 x^{\frac{1}{2}} m_1 m_2 (m_3^2 + n_3^2)^{\frac{1}{2}}}\right\} + O\left(\frac{L^2}{l}\right) = O\left\{\frac{x^{\frac{1}{2}} \rho^{10} \log x}{L^2 m_1 m_2 (m_3^2 + n_3^2)^{\frac{1}{2}}}\right\}.$$

The above argument does not apply to those squares of side l which are crossed by the circular boundary of S_3, if it has one. But the number of such squares is $O(L/l)$, and each square contributes $O(l^2)$, so that the total contribution of these is

$$O(lL) = O(L^5 x^{-\frac{1}{2}} \rho^{-5}).$$

Hence

$$S_3 = O\left\{\frac{x^{\frac{1}{2}}\rho^{10}\log x}{L^2 m_1 m_2 (m_3^2+n_3^2)^{\frac{1}{2}}}\right\} + O\left(\frac{L^5}{x^{\frac{1}{2}}\rho^5}\right) \tag{5}$$

with the restriction (3). A similar result, of course, holds for S_3'.

13. Substituting the first term on the right of (5) for S_3 in § 10 (10), and observing that

$$\sum_{m_3=1}^{\rho^2-1}\sum_{n_3=0}^{\rho^2-1}\frac{1}{(m_3^2+n_3^2)^{\frac{1}{2}}} < \sum_{m_3=1}^{\rho^2-1}\frac{1}{m_3} + 2\sum_{m_2=1}^{\rho^2-1}\sum_{n_3=1}^{m_3}\frac{1}{m_3} = O(\rho^2),$$

we obtain

$$O\left[\frac{L^7}{\rho^{\frac{5}{2}}}\left(\frac{x^{\frac{1}{2}}\rho^{10}\log x}{L^2}\right)^{\frac{1}{8}}\left\{\sum_{m_1=1}^{\rho-1}\left(\sum_{m_2=1}^{\rho^2-1}\frac{\rho}{m_1^{\frac{1}{2}}m_2^{\frac{1}{2}}}\right)^{\frac{1}{2}}\right\}^{\frac{1}{2}}\right] = O(L^3\rho^5 x^{\frac{1}{16}}\log^{\frac{1}{8}} x).$$

The second term on the right of (5) contributes

$$O\left\{L^7\left(\frac{L^5}{x^{\frac{1}{2}}\rho^5}\right)^{\frac{1}{8}}\right\} = O\left(\frac{L^{15}}{x^{\frac{1}{16}}\rho^5}\right).$$

Since $S_3 = O(L^2)$, the terms for which § 12 (3) is not satisfied contribute

$$O\left[\frac{L^7}{\rho^{\frac{5}{2}}}\left[\sum_{m_1=1}^{\rho-1}\left\{\sum_{m_2=1}^{\rho^2-1}\left(\sum\sum_{m_3^2+n_3^2<O\{L^6/(xm_1^2m_2^2)\}}L^2\right)^{\frac{1}{2}}\right\}^{\frac{1}{2}}\right]^{\frac{1}{2}}\right]$$

$$= O\left[\frac{L^7}{\rho^{\frac{5}{2}}}\left[\sum_{m_1=1}^{\rho-1}\left\{\sum_{m_2=1}^{\rho^2-1}\left(\frac{L^8}{xm_1^2m_2^2}\right)^{\frac{1}{2}}\right\}^{\frac{1}{2}}\right]^{\frac{1}{2}}\right] = O\left(\frac{L^{11}\log^{\frac{1}{4}}\rho}{x^{\frac{1}{8}}\rho^5}\right).$$

Hence

$$S = O\left(\frac{L^2}{\rho^{\frac{1}{2}}}\right) + O(L^3\rho^5 x^{\frac{1}{16}}\log^{\frac{1}{8}} x) + O\left(\frac{L^{15}}{x^{\frac{1}{16}}\rho^5}\right) + O\left(\frac{L^{11}\log^{\frac{1}{4}}\rho}{x^{\frac{1}{8}}\rho^5}\right). \tag{1}$$

The first two terms are of the same form if

$$\rho = [L^{\frac{4}{5}}x^{-\frac{1}{16}}\log^{-\frac{1}{5}} x],$$

and, if this is a permissible value of ρ, we obtain

$$S = O(L^{\frac{16}{9}}x^{\frac{3}{36}}\log^{\frac{1}{18}} x) + O(L^{\frac{151}{76}}x^{-\frac{1}{36}}\log^{\frac{5}{72}} x) + O(L^{\frac{33}{36}}x^{-\frac{1}{18}}\log^{\frac{7}{18}} x). \tag{2}$$

In the proof we have assumed that $m \leqslant n$, but clearly a similar proof gives the same result for $m > n$.

The condition (8) of § 10 is now

$$1 \leqslant L^{\frac{4}{5}}x^{-\frac{1}{16}}\log^{-\frac{1}{5}} x \leqslant \tfrac{1}{2}x^{\frac{3}{16}},$$

i.e.

$$x^{\frac{1}{6}} \log^{\frac{1}{4}} x \leqslant L \leqslant 2^{-\frac{3}{4}} x^{\frac{77}{164}} \log^{\frac{1}{4}} x, \tag{3}$$

and the condition arising from § 11 is that

$$\rho^6/L = L^{\frac{5}{3}} x^{-\frac{1}{3}} \log^{-\frac{2}{3}} x \tag{4}$$

should be small enough. Both conditions are satisfied if

$$x^{\frac{1}{5}} \leqslant L \leqslant x^{\frac{1}{3}}. \tag{5}$$

With this condition it is also easily seen that $l < L$.

If also $L \leqslant x^{\frac{1}{3}}/\log x$, the last term in (2) is negligible compared with the first. Hence

$$S = O(L^{\frac{16}{9}} x^{\frac{1}{36}} \log^{\frac{1}{18}} x) + O(L^{\frac{151}{72}} x^{-\frac{1}{36}} \log^{\frac{77}{72}} x) \quad (x^{\frac{1}{5}} \leqslant L \leqslant x^{\frac{1}{3}}/\log x). \tag{6}$$

14. Now consider the function $\phi_1(y)$ defined in § 9. We have

$$\phi_1(y) = \Sigma\Sigma \frac{e^{2\pi i \sqrt{\{(m^2+n^2)\, y\}}}}{(m^2+n^2)^{\frac{3}{4}}} = \sum_{p=1}^{P} \sum_{q=1}^{Q} \left\{ \sum_{2^{p-1}}^{2^p} \sum_{2^{q-1}}^{2^q} \frac{e^{2\pi i \sqrt{\{(m^2+n^2)\, y\}}}}{(m^2+n^2)^{\frac{3}{4}}} \right\},$$

where p and q run through $O(\log x)$ values, and all the sums are restricted to $m > x^{\frac{7}{46}}$, $n > x^{\frac{7}{46}}$, $m^2+n^2 \leqslant \lambda$. When we apply Lemma a to each of the inner sums, with

$$a_{m,n} = e^{2\pi i \sqrt{\{(m^2+n^2)\, y\}}} \quad (m^2+n^2 \leqslant \lambda), \qquad 0 \quad (m^2+n^2 > \lambda),$$

and $b_{m,n} = (m^2+n^2)^{-\frac{3}{4}}$, it follows from § 13 (6) that the inner sum is

$$O[\{\max(2^p, 2^q)\}^{\frac{5}{18}} y^{\frac{1}{36}} \log^{\frac{1}{18}} y] + O[\{\max(2^p, 2^q)\}^{\frac{43}{72}} y^{-\frac{1}{36}} \log^{\frac{5}{72}} y].$$

Now

$$\sum_{p=1}^{P} \sum_{q=1}^{Q} \{\max(2^p, 2^q)\}^k = O\{(P+Q)(2^P+2^Q)\} = O(\lambda^{\frac{1}{2}k} \log x),$$

assuming that $\lambda = O(x^A)$, for $k = \frac{5}{18}$ or $k = \frac{43}{72}$. Hence, since $x \leqslant y < 2x$,

$$\phi_1(y) = O(\lambda^{\frac{5}{36}} x^{\frac{1}{36}} \log^{\frac{1}{18}} x) + O(\lambda^{\frac{43}{144}} x^{-\frac{1}{36}} \log^{\frac{77}{72}} x).$$

Hence, by § 8 (2) and § 9 (1),

$$\Sigma_1 = O(\lambda^{\frac{5}{36}} x^{\alpha+\frac{5}{18}} \log^{\frac{1}{18}} x) + O(\lambda^{\frac{43}{144}} x^{\alpha+\frac{2}{5}} \log^{\frac{77}{72}} x) + O(x^{\alpha+\frac{15}{16}}).$$

Similar considerations apply to $\psi_1(x)$, except that there is an extra factor L in the denominator, and the second term on the right of § 13 (6),

arising from the curved boundary, occurs only if $L < A\sqrt{\lambda}$. We deduce that

$$\psi_1(x) = O(\lambda^{-\frac{13}{56}} x^{\frac{5}{56}} \log^{\frac{19}{28}} x) + O(\lambda^{-\frac{29}{114}} x^{-\frac{1}{56}} \log^{\frac{77}{57}} x),$$

and hence that

$$\Sigma_3 = O(\lambda^{-\frac{13}{56}} x^{\frac{7}{8}} \log^{\frac{19}{28}} x) + O(\lambda^{-\frac{29}{144}} x^{\frac{19}{28}} \log^{\frac{77}{57}} x) + O(x^{\frac{15}{23}}) + O(x^{\frac{21}{23}} \lambda^{-\frac{8}{23}}). \tag{2}$$

A similar result holds for Σ_2.

The first terms on the right of (1) and (2) are of the same order if $\lambda = x^{1-2\alpha}$, and then

$$\int_x^{x+x^\alpha} \{R(y) - \pi y\}\, dy = O(x^{\frac{5}{12}+\frac{13}{28}\alpha} \log^{\frac{19}{28}} x) + O(x^{\frac{25}{28}+\frac{57}{112}\alpha} \log^{\frac{77}{57}} x) + O(x^{\frac{15}{23}}) + O(x^{\frac{15}{23}+\frac{3}{23}\alpha}).$$

Since $R(x)$ is non-decreasing, the left-hand side is not less than

$$x^\alpha \{R(x) - \pi(x + x^\alpha)\},$$

and hence

$$R(x) - \pi x \leqslant \pi x^\alpha + O(x^{\frac{5}{12}-\frac{5}{28}\alpha} \log^{\frac{19}{28}} x) + O(x^{\frac{25}{28}-\frac{43}{112}\alpha} \log^{\frac{77}{57}} x) + O(x^{\frac{15}{23}-\alpha}) + O(x^{\frac{15}{23}+\frac{1}{23}\alpha}).$$

We obtain the same power of x in the first two terms by taking $\alpha = \frac{15}{46}$, and then

$$R(x) - \pi x < O(x^{\frac{15}{46}} \log^{\frac{77}{57}} x).$$

By considering integrals over $(x - x^\alpha, x)$ in the same way*, it can be proved that

$$R(x) - \pi x > O(x^{\frac{15}{46}} \log^{\frac{77}{57}} x).$$

Hence

$$R(x) = \pi x + O(x^{\frac{15}{46}} \log^{\frac{77}{57}} x).$$

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* *Cf.* Landau, *ibid.*, 207–8.

ON POLYNOMIALS IN A GALOIS FIELD: SOME FORMULAE INVOLVING DIVISOR FUNCTIONS

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1. Let $\mathfrak{D}$ denote the totality of polynomials in a single indeterminate x with coefficients in a fixed Galois (finite) field $GF(p^n)$, of order p^n, where p is any prime, and n is a positive integer*. A polynomial is called *primary* if the coefficient of the highest power of x is unity; a fundamental property of $\mathfrak{D}$ is that any polynomial of the domain can be expressed uniquely—except for order—as the product of a quantity of the underlying Galois field into a product of irreducible primary polynomials. From this it would be expected many of the theorems of ordinary arithmetic—particularly those of a multiplicative nature—would have analogues for $\mathfrak{D}$. This is actually the case for many theorems, and indeed the analogues are often considerably simpler than the original theorems†.

In this note I consider a property of $\tau(F)$, the number of (primary) divisors of a polynomial F:

$$\tau(F) = \sum_{M|F} 1.$$

The theorem in question is concerned with the sum

$$(1) \qquad \sum_{(\alpha+\beta)F=\alpha A+\beta B} \tau(A)\tau(B),$$

taken over all primary polynomials A, B of degree equal to that of F, such that $(\alpha+\beta)F = \alpha A+\beta B$, where α and β are elements of the $GF(p^n)$, such that

$$\alpha\beta(\alpha+\beta) \neq 0.$$

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* For the properties of Galois fields assumed here see, for example, L. E. Dickson, *Linear groups* (1901).

† See, for example, R. Dedekind, *Journal für Math.*, 54 (1857), 1–26; E. Artin, *Math. Zeitschrift*, 19 (1924), 153–246; L. Carlitz, *Amer. J. of Math.*, 54 (1932), 39–50.

It is found that, if the degree of F is odd, equal to $2f-1$, say, then (1) is

$$4\sum_{M|F}^{m\geqslant f}|M|\{(m-f+1)^2-(m-f)^2p^{-n}\}\ ; \tag{2}$$

if the degree of F is even, equal to $2f$, say, then (1) is

$$\sum_{M|F}^{m>f}|M|\{(2m-2f+1)^2-(2m-2f-1)^2p^{-n}\}+\sum_{M|F}^{m=f}|M|. \tag{3}$$

The summation in (2) and the first of (3) is over all primary M, dividing F, and of degree m, $m\geqslant f$ and $m>f$ respectively; similarly for the second summation in (3) ; $|M|$ denotes p^{nm}.

This theorem is the analogue for $\mathfrak{D}$ of a rather complicated asymptotic formula first proved by A. E. Ingham and later refined by T. Estermann.*

2. In the following a polynomial will always mean a primary polynomial; polynomials will be denoted by large italic letters. Ordinary integers will be denoted by small italic letters and elements of the Galois field by small Greek letters. If m is the degree of M (written $\deg M$), then $|M|$ is by definition p^{nm}. The symbol (A, B) denotes the " greatest " common divisor of A and B.

We begin by proving two lemmas, the first of which is almost obvious.

LEMMA 1. *If α and β are two quantities of $GF(p^n)$ such that*

$$\alpha\beta(\alpha+\beta)\neq 0,$$

and F, A, B are polynomials such that

$$(A, B)=1, \quad |AB|\leqslant|F|,$$

then the number of solutions of

$$(\alpha+\beta)F=\alpha AX+\beta BY, \quad |F|=|AX|=|BY|, \tag{4}$$

in polynomials X, Y is

$$t(F;\ A, B)=|F|/|AB|.$$

From (4) it is evident that

$$\begin{cases} X\equiv A' \pmod{B}, & \deg A'<\deg B, \\ Y\equiv B' \pmod{A}, & \deg B'<\deg A, \end{cases} \tag{5}$$

* A. E. Ingham, *Journal London Math. Soc.*, 2 (1927), 202–208 ; T. Estermann, *Journal London Math. Soc.*, 5 (1930), 131–137.

where A' and B' are not necessarily primary. But, since $\deg X \geqslant \deg B$, and $\deg Y \geqslant \deg A$, (5) may be written

$$X = A' + BX', \quad Y = B' + AY', \quad |X'| = |Y'|,$$

where now X' and Y' are primary. Then (4) becomes

$$\{(\alpha+\beta)F - \alpha AA' - \beta BB'\}/AB = \alpha X' + \beta Y';$$

the expression on the left is evidently of the form $(\alpha+\beta)H$, H being a polynomial of degree equal to

$$\deg F - \deg(AB).$$

When we observe that the number of polynomials of degree k is p^{nk}, the lemma follows at once.

3. LEMMA 2. *Let $\psi(a, b)$ be the number of pairs of polynomials $[A, B]$ satisfying*

$$\deg A = a, \quad \deg B = b, \quad (A, B) = 1.$$

Then

$$\psi(a, b) = \begin{cases} p^{n(a+b)} & \text{for} \quad ab = 0, \\ p^{n(a+b)}(1-p^{-n}) & \text{for} \quad ab > 0. \end{cases}$$

The first case is immediate: If $a = 0$, $A = 1$; and, since $(1, B) = 1$ for all B,

$$\psi(0, b) = p^{nb}.$$

For the second case, let us classify the $p^{n(a+b)}$ sets $[A, B]$, $\deg A = a$, $\deg B = b$, according to their g.c.d. Assuming, as we may, that $a \leqslant b$, it is clear that

$$p^{n(a+b)} = \sum_{m \leqslant a} p^{nm} \psi(a-m, b-m), \tag{6}$$

since, for fixed m, M may be chosen in $|M| = p^{nm}$ ways. But the expression on the right of (6) is equal to

$$\psi(a, b) + \sum_{m=1}^{a} p^{nm} \psi(a-m, b-m) = \psi(a, b) + p^{n} \sum_{m=0}^{a-1} p^{nm} \psi(a-1-m, b-1-m)$$

$$= \psi(a, b) + p^{n(a+b-1)},$$

and therefore

$$\psi(a, b) = p^{n(a+b)}(1-p^{-n}) \quad (ab > 0).$$

4. THEOREM 1. *If α, β are in $GF(p^n)$, $\alpha\beta(\alpha+b)\neq 0$, and F is an arbitrary polynomial of degree k, then*

$$(7)\qquad \sum_{(\alpha+\beta)F=\alpha A+\beta B} \tau(A)\tau(B)$$

$$= \sum_{M|F}^{m>\frac{1}{2}k} |M|\{(2m-k+1)^2-(2m-k-1)^2 p^{-n}\}+ \sum_{M|F}^{m=\frac{1}{2}k} |M|,$$

where the summation on the left is over all A, B of degree k satisfying $(\alpha+\beta)F=\alpha A+\beta B$, and on the right m is the degree of M.

The right member of (7) evidently includes both (3) and (4); when k is odd, the second sum is vacuous and is to be interpreted as zero.

To prove (7) it is convenient to take first k odd, equal to $2f-1$. Then

$$(8)\qquad \Sigma = \sum_{(\alpha+\beta)F=\alpha A+\beta B} \tau(A)\tau(B) = \sum_{(\alpha+\beta)F=\alpha AC+\beta BD} 1,$$

where, in the last sum,

$$\deg(AC) = a+c = \deg(BD) = b+d = \deg F.$$

Now the last sum in (8) may be broken into four parts, namely,

$$\sum_{a,\,b<f} + \sum_{a,\,d<f} + \sum_{c,\,b<f} + \sum_{c,\,d<f};$$

therefore, by symmetry,

$$\Sigma = 4 \sum_{\substack{(\alpha+\beta)F=\alpha AC+\beta BD \\ a,\,b<f}} 1$$

$$(9)\qquad = 4 \sum_{M|F}^{m<f} \sum_{\substack{(\alpha+\beta)FM^{-1}=\alpha A'C+\beta B'D \\ (A',\,B')=1 \\ a',\,b'<f-m}} 1 \quad (\deg M,\ A',\ B' = m,\ a',\ b').$$

Now the inner sum in (9) is, by Lemma 1,

$$(10)\qquad = \sum_{\substack{(A,\,B)=1 \\ a,\,b<f-m}} t(FM^{-1};\ A,\ B) = \sum_{\substack{(A,\,B)=1 \\ a,\,b<f-m}} \frac{|F|}{|MAB|}.$$

By the definition of $\psi(a, b)$, in Lemma 2, the expression on the right of (10) is clearly

$$(11)\qquad = \left|\frac{F}{M}\right| \sum_{a,\,b<f-m} \psi(a, b)\, p^{-n(a+b)}.$$

By means of Lemma 2, the sum in (11) is very easily evaluated thus:

$$\sum_{a,\,b=0}^{f-m-1}\psi(a,\,b)\,p^{-n(a+b)}$$

$$=\sum_{a=0}^{f-m-1}\psi(a,\,0)\,p^{-na}+\sum_{b=1}^{f-m-1}\psi(0,\,b)\,p^{-nb}+\sum_{a,\,b=1}^{f-m-1}\psi(a,\,b)\,p^{-n(a+b)}$$

$$=(f-m)+(f-m-1)+(1-p^{-n})(f-m-1)^2$$

$$=(f-m)^2-(f-m-1)^2p^{-n}. \tag{12}$$

Substituting from (9), ..., (12) in (8), we get

$$\Sigma=4\sum_{M\mid F}^{m<f}|FM^{-1}|\{(f-m)^2-(f-m-1)^2p^{-n}\}$$

$$=\sum_{M\mid F}^{m\geqq f}|M|\{(2m-k+1)^2-(2m-k-1)^2p^{-n}\},$$

which proves (7) for the case k odd.

5. The case k even is slightly more complicated. Taking $k=2f$, we begin with

$$\Sigma=\sum_{(\alpha+\beta)F=\alpha A+\beta B}\tau(A)\tau(B)=\sum_{(\alpha+\beta)F=\alpha AC+\beta BD}1$$

$$=4\sum_{a,\,b<f}+4\sum_{a<f,\ b=f}+\sum_{a,\,b=f}=4\Sigma_1+4\Sigma_2+\Sigma_3, \tag{13}$$

say. The calculation of Σ_1 is parallel to that of Σ in the case k odd, and we find that

$$\Sigma_1=\sum_{M\mid F}^{m<f}|FM^{-1}|\{(f-m)^2-(f-m-1)^2p^{-n}\}. \tag{14}$$

As for Σ_2, in exactly the same way,

$$\Sigma_2=\sum_{M\mid F}^{m<f}\sum_{\substack{(A,\,B)=1\\ a<f-m=b}}t(FM^{-1};\ A,\ B)$$

$$=\sum_{M}\sum_{A,\,B}|FM^{-1}|/|AB|$$

$$=\sum_{M}|FM^{-1}|\sum_{a<f-m}\psi(a,\,f-m)\,p^{-n(a+f-m)}, \tag{15}$$

The inner sum, by Lemma 2, is equal to

$$1+(1-p^{-n})(f-m-1)=(f-m)+(f-m-1)p^{-n},$$

so that, substituting in (15), we have

$$\Sigma_2 = \sum_{M|F}^{m<f} |FM^{-1}|\{(f-m)+(f-m-1)p^{-n}\}. \tag{16}$$

Finally, for Σ_3, which is the simplest of the three sums, it may be verified that

$$\Sigma_3 = \sum_{M|F}^{m\leqslant f} |FM^{-1}|\psi(f-m, f-m)\,p^{-2n(f-m)}$$

$$= \sum_{M|F}^{m<f} |FM^{-1}|(1-p^{-n}) + \sum_{M|F}^{m=f} |FM^{-1}|. \tag{17}$$

Returning now to (13), we have, using (14), (15), and (17),

$$\Sigma = \sum_{M|F}^{m<f} |FM^{-1}|\{(2f-2m+1)^2-(2f-2m+1)^2p^{-n}\} + \sum_{M|F}^{m=f} |FM^{-1}|$$

$$= \sum_{M|F}^{m>f} |M|\{(2m-k+1)^2-(2m-k-1)^2p^{-n}\} + \sum_{M|F}^{m=f} |M|,$$

which proves (7) in the case k even.

6. The sum Σ_3 above suggests an interesting formula for a variation of the τ function. We define $\tau'(F)$ by

$$\tau'(F) = \sum_{M|F}^{m=f} 1 \quad (\deg F = 2f).$$

[For F of odd degree, $\tau'(F)$ is identically zero.]

It is clear then that the identity of Σ_3 and (17) imply the following theorem.

THEOREM 2. *If $\alpha\beta(\alpha+\beta) \neq 0$, F is a polynomial of even degree equal to $2f$, then*

$$\sum_{(\alpha+\beta)F=\alpha A+\beta B} \tau'(A)\tau'(B) = \sum_{M|F}^{m>f} |M|(1-p^{-n}) + \sum_{M|F}^{m=f} |M|. \tag{18}$$

It is not difficult to see that a theorem like the above may be framed for more general functions similar to τ'. Indeed, if F is of degree k (odd or even), and $0 \leqslant r \leqslant k$, then we define the function

$$\tau^{(r)}(F) = \sum_{M|F}^{m=r} 1; \tag{19}$$

we may evidently assume, without any loss of generality, that $r \geqslant \frac{1}{2}k$. We may then replace Theorem 2 by the following more general theorem, which is useful in proving further results.

THEOREM 3. *If $\alpha\beta(\alpha+\beta) \neq 0$, F is of degree k, $\frac{1}{2}k \leqslant r \leqslant s < k$, then*

$$\sum_{(\alpha+\beta)F=\alpha A+\beta B} \tau^{(r)}(A)\tau^{(s)}(B) = \sum_{M|F}^{m>s} |M|(1-p^{-n}) + \sum_{M|F}^{m=s} |M|. \tag{20}$$

Since the proof is very much like that of Theorem 2, it will suffice to sketch it briefly. From the definition (19), it is evident that the expression on the left of (20) is equal to

$$\begin{aligned}
\sum_{\substack{(\alpha+\beta)F=\alpha AC+\beta BD \\ a=k-r,\ b=k-s}} 1 &= \sum_{M|F}^{m\leqslant k-s} \sum_{\substack{(A,B)=1 \\ a=k-r-m \\ b=k-s-m}} t(FM^{-1};\ A,\ B) \\
&= \sum_{M|F}^{m\leqslant k-s} |FM^{-1}|\psi(k-r-m,\ k-s-m)\, p^{2k-2m-r-s} \\
&= \sum_{M|F}^{m<k-s} |FM^{-1}|(1-p^{-n}) + \sum_{M|F}^{m=k-s} |FM^{-1}| \\
&= \sum_{M|F}^{m>s} |M|\ (1-p^{-n}) + \sum_{M|F}^{m=s} |M|,
\end{aligned}$$

which completes the proof of the theorem.

7. The method of this paper may be applied to a number of similar divisor functions. In the first place a formula can be derived for

$$\Sigma\, \tau(A_1)\tau(A_2)\dots\tau(A_s),$$

summed over all A_i such that

$$\alpha F = \alpha_1 A_1 + \dots + \alpha_s A_s \quad (|A_i| = |F|);$$

$$\alpha = \alpha_1 + \dots + \alpha_s, \quad \alpha\alpha_1\alpha_2\dots\alpha_s \neq 0;$$

the formula is, however, very complicated and of no great interest.

A second natural generalization of (1) is

$$\sum_{(\alpha+\beta)F=\alpha A+\beta B} \sigma_r(A)\sigma_s(B), \tag{21}$$

where, of course, $\alpha\beta(\alpha+\beta) \neq 0$, $|A| = |B| = |F|$; and

$$\sigma_r(A) = \sum_{M|A} |A|^r, \tag{22}$$

r and s being arbitrary (complex) quantities. The sum in (21) can be evaluated much as in Theorem 1; it is simpler, however, to use Theorem 3. For, using (22), (21) becomes

$$(23) \qquad \sum_{(\alpha+\beta)F=\alpha AC+\beta BD} |A|^r |B|^s = \sum_{a,\,b=0}^{k} p^{n(ar+bs)} g(a, b),$$

where k is the degree of F and $g(r, s)$ denotes the expression on the left of (20). Let us assume, first, k odd and equal to $2f-1$; then the expression on the right of (23) is equal to

$$\sum_{a,\,b=f}^{k} (p^{ar}+p^{(k-a)r})(p^{bs}+p^{(k-b)s})\, g(a, b)$$

[since, by (19), $g(a, b) = g(b, a) = g(k-a, b)$]

$$= \sum_{a,\,b=f}^{k} (p^{ar}+p^{(k-a)r})(p^{bs}+p^{(k-b)s}) \Big\{ \sum_{M|F}^{m \geq \mathrm{Max}(a,\,b)} |M| - p^{-n} \sum_{M|F}^{m > \mathrm{Max}(a,\,b)} |M| \Big\}$$

$$= \sum_{M|F}^{m \geqq f} |M| \sum_{a,\,b=f}^{m} - p^{-n} \sum_{M|F}^{m > f} |M| \sum_{a,\,b=f}^{m-1}$$

$$= \sum_{M|F}^{m \geqq f} |M|^{r+s+1} \{[2m-k+1]_r [2m-k+1]_s - p^{-n-r-s} [2m-k-1]_r [2m-k-1]_s\},$$

where, for brevity,

$$(24) \qquad [u]_r = \frac{1-p^{-nur}}{1-p^{-r}}.$$

This completes the proof, in the case k odd, of the following theorem.

THEOREM 4. *If* $\alpha\beta(\alpha+\beta) \neq 0$, $\deg F = k$, *while* $\sigma_r(F)$ *and* $[u]_r$ *are defined by* (22) *and* (24) *respectively, then*

$$(25) \qquad \sum_{(\alpha+\beta)F=\alpha A+\beta B} \sigma_r(A)\, \sigma_s(B)$$

$$= \sum_{M|F}^{m=\frac{1}{2}k} |M|^{r+s+1} + \sum_{M|F}^{m>\frac{1}{2}k} |M|^{r+s+1} \{[2m-k+1]_r [2m-k+1]_s - p^{-n-r-s} [2m-k-1]_r [2m-k-1]_s\}.$$

The proof of this theorem in the case k even, though somewhat more involved, presents no new difficulties, and will be omitted. It will be noticed that, for $r = s = 0$, (25) reduces to (7).

8. Rather more elegant results may be obtained for a class of functions which may be called "incomplete" divisor-functions. We define $\rho_s(F)$ by

$$\rho_s(F) = \sum_{M|F}^{m=f} |M|^s (1-p^{-ns}) + \sum_{M|F}^{m=f} |M|^s,$$

where F is assumed to be of even degree, $2f$. For example, $\tau'(F)$ of Theorem 2 is $\rho_0(F)$, the right member of (18) is $\rho_1(F)$. We give the following general theorem.

THEOREM 5. *If* $a = a_1 + \ldots + a_r$, $a\,a_1 \ldots a_r \neq 0$; $S_1, \ldots, S_r$ *are arbitrary* (*complex*) *quantities*, $S = S_1 + \ldots + S_r + r - 1$; F *is of even degree*; *then*

$$\sum_{aF = a_1 A_1 + \ldots + a_r A_r} \rho_{s_1}(A_1) = \rho_{s_r}(A) = \rho_s(F) \quad (|A_i| = |F|). \tag{26}$$

As (26) is of importance in another investigation which I shall consider in detail elsewhere*, I shall not include a proof here.

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* *Trans. American Math. Soc.*, 35 (1933), 397–410. Making use of Lemma 2, case 2, on p. 400 of that paper, it will be evident how the theorems of the present paper may be generalized.

ON POSITIVE HARMONIC FUNCTIONS: A CONTRIBUTION TO THE ALGEBRA OF FOURIER SERIES

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I. *Introduction.*

1. In this paper, a set of numbers γ_n ($n = 0, \pm 1, \ldots, \pm m$, or $n = 0, \pm 1, \ldots$) will have the property $\gamma_{-n} = \bar{\gamma}_n$, where $\bar{\gamma}_n$ denotes the conjugate of γ_n.

A determinant

$$\begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix}$$

in which $a_{rs} = \gamma_{r-s}$ is called a Laurentian determinant. The elements in the principal diagonal are all equal to the real number γ_0.

Given the trigonometric series

$$1 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

we write $c_0 = 1, \quad c_n = \frac{1}{2}(a_n - ib_n), \quad c_{-n} = \bar{c}_n \qquad (n = 1, 2, \ldots),$

$$D_n = \begin{vmatrix} 1 & c_1 & c_2 & \ldots & c_n \\ c_{-1} & 1 & c_1 & \ldots & c_{n-1} \\ \ldots & \ldots & \ldots & & \ldots \\ c_{-n} & c_{-(n-1)} & \ldots & & 1 \end{vmatrix},$$

so that D_n is a Laurentian determinant of order $n+1$.

Consider the class of functions which are harmonic in the unit circle $r < 1$ and positive in that domain, and which have the value 1 at the origin. Every such function can be uniquely represented in the form

$$1 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) r^n. \tag{1}$$

We call the sequence $1, c_1, c_2, \ldots$ the sequence of complex Fourier constants of (1). The $n+1$ numbers $1, c_1, \ldots, c_n$ we call the $n+1$ initial constants of (1).

Let $z_1, z_2, \ldots$ be a sequence of independent complex parameters, subject to the conditions

$$|z_n| \leqslant 1 \quad (n = 1, 2, \ldots).$$

The above class of harmonic functions defines a certain class P of sequences $c_1, c_2, \ldots$. The main object of this paper is to show that there exists a sequence p_n $(n = 1, 2, \ldots)$ of functions, where p_n is a polynomial in $2n-1$ variables $x_1, y_1, \ldots, x_{n-1}, y_{n-1}, x_n$, with integer coefficients

$$p_n \equiv p_n(x_1, y_1, \ldots, x_{n-1}, y_{n-1}, x_n),$$

such that the class P is identical with the class defined by

$$c_n = p_n(z, \bar{z}_1, \ldots, z_{n-1}, \bar{z}_{n-1}, z_n) \quad (n = 1, 2, \ldots). \tag{2}$$

More precisely, every sequence (c_n) obtained by substituting values of $z_1, z_2, \ldots$ in (2) is a sequence of class P; every sequence of class P can be obtained by such a substitution, and, subject to exceptions of a trivial kind, in only one way.

To prove this, we use the fundamental results of that theory of positive harmonic functions which is due in the first place to Carathéodory* and Toeplitz†, and which is further associated with the names of Fejér‡, Fischer§, Herglotz ‖, F. Riesz¶, I. Schur**. We have judged it desirable to give a simple exposition of as much of the theory as we shall use.

To follow this exposition (§ II), all that the reader need know about Hermitian forms

$$\sum_{j,k=0}^{n} a_{jk} x_j \bar{x}_k \quad (a_{jk} = \bar{a}_{kj})$$

* *Rend. di Palermo*, 32 (1911), 193–217.
† *Rend. di Palermo*, 32 (1911), 191–192.
‡ *Rend. di Palermo*, 32 (1911), 218–239.
§ *Rend. di Palermo*, 32 (1911), 240–256.
‖ *Leipziger Berichte*, 63 (1911), 501–511.
¶ *Ann. Sci. de l'Ecole Normale*, 28 (1911), 33–62.
** *Journal für Math.*, 147 (1917), 205–232.

is that such a form is positive definite if, and only if, the $n+1$ determinants of its "sections"

$$\overset{0}{\underset{0}{\Sigma}}, \quad \overset{1}{\underset{0}{\Sigma}}, \quad \ldots, \quad \overset{n}{\underset{0}{\Sigma}}$$

are positive, and that, if the first n determinants are positive and the $(n+1)$-th is zero, then the form is positive semi-definite. We shall be concerned with a special kind of Hermitian form, namely one whose determinant is a Laurentian determinant. Such a form is called a Laurentian form. Laurentian determinants and forms were introduced into analysis by Toeplitz, who gave them the name.

In § III, we prove the existence theorem mentioned above, and we examine the properties of the polynomials p_n. An explicit expression for p_n would probably be too complicated for any applications, even if it was not too complicated to be written down. We can, however, give a method for finding p_{n+1} when $p_1, \ldots, p_n$ are known. It will be seen that $p_1(x_1) = x_1$, so that the p_n may be regarded as known.

Noteworthy sub-classes of the class P are obtained by putting the z's from and after a certain one equal to zero. If we put $z_{n+1} = z_{n+2} = \ldots = 0$, we obtain the sub-class P_n, depending on n parameters. We show that, when $p_1, \ldots, p_n$ are known, then the class P_n is defined by a recurrence formula

$$c_{n+\nu} + X_1 c_{n+\nu-1} + X_2 c_{n+\nu-2} + \ldots + X_n c_\nu = 0 \quad (\nu = 1, 2, \ldots),$$

where the $X_1, \ldots X_n$ are known polynomials in $z_1, \bar{z}_1, \ldots, z_{n-1}, \bar{z}_{n-1}, z_n$.

II. *The theory of Carathéodory-Toeplitz.*

2. This can be summarised in the following three theorems.

THEOREM A. *The following four propositions are equivalent:*

(*a*) $$1 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) r^n > 0 \quad (0 \leqslant r < 1);$$

(*b*) *there is a non-diminishing function* $g(\theta)$ *such that*

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} \, dg(\theta) \quad (n = 0, 1, \ldots); \tag{3}$$

(*c*) $$1 + \sum_{m=1}^{n} A_m(x) \left(1 - \frac{m}{n+1}\right) \geqslant 0 \quad (n = 1, 2, \ldots),$$

for all x, where $A_n(x) = a_n \cos nx + b_n \sin nx$;

(*d*) $$\sum_{j,k=0}^{n} c_{j-k} x_j \bar{x}_k \geqslant 0 \quad (n = 0, 1, \ldots).$$

Proof. (α) First (b) implies (a) and (c). For if (3) holds, then

$$1 + \sum_{n=1}^{\infty} A_n(x) r^n = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-x)+r^2}\, dg(\theta),$$

$$1 + \sum_{m=1}^{n} A_m(x) \left(1 - \frac{m}{n+1}\right) = \frac{1}{2\pi(n+1)} \int_0^{2\pi} \left[\frac{\sin \frac{1}{2}(n+1)(\theta-x)}{\sin \frac{1}{2}(\theta-x)}\right]^2 dg(\theta).$$

(β) Next (a) implies (b), and therefore, by (α), implies (c). For

$$P(r, x) = x - \sum_{1}^{\infty} \frac{B_n(x)}{n} r^n \quad [B_n(x) = b_n \cos nx - a_n \sin nx]$$

is an increasing function of x. Hence

$$0 \leqslant P(r, x) - P(r, 0) \leqslant P(r, 2\pi) - P(r, 0) = 2\pi \quad (0 \leqslant x \leqslant 2\pi),$$

and

$$2\pi P(r, 0) = \int_0^{2\pi} P(r, x)\, dx - \int_0^{2\pi} \{P(r, x) - P(r, 0)\}\, dx.$$

If $P(r, 0) \geqslant 0$, this gives $2\pi P(r, 0) \leqslant 2\pi^2$; if $P(r, 0) < 0$, this gives

$$-2\pi P(r, 0) \leqslant 4\pi^2 - 2\pi^2 = 2\pi^2.$$

Thus $$|P(r, 0)| \leqslant \pi, \quad |P(r, x)| \leqslant 3\pi \quad (0 \leqslant x \leqslant 2\pi).$$

We can find a sequence of values of r, $r_n \to 1$, such that $P(r_n, x)$ converges as $n \to \infty$ to $g(x)$ say, whenever x is a rational multiple of 2π in $0 \leqslant x \leqslant 2\pi$. Then $g(x)$ is non-diminishing on this set. It can, therefore, be extended in a unique manner so as to be defined except possibly at an enumerable set E, and to be continuous and non-diminishing on CE. Manifestly, $P(r_n, x) \to g(x)$ for $x \subset CE$. We can find a subsequence of r_n, $\rho_\nu \to 1$, such that $P(\rho_\nu, x)$ converges as $\nu \to \infty$ for $x \subset E$. Then $P(\rho_\nu, x)$ converges for all x, and its limit defines $g(x)$. The convergence being bounded, we have

$$\frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta}\, dg(\theta) = \frac{g(2\pi) - g(0)}{2\pi} + \frac{in}{2\pi} \int_0^{2\pi} g(\theta)\, e^{-in\theta}\, d\theta$$

$$= \lim_{\nu \to \infty} \left[\frac{P(\rho_\nu, 2\pi) - P(\rho_\nu, 0)}{2\pi} + \frac{in}{2\pi} \int_0^{2\pi} \left[\theta - \sum_{-\infty}^{\infty} \frac{c_m}{im} \rho_\nu^{\,m} e^{im\theta}\right] e^{-in\theta}\, d\theta \right]$$

$$= c_n.$$

(γ) Next (c) implies (b), and therefore, by (a), implies (a). The proof is precisely similar to that of (β).

(δ) We have proved that (a), (b), (c) are equivalent. To prove that each is equivalent to (d), we show that (b) implies (d), and that (d) implies (c). The first statement is immediate, for if

$$p(x) = \sum_{j=0}^{n} x_j e^{-ijx},$$

then

$$0 \leqslant \frac{1}{2\pi}\int_0^{2\pi} |p(t)|^2 dg(t) = \frac{1}{2\pi}\int_0^{2\pi}\Big[\sum_{j,k=0}^{n} x_j \bar{x}_k e^{-i(j-k)t}\Big] dg(t)$$

$$= \sum_{j,k=0}^{n} c_{j-k} x_j \bar{x}_k.$$

To prove the second statement, we observe that, for any assigned n and x,

$$0 \leqslant \sum_{j,k=0}^{n} c_{j-k} e^{i(j-k)x} = \frac{1}{2\pi}\int_0^{2\pi}\Big[\sum_{m=-n}^{n} c_m e^{im(x+t)}\Big]\Big[\sum_{j,k=0}^{n} e^{-i(j-k)t}\Big] dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\Big[\sum_{m=-n}^{n} c_m e^{im(x+t)}\Big]\Big[\frac{\sin(n+1)t/2}{\sin t/2}\Big]^2 dt$$

$$= (n+1)\Big[1+\sum_{m=1}^{n} A_m(x)\Big(1-\frac{m}{n+1}\Big)\Big].$$

3. THEOREM B. *Given the numbers $c_0 = 1$, c_1, ..., c_m, a necessary and sufficient condition that these shall be the $m+1$ initial constants of a positive harmonic function is that*

$$\sum_{j,k-0}^{m} c_{j-k} x_j \bar{x}_k \geqslant 0.$$

Proof. The necessity is immediate, since, in Theorem A, (a) implies (d). To prove the sufficiency, we consider two cases.

(i) Suppose that $$\sum_{j,k=0}^{m} c_{j-k} x_j \bar{x}_k > 0 \quad \Big(\sum_{0}^{m} x_j \bar{x}_j = 1\Big). \tag{4}$$

It is sufficient to show that we can find a c_{m+1} such that

$$\sum_{j,k=0}^{m+1} c_{j-k} x_j \bar{x}_k > 0 \quad \Big(\sum_{0}^{m+1} x_j \bar{x}_j = 1\Big). \tag{5}$$

For, by repeating the argument, we obtain a sequence c_n for which (d) of Theorem A holds.

If (4) holds, then $D_0, \ldots, D_m$ are positive. If we can choose c_{m+1} so that $D_{m+1} > 0$, then (5) will hold. In D_{m+1}, the minors of the elements in the top left-hand corner and the bottom right-hand corner are both D_m; while the minors of the elements in the top right-hand corner and bottom left-hand corner are conjugate numbers, Δ_{m+1} and $\bar{\Delta}_{m+1}$. By an elementary property of determinants,

$$D_{m+1} D_{m-1} = D_m^2 - \Delta_{m+1} \Delta_{m-1}, \tag{6}$$

where

$$\Delta_{m+1} = \begin{vmatrix} c_{-1} & 1 & c_1 & \ldots & c_{m-1} \\ c_{-2} & c_{-1} & 1 & \ldots & c_{m-2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ c_{-m} & c_{-(m-1)} & & \ldots & 1 \\ c_{-(m+1)} & c_{-m} & & \ldots & c_{-1} \end{vmatrix}.$$

But

$$\Delta_{m+1} = (-1)^m c_{-(m+1)} D_{m-1} + \delta_{m+1}, \tag{7}$$

where δ_{m+1} is independent of c_{m+1}. Since $D_{m-1} > 0$, we can choose c_{m+1} so as to make (7) zero, in which case (6) gives $D_{m+1} > 0$.

(ii) In the general case,

$$\sum_{j,k=0}^{m} c_{j-k} x_j \bar{x}_k + \epsilon \sum_{j=0}^{m} x_j \bar{x}_j > 0 \quad \left(\sum_0^m x_j \bar{x}_j = 1 \right)$$

for all $\epsilon > 0$. By (i), there is a non-diminishing g_ϵ such that

$$1 + \epsilon = \frac{1}{2\pi} \int_0^{2\pi} dg_\epsilon, \qquad c_j = \frac{1}{2\pi} \int_0^{2\pi} dg_\epsilon \, e^{-ij\theta} \quad (j = 1, \ldots, m). \tag{8}$$

We may suppose that $g_\epsilon(0) = 0$. Then, by an argument used in (β), § 2, we can find a sequence $\epsilon_\nu \to 0$ such that

$$g_{\epsilon_\nu}(\theta) \to g(\theta) \quad (0 \leqslant \theta \leqslant 2\pi).$$

Then (8) gives

$$c_j = \frac{1}{2\pi} \int_0^{2\pi} dg \, e^{-ij\theta} \quad (j = 0, 1, \ldots, m).$$

By Theorem A, this is the desired result.

4. In the next theorem, the enunciation (b) seems to be new. It is of importance for the investigations of § III.

THEOREM C. *Given the numbers* $c_0 = 1, c_1, \ldots, c_n$, *such that*

$$D_0, \ldots, D_{n-1} > 0, \quad D_n = 0, \tag{9}$$

then (*a*) *there is one and only one positive harmonic function of which these are the initial* $n+1$ *constants. This function is of the form*

$$\sum_{k=1}^{n} d_k \frac{1-r^2}{1-2r\cos(x-x_k)+r^2} \quad (d_k > 0),$$

where the x_k *are distinct to* mod 2π;

(*b*) *if* $C_0, C_1, \ldots, C_n$ *denote the cofactors of the elements from left to right in the last row of* D_n, *then the numbers* x_k *are determined by the numbers* $\xi_k = e^{-ix_k}$, *which are the necessarily distinct roots of*

$$C_0 + C_1\xi + C_2\xi^2 + \ldots + C_n\xi^n = 0. \tag{10}$$

The numbers d_k *are the unique and necessarily positive system of solutions of the set of linear equations*

$$\sum_{j=1}^{n} d_j = 1, \quad \Sigma d_j \xi_j = c_1, \quad \Sigma d_j \xi_j^2 = c_2, \quad \ldots, \quad \Sigma d_j \xi_j^n = c_n. \tag{11}$$

The condition (9) implies that the condition of Theorem B (with $m = n$) is satisfied. By that theorem, we have

$$\sum_{j,k=0}^{n} c_{j-k} x_j \bar{x}_k = \frac{1}{2\pi}\int_0^{2\pi} dg \left(\sum_0^n x_j e^{-ij\theta}\right)\left(\sum_0^n \bar{x}_j e^{ij\theta}\right),$$

with a non-diminishing $g(\theta)$. Since $D_n = 0$, there is a set X_j of x_j, not all zero, such that

$$0 = \sum_{j,k=0}^{n} c_{j-k} X_j \bar{X}_k = \frac{1}{2\pi}\int_0^{2\pi} dg\, P_n(\theta)\,\bar{P}_n(\theta), \tag{12}$$

where

$$P_n(\theta) = \sum_0^n X_j e^{-ij\theta}.$$

The numbers X_j can be taken as a non-trivial solution of the system of linear equations

$$c_0 X_0 + c_1 X_1 + \ldots + c_n X_n = 0,$$

$$c_{-1} X_0 + c_0 X_1 + \ldots + c_{n-1} X_n = 0,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$c_{-n} X_0 + c_{-(n-1)} X_1 + \ldots + c_0 X_n = 0.$$

Since $D_n = 0$, $C_n = D_{n-1} > 0$, we can take

$$X_0 = C_0, \quad X_1 = C_1, \quad \ldots, \quad X_n = C_n,$$

then

$$P_n(\theta) = \sum_0^n C_j e^{-ij\theta}. \tag{13}$$

Since $g(\theta)$ is non-diminishing, (12) holds if, and only if, g is constant in the intervals in which the non-negative function $P_n(\theta)\,\bar{P}_n(\theta)$ is positive. Thus the discontinuities of g can occur only at the zeros of $P_n(\theta)$.

Let $x_1 < x_2 < \ldots < x_\nu$ be the distinct zeros of $P_n(\theta)$ in $0 \leqslant \theta \leqslant 2\pi$. If $x_1 > 0$, so that $x_\nu < 2\pi$, we write $d_r = g(x_r+0) - g(x_r-0)$ for $r = 1, 2, \ldots \nu$. If $x_1 = 0$, so that $x_\nu = 2\pi$, we write $d_1 = g(2\pi) - g(2\pi-0) + g(+0) - g(0)$, and we define $d_2, \ldots, d_{\nu-1}$ as before. Let s denote ν in the first case and $\nu - 1$ in the second case. Since

$$c_j = \frac{1}{2\pi}\int_0^{2\pi} dg\, e^{-ij\theta} \quad (j = 0, 1, \ldots, n),$$

we have the equations

$$\sum_{j=1}^s d_s = 1, \quad \sum_{j=1}^s d_j \xi_j = c_1, \quad \ldots, \quad \sum_{j=1}^s d_j \xi_j{}^n = c_n.$$

The d's are non-negative. If some are zero, we can neglect them, alter the notation, and replace s by a smaller number σ. We then have

$$\sum_{j=1}^\sigma d_j = 1, \quad \sum_{j=1}^\sigma d_j \xi_j = c_1, \quad \ldots, \quad \sum_{j=1}^\sigma d_j \xi_j{}^n = c_n.$$

If $\sigma < n$, then $\sigma \leqslant n-1$, and if we substitute the preceding expressions in D_{n-1}, we can express this determinant as the sum of n^σ determinants each of which vanishes because some one column is a multiple of some other. This contradicts $D_{n-1} > 0$. Hence $\sigma = n$, and so $s = n$. Remembering the meaning of s, we see that $P_n(\theta)$ has n non-equivalent zeros x_k, and that, by (13), these are determined as stated in the enunciation.

We have proved that (11) has a solution with $d_k > 0$; and, since the determinant of the first n equations of that system,

$$\prod_{j>k} (\xi_j - \xi_k),$$

is different from zero, there is only one solution. This proves (b). Further, we know enough about g to write down the equations

$$c_{n+r} = \frac{1}{2\pi}\int_0^{2\pi} dg\, e^{-i(n+r)\theta}$$

$$= \sum_{k=1}^n d_k e^{-i(n+r)x_k} \quad (r = 1, 2, \ldots),$$

which proves (a).

III. *The parametric representation of positive harmonic functions.*

5. In all that follows, we denote by $z_n = \rho_n e^{i\theta_n}$ $(n = 1, 2, \ldots)$ a sequence of independent parameters which are subject to the condition

$$\rho_n^2 \leqslant 1 \quad (n = 1, 2, \ldots). \tag{14}$$

A sequence $c_1, \ldots, c_n, \ldots$ of numbers, where $c_n = \frac{1}{2}(a_n - ib_n)$ will be said to be of class P if, and only if,

$$1 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) r^n > 0 \quad (0 \leqslant r < 1).$$

THEOREM I. (i) *To each integer $n > 1$ there corresponds $n-1$ polynomials with integer coefficients,*

$$p_n^{(1)}(x_1, y_1), \quad p_n^{(2)}(x_1, y_1, x_2, y_2), \quad \ldots, \quad p_n^{(n-1)}(x_1, y_1, \ldots, x_{n-1}, y_{n-1}), \tag{15}$$

such that

$$p_n^{(r)}(x_1, y_1, \ldots, x_{r-1}, y_{r-1}, 0, 0) \equiv 0 \tag{16}$$

$$(r = 1, \ldots, n-1),$$

and such that any sequence of class P admits of the representation

$$c_1 = z_1, \tag{17}$$

$$\begin{aligned} c_n = {} & p_n^{(1)}(z_1, \bar{z}_1) + (1-\rho_1^2) p_n^{(2)}(z_1, \bar{z}_1, z_2, \bar{z}_2) \\ & + (1-\rho_1^2)(1-\rho_2^2) p_n^{(3)}(z_1, \bar{z}_1, z_2, \bar{z}_2, z_3, \bar{z}_3) + \ldots \\ & + (1-\rho_1^2)\ldots(1-\rho_{n-2}^2) p_n^{(n-1)}(z_1, \bar{z}_1, \ldots, z_{n-1}, \bar{z}_{n-1}) \\ & + z_n \prod_{j=1}^{n-1} (1-\rho_j^2) \qquad (n > 1). \end{aligned} \tag{18}$$

(ii) *Conversely, a sequence which is calculated from* (17) *and* (18), *by giving assigned values to the z_n, is of class P.*

(iii) *Given a sequence of class P, the parameters z_n, on which the sequence actually depends, are uniquely determined.*

(iv) $$p_n^{(1)}(x_1, y_1) = x_1^n.$$

The proposition (i) will subsequently be made more precise by a proof that in each of the polynomials (15) the last variable does not occur. The precise meaning of (iii) will appear in the following proof.

A sequence $c_1, \ldots, c_n, \ldots$ of numbers will be said to be of class P_0 if, and only if,

$$D_n > 0 \quad (n = 0, 1, \ldots). \tag{19}$$

We write $(c_n) \subset P_0$ to express this event. If $(c_n) \subset P_0$, then, by (6) and (7),

$$D_n D_{n-2} = D_{n-1}^2 - [\delta_n - (-1)^n c_{-n} D_{n-2}][\overline{\delta}_n - (-1)^n c_n D_{n-2}] > 0 \quad (20)$$

$$(n \geqslant 2).$$

Let $\zeta_n = \sigma_n e^{i\phi_n}$ denote a sequence of independent parameters which are subject to the conditions

$$\sigma_n^2 < 1 \quad (n = 1, 2, \ldots). \quad (21)$$

By (20), we must have an equation of the form

$$c_n D_{n-2} - (-1)^n \overline{\delta}_n = \sigma_n e^{i\phi_n} D_{n-1} \quad (n = 2, 3, \ldots). \quad (22)$$

Conversely, if $c_1, \ldots, c_{n-1}$ have values such that $D_0, \ldots, D_{n-1} > 0$, and c_n satisfies (22), then $D_n > 0$.

For $n = 1$, we must have

$$D_1 = 1 - c_1 c_{-1} > 0,$$

i.e.

$$c_1 = \sigma_1 e^{i\phi_1} = \zeta_1. \quad (23)$$

If now we use the equation (22) successively for $n = 2, 3, \ldots$, we obtain relations of the form

$$c_n = R_n(\zeta_1, \overline{\zeta}_1, \ldots, \zeta_n, \overline{\zeta}_n), \quad (24)$$

where

$$R_n(x_1, y_1, \ldots, x_n, y_n)$$

is a uniquely determined rational function of its arguments with integer coefficients. We thus have

LEMMA 1. *The class P_0 is identical with the class represented by* (23), (24).

LEMMA 2. *With the values* (23), (24), *we have*

$$D_n = (1-\sigma_1^2)^n (1-\sigma_2^2)^{n-1} \ldots (1-\sigma_n^2).$$

We prove this by induction. It is true for $n = 1$. Suppose that it is true for suffixes less than n. Then, by substituting (22) in (20), we get

$$D_n D_{n-2} = D_{n-1}^2 (1-\sigma_n^2).$$

Since the lemma holds for D_{n-1}, D_{n-2}, this equation shows that it holds for D_n.

The rational function $R_n(\zeta_1, \overline{\zeta}_1, \ldots, \zeta_n, \overline{\zeta}_n)$ is a polynomial in its arguments, with integer coefficients. This is true for $n = 1$ with $R_1(\zeta_1, \zeta_1) = \zeta_1$.

For $n > 1$, we have, by (22),

$$c_n = (-1)^n \frac{\bar{\delta}_n}{D_{n-2}} + \zeta_n \frac{D_{n-1}}{D_{n-2}}.$$

Suppose that $R_1, \ldots, R_{n-1}$ are polynomials with integer coefficients. Then $\bar{\delta}_n$, which is a polynomial in $c_1, \bar{c}_1, \ldots, c_{n-1}, \bar{c}_{n-1}$ with integer coefficients, is a polynomial in $\zeta_1, \bar{\zeta}_1, \ldots, \zeta_{n-1}, \bar{\zeta}_{n-1}$ with integer coefficients, say g_n. Using Lemma 2, we have

$$c_n = \frac{(-1)^n g_n(\zeta_1, \bar{\zeta}_1, \ldots, \zeta_{n-1}, \bar{\zeta}_{n-1})}{(1-\zeta_1\bar{\zeta}_1)^{n-2}(1-\zeta_2\bar{\zeta}_2)^{n-3}\ldots(1-\zeta_{n-2}\bar{\zeta}_{n-2})} + \zeta_n \prod_{j=1}^{n-1}(1-\zeta_j\bar{\zeta}_j). \quad (25)$$

If now g_n was not divisible by D_{n-2}, we could choose the ζ_j $(j = 1, \ldots, n-2)$ so as to make c_n arbitrarily large in absolute value. This, however, is impossible, since c_n is given by

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}\, dg(\theta)$$

for some non-diminishing $g(\theta)$ which satisfies

$$\frac{1}{2\pi}\int_0^{2\pi} dg(\theta) = 1.$$

Hence we can divide g_n by the denominator in (25). Since the denominator has the constant term 1, the quotient has integer coefficients. We thus have

LEMMA 3. *The class P_0 is identical with the class defined by*

$$c_1 = \zeta_1, \quad c_n = q_n(\zeta_1, \bar{\zeta}_1, \ldots, \zeta_{n-1}, \bar{\zeta}_{n-1}) + \zeta_n \prod_{j=1}^{n-1}(1-\sigma_j^2) \quad (n > 1), \quad (26)$$

where $$q_n(x_1, y_1, \ldots, x_{n-1}, y_{n-1})$$

is a polynomial with integer coefficients.

Suppose now that $c_1, \ldots, c_n, \ldots$ is a sequence of class P. The Laurentian

$$\sum_{j,k=0}^{n} c_{j-k} x_j \bar{x}_k$$

is non-negative for all n; and, conversely, if this Laurentian is non-negative for all n, then (c_n) is of class P. Consequently, if $\epsilon > 0$, the Laurentian

$$\frac{1}{1+\epsilon}\left[\sum_0^n c_{j-k} x_j \bar{x}_k + \epsilon \sum_0^n x_j \bar{x}_j\right]$$

is positive definite; *i.e.* the sequence $c_n/(1+\epsilon)$ is of class P_0. By Lemma 3, we have

$$\frac{c_n}{1+\epsilon} = q_n(\zeta_1, \bar{\zeta}_1, \ldots, \zeta_{n-1}, \bar{\zeta}_{n-1}) + \zeta_n \prod_{j=1}^{n-1} (1-\sigma_j^2) \quad (n>1),$$

$$\frac{c_1}{1+\epsilon} = \zeta_1,$$

where the ζ's depend on ϵ. Let $\epsilon \to 0$. By the method of selection, we can find numbers z_n such that $\zeta_n \to z_n$, and then

$$c_1 = z_1, \quad c_n = q_n(z_1, \bar{z}_1, \ldots, z_{n-1}, \bar{z}_{n-1}) + z_n \prod_{j=1}^{n-1} (1-\rho_j^2) \quad (n>1). \quad (27)$$

Further, since, by Lemma 2,

$$\begin{vmatrix} 1 & \frac{c_1}{1+\epsilon} & \ldots & \frac{c_n}{1+\epsilon} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{c_{-n}}{1+\epsilon} & & \ldots & 1 \end{vmatrix} = (1-\sigma_1^2)^n \ldots (1-\sigma_n^2),$$

it follows that

$$D_n = (1-\rho_1^2)^n \ \ldots \ (1-\rho_n^2). \quad (28)$$

Conversely, let (c_n) be a sequence defined by (27). We know that when the values (26) are substituted in D_n, the result is

$$(1-\sigma_1^2)^n \ldots (1-\sigma_n^2).$$

We can find sequences $\zeta_n^{(\nu)} \to z_n$ such that $\zeta_n^{(\nu)} \bar{\zeta}_n^{(\nu)} < 1$. It follows, by continuity, that when the values (27) are substituted in D_n, the result is (28). Further, since the sequence defined by

$$c_1^{(\nu)} = \zeta_1^{(\nu)}, \quad c_n^{(\nu)} = q_n(\zeta_1^{(\nu)}, \bar{\zeta}_1^{(\nu)}, \ldots, \zeta_{n-1}^{(\nu)}, \bar{\zeta}_{n-1}^{(\nu)}) + \zeta_n^{(\nu)} \prod_{j=1}^{n-1} (1-\sigma_j^{(\nu)2})$$

is of class P_0, the Laurentians

$$\sum_{j,k=0}^{n} c_{j-k}^{(\nu)} x_j \bar{x}_k$$

are positive definite. Hence, by continuity, the Laurentians

$$\sum_{j,k=0}^{n} c_{j-k} x_j \bar{x}_k$$

are non-negative and so (c_n) belongs to P. We thus have

LEMMA 4. *The class P is identical with the class defined by*

$$c_1 = z_1, \quad c_n = q_n(z_1, \bar{z}_1, \ldots, z_{n-1}, \bar{z}_{n-1}) + z_n \prod_{j=1}^{n-1} (1-\rho_j^2) \quad (n>1), \tag{29}$$

where q_n is the polynomial of Lemma 3; and these values give

$$D_n = (1-\rho_1^2)^n \ldots (1-\rho_n^2).$$

LEMMA 5. (α) *The polynomials q_n of Lemma 4 can be written in the form*

$$\begin{aligned} q_n = {} & p_n^{(1)}(z_1, \bar{z}_1) + (1-\rho_1^2) p_n^{(2)}(z_1, \bar{z}_1, z_2, \bar{z}_2) \\ & + (1-\rho_1^2)(1-\rho_2^2) p_n^{(3)}(z_1, \bar{z}_1, z_2, \bar{z}_2, z_3, \bar{z}_3) + \ldots \\ & + (1-\rho_1^2) \ldots (1-\rho_{n-2}^2) p_n^{(n-1)}(z_1, \bar{z}_1, \ldots, z_{n-1}, \bar{z}_{n-1}), \end{aligned} \tag{30}$$

where the $p_n^{(r)}$ are polynomials with integer coefficients.

(β) *This decomposition of q_n can be so effected that*

$$p_n^{(r)}(z_1, \bar{z}_1, \ldots, z_{r-1}, \bar{z}_{r-1}, 0, 0) \equiv 0 \quad (r = 1, \ldots, n-1).$$

(γ) *The decomposition of q_n in (α) subject to (β) is unique.*

Proof of (α). For $n=2$, we can write $q_2(z_1, \bar{z}_1) = p_2^{(1)}(z_1, \bar{z}_1)$. For $n>2$, we write

$$\begin{aligned} c_n &= q_n(z_1, \bar{z}_1, \ldots, z_{n-1}, \bar{z}_{n-1}) + z_n \prod_{j=1}^{n-1} (1-\rho_j^2) \\ &= q_n'(z_1, \bar{z}_1) + q_n''(z_1, \bar{z}_1, \ldots, z_{n-1}, \bar{z}_{n-1}) + z_n \prod_{j=1}^{n-1} (1-\rho_j^2), \end{aligned}$$

where q_n' consists of those terms of q_n which involve only z_1 or $\bar{z}_1$, and q_n'' consists of the remaining terms, We now observe that, by Theorem C, if

$$D_0, \quad \ldots, \quad D_{m-1} > 0, \quad D_m = 0, \tag{31}$$

then $c_{m+1}, c_{m+2}, \ldots$ are uniquely determined by $c_1, \ldots, c_m$, *i.e.* by $z_1, \ldots, z_m$. Further, by the expression for D_n in Lemma 4, the event (31) happens if, and only if, z_m is the first z of modulus 1.

Take $m=1$. Then, whatever values we give to $z_2, \ldots, z_n$, the numbers $c_n (n>1)$ will have the same value for an assigned $z_1 = e^{i\theta_1}$. Hence, since either every term of q_n'' contains some of the variables $z_2, \bar{z}_2, \ldots, z_{n-1}, \bar{z}_{n-1}$, or else $q_n'' \equiv 0$, we must have

$$q_n'' = 0 \text{ for } z_1 \bar{z}_1 = 1.$$

Thus q_n'' has $1-z_1\bar{z}_1$ as a factor, and, since this factor has the constant term 1, we have $q_n'' = (1-\rho_1^2)\kappa_n$, where κ_n is a polynomial with integer coefficients. Thus

$$c_n = p_n^{(1)}(z_1, \bar{z}_1)+(1-\rho_1^2)\kappa_n(z_1, \bar{z}_1, \ldots, z_{n-1}, \bar{z}_{n-1})+z_n \prod_{j=1}^{n-1}(1-\rho_j^2).$$

If $n = 3$, this is the desired result. If $n > 3$, we write

$$\kappa_n = \kappa_n'(z_1, \bar{z}_1, z_2, \bar{z}_2)+\kappa_n''(z_1, \bar{z}_1, \ldots, z_{n-1}, \bar{z}_{n-1}),$$

where κ_n' consists of those terms of κ_n which involve only $z_1, \bar{z}_1, z_2, \bar{z}_2$, and κ_n'' consists of the remaining terms. Writing $\kappa_n' = p_n^{(2)}$, we have

$$c_n = p_n^{(1)}(z_1, \bar{z}_1)+(1-\rho_1^2)p_n^{(2)}(z_1, \bar{z}_1, z_2, \bar{z}_2) + (1-\rho_1^2)\kappa_n''(z_1, \bar{z}_1, \ldots, z_{n-1}, \bar{z}_{n-1})+z_n \prod_{j=1}^{n-1}(1-\rho_j^2).$$

Let now $z_1 = \rho_1 e^{i\theta_1}$ with $\rho_1^2 < 1$. If $z_2 = e^{i\theta_2}$, then, whatever values we give to $z_3, \ldots, z_n$, the numbers c_n $(n \geqslant 3)$ remain the same. Hence, since either every term in κ_n'' contains some one of the variables $z_3, \bar{z}_3, \ldots, z_{n-1}, \bar{z}_{n-1}$, or else $\kappa_n'' \equiv 0$, we must have

$$\kappa_n'' = 0 \text{ for } z_2\bar{z}_2 = 1.$$

Thus κ_n'' has $1-z_2\bar{z}_2$ as a factor, and, since this factor has the constant term 1, we have $\kappa_n'' = (1-\rho_2^2)\lambda_n$, where λ_n is a polynomial with integer coefficients. Thus

$$c_n = p_n^{(1)}(z_1, \bar{z}_1)+(1-\rho_1^2)p_n^{(2)}(z_1, \bar{z}_1, z_2, \bar{z}_2) + (1-\rho_1^2)(1-\rho_2^2)\lambda_n(z_1, \bar{z}_1, \ldots, z_{n-1}, \bar{z}_{n-1})+z_n \prod_{j=1}^{n-1}(1-\rho_j^2).$$

If $n = 4$, this is the desired result. If $n > 4$, we repeat the above argument. We manifestly arrive at (30).

Proof of (β). If, in (30), a term

$$(1-\rho_1^2)\ldots(1-\rho_{j-1}^2)p_n^{(j)}(z_1, \bar{z}_1, \ldots, z_j, \bar{z}_j) \quad (j = 2, \ldots, n-1)$$

is such that

$$p_n^{(j)}(z_1, \bar{z}_1, \ldots, z_{j-1}, \bar{z}_{j-1}, 0, 0) \not\equiv 0,$$

then we can write

$$p_n^{(j)} = \pi_n^{(j)}(z_1, \bar{z}_1, \ldots, z_{j-1}, \bar{z}_{j-1})+P_n^{(j)}(z_1, \bar{z}_1, \ldots, z_j, \bar{z}_j)$$

with

$$P_n^{(j)}(z_1, \bar{z}_1, \ldots, z_{j-1}, \bar{z}_{j-1}, 0, 0) \equiv 0.$$

We can then add

$$(1-\rho_1^2)\dots(1-\rho_{j-1}^2)\,\pi_n^{(j)} = (1-\rho_1^2)\dots(1-\rho_{j-2}^2)(1-z_{j-1}\bar{z}_{j-1})\,\pi_n^{(j)}$$

to the preceding term

$$(1-\rho_1^2)\dots(1-\rho_{j-2}^2)\,p_n^{(j-1)}$$

without altering the form of the latter. If we perform this operation first for $j=n-1$, and work to the left in (30), we see that we may suppose that (β) is satisfied for $r>1$. It remains then to show that in these circumstances, *i.e.* when the $p_n^{(r)}$ for $r>1$ satisfy (β), then $p_n^{(1)}(0,0)\equiv 0$. This means that we must show that $c_n=0$ $(n>1)$ when $z_1=z_2=\dots=0$. We prove it for $n\geqslant 1$ by induction. It is true for c_1, since $c_1=z_1$. If $c_1=c_2=\dots=c_{n-1}=0$ when all the z's vanish, then, since by Lemma 4 we must have $D_n=1$, *i.e.* $1-c_n c_{-n}=1$, we must have $c_n=0$.

Proof of (γ). If we have an identity

$$p_n^{(1)}(z_1,\bar{z}_1)+(1-\rho_1^2)\,p_n^{(2)}(z_1,\bar{z}_1,z_2,\bar{z}_2)+\dots \equiv \pi_n^{(1)}(z_1,\bar{z}_1)+(1-\rho_1^2)\,\pi_n^{(2)}(z_1,\bar{z}_1,z_2,\bar{z}_2)+\dots, \quad (32)$$

where the p's and π's satisfy (β), then, putting $z_2=\dots=z_n=0$, we get

$$p_n^{(1)}(z_1,\bar{z}_1)\equiv\pi_n^{(1)}(z_1,\bar{z}_1).$$

Subtracting this from (32), and putting $z_3=\dots=z_n=0$, we get

$$p_n^{(2)}=\pi_n^{(2)},$$

and so on.

6. We have now proved (i) and (ii) of Theorem I. Concerning (iii), if we are given a sequence (c_n) of class P, we have

$$c_1=z_1. \quad (33)$$

If (33) makes $z_1\bar{z}_1=1$, then, by (18),

$$c_n=p_n^{(1)}(z_1,\bar{z}_1) \quad (n>1),$$

and the sequence depends only on z_1. If, however, $z_1\bar{z}_1<1$, then, by (18), $n=2$,

$$c_2=p_2^{(1)}(z_1,\bar{z}_1)+z_2(1-z_1\bar{z}_1), \quad (34)$$

which gives the value of z_2. If (34) makes $z_2\bar{z}_2=1$, then (18) gives

$$c_n=p_n^{(1)}(z_1,\bar{z}_1)+(1-z_1\bar{z}_1)\,p_n^{(2)}(z_1,\bar{z}_1,z_2,\bar{z}_2) \quad (n>2),$$

so that the sequence depends only on z_1, z_2. If, however, $z_2\bar{z}_2 < 1$, then, by (18), $n = 3$,

$$c_3 = p_3^{(1)}(z_1, \bar{z}_1) + (1-z_1\bar{z}_1)\, p_3^{(2)}(z_1, \bar{z}_1, z_2, \bar{z}_2) + z_3(1-z_1\bar{z}_1)(1-z_2\bar{z}_2),$$

which gives the value of z_3. Proceeding in this manner, we obtain (iii).

To prove (iv), *i.e.* to prove that

$$p_n^{(1)}(z_1, \bar{z}_1) = z_1^n \quad (n > 1),$$

we must show, by (i), Theorem I, that

$$c_n = z_1^n \quad (z_2 = z_3 = \ldots = 0) \quad (n > 1). \tag{35}$$

We prove this for $n > 1$ by induction. It is true for $n = 1$. Let it be true for indices less than n. By Lemma 4, we must now have

$$D_n = (1-\rho_1^2)^n.$$

By the definition of D_n, we must accordingly have

$$\begin{vmatrix} 1 & z_1 & z_1^2 & \ldots & z_1^{n-1} & c_n \\ \bar{z}_1 & 1 & z_1 & \ldots & & z_1^{n-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ c_{-n} & \bar{z}_1^{n-1} & & \ldots & & 1 \end{vmatrix} = (1-\rho_1^2)^n.$$

Multiply the second row by z_1 and subtract from the first; multiply the n-th row by $\bar{z}_1$ and subtract from the last. We get

$$\begin{vmatrix} 1-\rho_1^2 & 0 & 0 & \ldots & 0 & c_n - z_1^n \\ \bar{z}_1 & 1 & z_1 & \ldots & z_1^{n-2} & z_1^{n-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \bar{z}_1^{n-1} & \bar{z}_1^{n-2} & & \ldots & 1 & \bar{z}_1 \\ c_{-n} - \bar{z}_1^n & 0 & 0 & \ldots & 0 & 1-\rho_1^2 \end{vmatrix} = (1-\rho_1^2)^n.$$

Expanding this determinant, we get

$$(1-\rho_1^2)^2 D_{n-2} - (c_n - z_1^n)(c_{-n} - \bar{z}_1^n) D_{n-2} = (1-\rho_1^2)^n.$$

But $D_{n-2} = (1-\rho_1^2)^{n-2} \not\equiv 0$, and so the preceding relation gives $c_n = z_1^n$, the desired result.

7. Let r be a positive integer. The class of sequences which are obtained from (17), (18) by putting $z_{r+1} = z_{r+2} = \ldots = 0$ will be called the

class P_r. The class P_1 is, by Theorem I, defined by

$$c_1 = z_1^n \quad (n \geqslant 1).$$

The corresponding class of positive harmonic functions is the elementary class

$$\frac{1-r^2\rho_1^2}{1-2r\rho_1\cos(\theta_1+x)+r^2\rho_1^2} \quad (\rho_1^2 \leqslant 1).$$

THEOREM II. *Suppose that the general values of $c_1, \ldots, c_r$ have been determined; namely*

$$\left.\begin{aligned} &c_1 = z_1, \\ &c_2 = z_1^2+(1-\rho_1^2)z_2, \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \\ &c_r = z_1^r+(1-\rho_1^2)\,p_r^{(2)}(z_1, \bar{z}_1, z_2, \bar{z}_2)+\ldots+z_r\prod_{j=1}^{r-1}(1-\rho_j^2). \end{aligned}\right\} \tag{36}$$

In the determinant

$$\begin{vmatrix} 1 & c_1 & \ldots & c_r \\ c_{-1} & 1 & \ldots & c_{r-} \\ \ldots & \ldots & \ldots & \ldots \\ c_{-r} & c_{-(r-1)} & \ldots & 1 \end{vmatrix}$$

let $\gamma_0, \gamma_1, \ldots, \gamma_r$ be the cofactors of the elements in the last row, expressed as functions of the z's. Then

(i) $\frac{\gamma_0}{\gamma_r}, \frac{\gamma_1}{\gamma_r}, \ldots, \frac{\gamma_{r-1}}{\gamma_r}$ *are polynomials in $z_1, \bar{z}_1, \ldots, z_{r-1}, \bar{z}_{r-1}, z_r$ with integer coefficients, and do not involve $\bar{z}_r$;*

(ii) *the class P_r is defined by the equations* (36) *together with*

$$c_n+\frac{\gamma_{r-1}}{\gamma_r}c_{n-1}+\frac{\gamma_{r-2}}{\gamma_r}c_{n-2}+\ldots+\frac{\gamma_0}{\gamma_r}c_{n-r}=0 \quad (n>r); \tag{37}$$

(iii) *in the class P_r, c_n does not involve $\bar{z}_r$.*

We require the following lemma.

LEMMA 6. *If $c_1, \ldots, c_r$ are defined by* (36), *then the general values of c_n $(n>r)$ in the class P_r are uniquely determined by the condition that they shall satisfy the equations*

$$D_n = (1-\rho_1^2)^n(1-\rho_2^2)^{n-1}\ldots(1-\rho_r^2)^{n-r+1} \quad (n>r). \tag{38}$$

By the definition of the class P_r, and by the expression for D_n in Lemma 4, we know that (38) must hold. We have, by (6) and (7),

$$D_{r+1}D_{r-1} = D_r^2 - [c_{r+1}D_{r-1} - (-1)^r \overline{\delta}_{r+1}][c_{-(r+1)}D_{r-1} - (-1)^r \delta_{r+1}].$$

The relations (36) give values for D_r, D_{r-1}, which are obtained from (38) by putting $n = r$, $n = r-1$ respectively in that equation. By hypothesis, D_{r+1} is given by (38) on putting $n = r+1$. Substituting these values, we have

$$[c_{r+1}D_{r-1} - (-1)^r \overline{\delta}_{r+1}][c_{-(r+1)}D_{r+1} - (-1)^r \delta_{r+1}] = 0.$$

Hence
$$c_{r+1} = (-1)^r \overline{\delta}_{r+1}/D_{r-1},$$

which determines c_{r+1} uniquely. The proof of the lemma now proceeds by induction, the same argument being used.

Turning to the proof of the theorem, we first consider (ii). By Lemma 6, we need only show that when (36), (37) are substituted in D_n, the result is

$$(1-\rho_1^2)^n (1-\rho_2^2)^{n-1} \dots (1-\rho_r^2)^{n-r+1} \quad (n > r).$$

We shall prove this for $n \geqslant r$ by induction. It is true for $n = r$. We may suppose then that $n > r$, and that when (36), (37) are substituted in D_{n-1}, the result is

$$D_{n-1} = (1-\rho_1^2)^{n-1}(1-\rho_2^2)^{n-2} \dots (1-\rho_r^2)^{n-r}. \tag{39}$$

Consider now D_n. We add to the first row γ_{r-1}/γ_r times the second row, γ_{r-2}/γ_r times the third row, ..., γ_0/γ_r times the $(r+1)$-th row. Then, in the resulting determinant, we have, in the first row, $r+1$ elements whose values we must calculate, and these are succeeded by $n-r$ zeros in virtue of (37). The $r+1$ elements are, written vertically for convenience,

$$\left.\begin{array}{l} c_0 + \frac{\gamma_{r-1}}{\gamma_r} c_{-1} + \frac{\gamma_{r-2}}{\gamma_r} c_{-2} + \dots \frac{\gamma_0}{\gamma_r} c_{-r}, \\ c_1 + \frac{\gamma_{r-1}}{\gamma_r} c_0 + \frac{\gamma_{r-2}}{\gamma_r} c_{-1} + \dots \frac{\gamma_0}{\gamma_r} c_{-(r-1)}, \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ c_r + \frac{\gamma_{r-1}}{\gamma_r} c_{r-1} + \frac{\gamma_{r-2}}{\gamma_r} c_{r-2} + \dots \frac{\gamma_0}{\gamma_r} c_0. \end{array}\right\} \tag{40}$$

Now the numbers $\gamma_0, \dots, \gamma_r$ are the cofactors of the elements in the last row of D_r; from the structure of a Laurentian determinant, we see that they are the cofactors of the elements in the first column, starting

from the bottom and going upwards. Hence the numbers (40) are

$$D_r/\gamma_r,\ 0,\ 0,\ \ldots,\ 0.$$

Since $\gamma_r = D_{r-1}$, we have for the first row of the determinant, obtained by transforming D_n,

$$(1-\rho_1^2)\ \ldots\ (1-\rho_r^2)\ 0\ 0\ \ldots\ 0.$$

Hence
$$D_n = (1-\rho_1^2)\ \ldots\ (1-\rho_r^2)\, D_{n-1}.$$

By (39), this is the desired result.

To prove (i), let $z_1, \ldots, z_{r-1}$ have values of modulus less than 1. Let $z_r = e^{i\theta_r}$. Denote the special value of c_r which thus arises by $c_r(\theta)$. We know by Theorem C that, if in

$$\begin{vmatrix} 1 & c_1 & c_2 & \ldots & c_{r-1} & c_r(\theta) \\ c_{-1} & 1 & & \ldots & & c_{r-1} \\ \ldots & \ldots & \ldots & & \ldots & \ldots \\ c_{-r}(\theta) & & & \ldots & & 1 \end{vmatrix}$$

we take the cofactors of the elements in the last row and call them $G_0, G_1, \ldots, G_r$, then the equation

$$\frac{G_0}{G_r} + \frac{G_1}{G_r}\,\xi + \ldots + \xi^r = 0 \tag{41}$$

has r distinct roots, each of modulus 1. Now

$$G_r = D_{r-1} = (1-\rho_1^2)^{r-1} \ldots (1-\rho_{r-1}^2).$$

If, then, not all of $G_0, \ldots, G_{r-1}$ were divisible by G_r, we could choose $\rho_1 < 1, \ldots, \rho_{r-1} < 1$ so that some of the coefficients in (41) were arbitrarily large in absolute value. This, however, is impossible. For, since the roots of (41) are of modulus 1, we must have

$$\left|\frac{G_{r-1}}{G_r}\right| \leqslant r, \quad \left|\frac{G_{r-2}}{G_r}\right| \leqslant \binom{r}{2}, \quad \ldots, \quad \left|\frac{G_0}{G_r}\right| \leqslant 1.$$

Hence $G_0, \ldots, G_{r-1}$ are divisible by G_r, and, since the constant term in G_r is 1, the quotients have integer coefficients.

Let us now see how $e^{i\theta_r}$ occurs in $G_0, \ldots, G_r$. In the first place, by (36), it does not occur in $c_1, \ldots, c_{r-1}$, while $c_r(\theta)$ is a linear function of

$e^{i\theta_r}$. Thus, in the elements of D_r, $e^{i\theta_r}$ occurs as a linear function in the element at the top right-hand corner. It also occurs, not as a linear function, in the element c_{-r} in the bottom left-hand corner. We have chosen $G_0, \ldots, G_r$ as the cofactors of the elements of the last row. Hence these cofactors involve $e^{i\theta_r}$, if at all, as a linear function.

Further, $G_r = D_{r-1}$ does not involve $e^{i\theta_r}$. Hence, if we replace $e^{i\theta_r}$ by z_r, so that $G_0, \ldots, G_{r-1}$ become $\gamma_0, \ldots, \gamma_{r-1}$ respectively, then these will be divisible by $G_r = \gamma_r$; the quotients will be linear functions of z_r and will not involve $\bar{z}_r$; they will have integer coefficients. This proves (i).

To prove (iii), we have only to observe that $c_1, \ldots, c_r$ do not involve $\bar{z}_r$, and that, by (i), the coefficients in the recurrence formula (37) do not involve $\bar{z}_r$.

8. THEOREM III. (i) *The polynomials* $p_n^{(r)}(z_1, \bar{z}_1, \ldots, z_r, \bar{z}_r)$ *in* (18) *do not involve the last variable, and can therefore be written in the form* $p_n^{(r)}(z_1, \bar{z}_1, \ldots, z_{r-1}, \bar{z}_{r-1}, z_r)$;

(ii) $$p_n^{(2)}(1, 1, z_2) \text{ is of degree } n-1,$$

$$p_n^{(3)}(1, 1, -1, -1, z_3) \text{ is of degree } n-2,$$

and generally

$$p_n^{(r)}\{1, 1, -1, -1, \ldots, (-1)^r, (-1)^r, z_r\} \text{ is of degree } n-r+1.$$

Concerning (ii), we must remember that $p_n^{(r)}$ has been defined only for $n > r$, so that (ii) shows that none of the $p_n^{(r)}$ vanishes identically, a possibility which was not excluded in the proof of Theorem I.

Proof of (i). The class P_r is defined by

$$c_1 = z_1,$$

$$c_2 = z_1^2 + (1-\rho_1^2) z_2,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$c_r = z_1^r + (1-\rho_1^2) p_r^{(2)}(z_1, \bar{z}_1, z_2, \bar{z}_2) + \ldots$$
$$+ (1-\rho_1^2)\ldots(1-\rho_{r-2}^2) p_r^{(r-1)}(z_1, \bar{z}_1, \ldots, z_{r-1}, \bar{z}_{r-1}) + z_r \prod_{j=1}^{r-1} (1-\rho_j^2),$$

$$c_n = z_1^n + (1-\rho_1^2) p_n^{(2)} + \ldots + (1-\rho_1^2)\ldots(1-\rho_{r-2}^2) p_n^{(r-1)}$$
$$+ p_n^{(r)}(z_1, \bar{z}_1, \ldots, z_r, \bar{z}_r) \prod_{j=1}^{r} (1-\rho_j^2) \quad (n > r).$$

To prove that $p_n^{(r)}$ does not involve $\bar{z}_r$, we observe that, in P_r, $c_n (n > r)$ is the

sum of r terms of which the first $r-1$ certainly do not involve $\bar{z}_r$. The last term is $p_n^{(r)}$ times a function which does not involve $\bar{z}_r$. Since, by Theorem II (iii), c_n does not involve $\bar{z}_r$, the same is true of $p_n^{(r)}$.

Proof of (ii). We need two lemmas.

LEMMA 7. *Let u_1 be a complex variable. Let $u_2, u_3, \ldots$ be a sequence of functions of u_1 such that*

$$\begin{vmatrix} 1 & u_1 & u_2 & \ldots & u_n \\ u_{-1} & 1 & u_1 & \ldots & u_{n-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ u_{-n} & u_{-(n-1)} & & \ldots & 1 \end{vmatrix} = (1-u_1u_{-1})^n \quad (n=0, 1, \ldots).$$

Then $u_n = u_1^n$.

The proof, by induction, is precisely similar to that of (iv), Theorem I, § 6.

LEMMA 8. *Let $\zeta_1, \ldots, \zeta_r$ be independent complex variables. Let $\gamma_1, \ldots, \gamma_n, \ldots$ be a sequence of functions defined by*

$$\gamma_1 = \zeta_1,$$

$$\gamma_2 = \zeta_1^2 + (1-\zeta_1\bar{\zeta}_1)\zeta_2,$$

$$\gamma_3 = \zeta_1^3 + (1-\zeta_1\bar{\zeta}_1)P_3^{(2)}(\zeta_1, \bar{\zeta}_1, \zeta_2) + \zeta_3(1-\zeta_1\bar{\zeta}_1)(1-\zeta_2\bar{\zeta}_2),$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\gamma_r = \zeta_1^r + (1-\zeta_1\bar{\zeta}_1)P_r^{(2)}(\zeta_1, \bar{\zeta}_1, \zeta_2) + \ldots + (1-\zeta_1\bar{\zeta}_1)\ldots(1-\zeta_{r-2}\bar{\zeta}_{r-2}) \times P_r^{(r-1)}(\zeta_1, \bar{\zeta}_1, \ldots, \zeta_{r-2}, \bar{\zeta}_{r-2}, \zeta_{r-1}) + \zeta_r \prod_{j=1}^{r-1}(1-\zeta_j\bar{\zeta}_j),$$

$$\gamma_n = \zeta_1^n + (1-\zeta_1\bar{\zeta}_1)P_n^{(2)}(\zeta_1, \bar{\zeta}_1, \zeta_2) + \ldots + (1-\zeta_1\bar{\zeta}_1)\ldots(1-\zeta_{r-1}\bar{\zeta}_{r-1}) \times P_n^{(r)}(\zeta_1, \bar{\zeta}_1, \ldots, \zeta_{r-1}, \bar{\zeta}_{r-1}, \zeta_r) \quad (n>r),$$

where the P's are polynomials in the variables indicated. For brevity let $[\nu] = \max(\nu, 0)$. If

$$\begin{vmatrix} 1 & \gamma_1 & \ldots & \gamma_n \\ \gamma_{-1} & 1 & \ldots & \gamma_{n-1} \\ \ldots & \ldots & \ldots & \ldots \\ \gamma_{-n} & & \ldots & 1 \end{vmatrix} = (1-\zeta_1\bar{\zeta}_1)^{[n]}(1-\zeta_2\bar{\zeta}_2)^{[n-1]}\ldots(1-\zeta_r\bar{\zeta}_r)^{[n-r+1]} \quad (n=0, 1, \ldots), \tag{42}$$

then $P_n^{(r)}\{1, 1, -1, -1, 1, 1, \ldots, (-1)^r, (-r)^r, \zeta_r\}$

is of degree $n-r+1$.

The first significant value of r in the lemma is $r = 2$. We first prove the lemma in this case. We have

$$\gamma_1 = \zeta_1,$$

$$\gamma_2 = \zeta_1^2 + (1-\zeta_1\bar{\zeta}_1)\,\zeta_2,$$

$$\gamma_n = \zeta_1^n + (1-\zeta_1\bar{\zeta}_1)\,P_n^{(2)}(\zeta_1, \bar{\zeta}_1, \zeta_2) \quad (n > 2).$$

The condition (42) now becomes

$$\begin{vmatrix} 1 & \gamma_1 & \ldots & \gamma_n \\ \gamma_{-1} & 1 & \ldots & \gamma_{n-1} \\ \ldots & \ldots & \ldots & \ldots \\ \gamma_{-n} & & \ldots & 1 \end{vmatrix} = (1-\zeta_1\bar{\zeta}_1)^n(1-\zeta_2\bar{\zeta}_2)^{n-1} \quad (n = 1, 2, \ldots). \tag{43}$$

Multiply the second row by ζ_1 and subtract from the first; multiply the third row by ζ_1 and subtract from the second, and so on; finally, multiply the last row by ζ_1 and subtract from the n-th. Then every element of the first n rows will be divisible by $1-\zeta_1\bar{\zeta}_1$. The transformed determinant equals the second member of (43). We divide both sides by $(1-\zeta_1\bar{\zeta}_1)^n$, and then put $\zeta_1 = 1$. Writing

$$P_n^{(2)}(1, 1, \zeta_2) = \pi_n, \quad \pi_2 = \zeta_2,$$

we get

$$\begin{vmatrix} 1 & 0 & \pi_2 & \pi_3-\pi_2 & \pi_4-\pi_3 & \ldots & \pi_n-\pi_{n-1} \\ 1-\bar{\pi}_2 & 1 & 0 & \pi_2 & \pi_3-\pi_2 & \ldots & \pi_{n-1}-\pi_{n-2} \\ 1+\bar{\pi}_2-\bar{\pi}_3 & 1-\bar{\pi}_2 & 1 & 0 & \pi_2 & \ldots & \\ 1+\bar{\pi}_3-\bar{\pi}_4 & 1+\bar{\pi}_2-\bar{\pi}_3 & 1-\bar{\pi}_2 & 1 & 0 & \ldots & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 1 & 1 & 1 & 1 & \ldots & 1 \end{vmatrix} = (1-\zeta_2\bar{\zeta}_2)^{n-1}$$

$$(n = 1, 2, \ldots).$$

In this determinant we subtract the second column from the first, the third column from the second, and so on; finally, the last from the n-th. Then in the resulting determinant there are n zeros in the last row followed by 1, so that the determinant is equal to the minor of the element in the bottom right-hand corner. We thus have

$$\begin{vmatrix} 1 & & & -\pi_2 & \pi_2-(\pi_3-\pi_2) \\ -\bar{\pi}_2 & & & 1 & -\pi_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \bar{\pi}_{n-1}-\bar{\pi}_{n-2}-(\bar{\pi}_n-\bar{\pi}_{n-1}) & & & & \end{vmatrix}$$

$$\begin{vmatrix} \pi_3-\pi_2-(\pi_4-\pi_3) & \cdots & \pi_{n-1}-\pi_{n-2}-(\pi_n-\pi_{n-1}) \\ \pi_2-(\pi_3-\pi_2) & \cdots & \\ \cdots & \cdots & \cdots \\ & \cdots & 1 \end{vmatrix} = (1-\zeta_2\bar{\zeta}_2)^{n-1}$$

$(n=2, 3, \ldots)$,

with the convention $\pi_1=\pi_0=0$. This is a Laurentian determinant of order n. If we write

$$k_n=\pi_n-\pi_{n-1}-(\pi_{n+1}-\pi_n) \quad (n=1, 2, \ldots)$$

then the above condition is the same as

$$\begin{vmatrix} 1 & k_1 & \cdots & k_n \\ k_{-1} & 1 & \cdots & k_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ k_{-n} & & \cdots & 1 \end{vmatrix} = (1-\zeta_2\bar{\zeta}_2)^n \quad (n=0, 1, \ldots).$$

Now $k_1=-\pi_2=-\zeta_2$. By Lemma 7, $k_n=(-\zeta_2)^n$. Thus

$$\pi_{n-1}-\pi_{n-2}-(\pi_n-\pi_{n-1})=(-\zeta_2)^{n-1}. \tag{44}$$

This manifestly gives for π_n, *i.e.* for $P_n^{(2)}(1, 1, \zeta_2)$, a polynomial of degree $n-1$.

For $r>2$, the proof is by induction. We assume the lemma for $2, \ldots, r-1$. In (42), the γ's have the values assigned in the enunciation of the lemma. We treat the determinant in (42) in exactly the same way as we treated the determinant in (43). We shall obtain a Laurentian

determinant of order n which must equal

$$(1-\zeta_2\bar{\zeta}_2)^{[n-1]}\dots(1-\zeta_r\bar{\zeta}_r)^{[n-r+1]}.$$

To find what now takes the place of k_n, we must form the expression

$$\frac{\gamma_n-\zeta_1\gamma_{n-1}-(\gamma_{n+1}-\zeta_1\gamma_n)}{1-\zeta_1\bar{\zeta}_1}$$

and then put $\zeta_1=1$. Call the result κ_n. For $n>r-1$, we manifestly get an expression of the form

$$[\pi_n-\pi_{n-1}-(\pi_{n+1}-\pi_n)]$$
$$+(1-\zeta_2\bar{\zeta}_2)q_n^{(2)}(\zeta_2,\bar{\zeta}_2,\zeta_3)+(1-\zeta_2\bar{\zeta}_2)(1-\zeta_3\bar{\zeta}_3)q_n^{(3)}(\zeta_2,\bar{\zeta}_2,\zeta_3,\bar{\zeta}_3,\zeta_4)+\dots$$
$$+(1-\zeta_2\bar{\zeta}_2)\dots(1-\zeta_{r-1}\bar{\zeta}_{r-1})q_n^{(r-1)}(\zeta_2,\bar{\zeta}_2,\dots,\zeta_{r-1},\bar{\zeta}_{r-1},\zeta_r),$$

where the q's are polynomials. For $n=1, 2, \dots, r-1$, we get, in virtue of (44), the respective values,

$$-\zeta_2,$$
$$(-\zeta_2)^2-\zeta_3(1-\zeta_2\bar{\zeta}_2),$$
$$(-\zeta_2)^3+(1-\zeta_2\bar{\zeta}_2)q_3^{(2)}(\zeta_2,\bar{\zeta}_2,\zeta_3)-\zeta_4(1-\zeta_2\bar{\zeta}_2)(1-\zeta_3\bar{\zeta}_3),$$
$$\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots$$
$$(-\zeta_2)^{r-1}+(1-\zeta_2\bar{\zeta}_2)q_{r-1}^{(2)}(\zeta_2,\bar{\zeta}_2,\zeta_3)+\dots-\zeta_r(1-\zeta_2\bar{\zeta}_2)\dots(1-\zeta_{r-1}\bar{\zeta}_{r-1}).$$

We therefore write

$$-\zeta_2=w_1,\quad -\zeta_3=w_2,\quad \dots,\quad -\zeta_r=w_{r-1}.$$

Then κ_n is a function of $w_1,\dots,w_{r-1}$ of the same form as is γ_n of $\zeta_1,\dots,\zeta_{r-1}$ in the enunciation of the lemma, if we replace r by $r-1$ in that enunciation. We further have the condition

$$\begin{vmatrix} 1 & \kappa_1 & \dots & \kappa_n \\ \kappa_{-1} & 1 & \dots & \kappa_{n-1} \\ \dots & \dots & \dots & \dots \\ \kappa_{-n} & & \dots & 1 \end{vmatrix} = (1-w_1\bar{w}_1)^{[n]}\dots(1-w_{r-1}\bar{w}_{r-1})^{[n-r]} \qquad (n=1, 2, \dots).$$

By the lemma for $r-1$, it follows that, if we write

$$q_n^{(r-1)}(\zeta_2,\bar{\zeta}_2,\dots,\zeta_{r-1},\bar{\zeta}_{r-1},\zeta_r)=Q_n^{(r-1)}(w_1,\bar{w}_1,\dots,w_{r-2},\bar{w}_{r-2},w_{r-1}),$$

then $$Q_n^{(r-1)}\{1, 1, -1, -1, \ldots (-1)^{r-1}, (-1)^{r-1}, w_{r-1}\}$$

is of degree $$n-(r-1)+1 = n-r+2.$$

Hence $q_n^{(r-1)}\{-1, -1, 1, 1, \ldots, (-1)^r, (-1)^r, -\zeta_r\}$, and therefore $q_n^{(r-1)}\{-1, -1, 1, 1, \ldots, (-1)^r, (-1)^r, \zeta_r\}$ is of degree $n-r+2$. But

$$q_n^{(r-1)} = [P_n^{(r)} - P_{n-1}^{(r)} - (P_{n+1}^{(r)} - P_n^{(r)})]_{\zeta_1=1}$$

$$(n > r-1, \quad P_r^{(r)} = \zeta_r, \quad P_{r-1}^{(r)} = 0).$$

Hence $$P_{n+1}^{(r)}\{1, 1, -1, -1, \ldots, (-1)^r, (-1)^r, \zeta_r\} \quad (n+1 > r)$$

is of degree $n-r+2$. This proves the lemma and manifestly includes (ii).

9. An examination of Theorem II shows that it contains an inductive method for determining the general value of c_n in (18). For, if we know the general values of $c_1, \ldots, c_r$, then Theorem II gives us the expression for c_{r+1} in P_r. To obtain the general value of c_{r+1}, we merely add $z_{r+1} \prod_{j=1}^{r} (1-\rho_j^2)$ to that expression.

The class P_2.

This class is defined by $c_1 = z_1$, $c_2 = z_1^2 + (1-\rho_1^2) z_2$, and by a recurrence formula which we find by Theorem II. The cofactors of the elements in the last row in

$$\begin{vmatrix} 1 & c_1 & c_2 \\ c_{-1} & 1 & c_1 \\ c_{-2} & c_{-1} & 1 \end{vmatrix}$$

are $$\gamma_0 = c_1^2 - c_2, \quad \gamma_1 = c_{-1}c_2 - c_1, \quad \gamma_2 = 1 - c_1 c_{-1}.$$

Substituting the values of c_1 and c_2, we get

$$\gamma_0 = -(1-\rho_1^2) z_2, \quad \gamma_1 = (1-\rho_1^2)(-z_1 + z_1 \bar{z}_2), \quad \gamma_2 = 1-\rho_1^2.$$

The recurrence formula is therefore

$$c_n + (\bar{z}_1 z_2 - z_1) c_{n-1} - z_2 c_{n-2} = 0 \quad (n > 2). \tag{45}$$

If we choose A and B so that

$$\frac{A+Bx}{1+(\bar{z}_1 z_2 - z_1)x - z_2 x^2} = c_1 + c_2 x + \sum_{n=3}^{\infty} u_n x^{n-1},$$

we shall have $u_n = c_n$. We find that $A = z_1$, $B = z_2$. Thus

$$\frac{z_1 + z_2 x}{1 + (\bar{z}_1 z_2 - z_1)x - z_2 x^2} = \sum_{n=1}^{\infty} c_n x^{n-1}.$$

If we decompose the first member into partial fractions, we can easily obtain an explicit expression for c_n.

We have $$c_n = z_1^n + p_n^{(2)}(z_1, \bar{z}_1, z_2)(1-\rho_1^2) \quad (n > 2). \tag{46}$$

In considering the class P_2, we can simplify the notation by writing $p_n = p_n^{(2)}(z_1, \bar{z}_1, z_2)$. We further write $p_2 = z_2$. From (45) and (46) we get

$$p_n = (z_1 - \bar{z}_1 z_2) p_{n-1} + z_2 p_{n-2} + z_2 z_1^{n-2}, \tag{47}$$

a formula which enables us to write down the first few values of p_n without trouble. In Table I, we give the values for $n \leqslant 10$, arranged in powers of z_2.

The class P_3.

In the class P_2 the value of c_3 is $z_1^3 + (1-\rho_1^2) p_3^{(2)}(z_1, \bar{z}_1, z_2)$. By Table I, $p_3^{(2)} = 2z_1 z_2 - \bar{z}_1 z_2^2$. Hence the general value of c_3 is given by

$$c_3 = z_1^3 + (1-\rho_1^2)(2z_1 z_2 - z_1 z_2^2) + z_3(1-\rho_1^2)(1-\rho_2^2).$$

To find the recurrence formula for the class P_3, we must write down the cofactors of the elements of the last row of

$$\begin{vmatrix} 1 & c_1 & c_2 & c_3 \\ c_{-1} & 1 & c_1 & c_2 \\ c_{-2} & c_{-1} & 1 & c_1 \\ c_{-3} & c_{-2} & c_{-1} & 1 \end{vmatrix}.$$

We call them $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ respectively. In the first place

$$\gamma_3 = D_2 = (1-\rho_1^2)^2(1-\rho_2^2).$$

If we write $$c_3 = c_3^{(2)} + z_3(1-\rho_1^2)(1-\rho_2^2),$$

so that $c_3^{(2)}$ is the value of c_3 in P_2, then by (45),

$$c_3^{(2)} + (\bar{z}_1 z_2 - z_1) c_2 - z_2 c_1 = 0.$$

Further, since by the analysis of P_2, the numbers 1, $\bar{z}_1 z_2 - z_1$, $-z_2$ are $(1-\rho_1^2)^{-1}$ times the cofactors of the elements in the last row of D_2, from

right to left, these numbers are $(1-\rho_1^2)^{-1}$ times the cofactors of the elements in the first column of D_2, from top to bottom. Hence we have the three relations

$$c_2+(\bar{z}_1 z_2-z_1)c_1-z_2=0,$$

$$c_1+(\bar{z}_1 z_2-z_1)c_0-z_2 c_{-1}=0,$$

$$c_0+(\bar{z}_1 z_2-z_1)c_{-1}-z_2 c_{-2}=D_2/(1-\rho_1^2)=(1-\rho_1^2)(1-\rho_2^2).$$

These relations enable us to calculate γ_0, γ_1, γ_2 at once. For example,

$$\gamma_2=-\begin{vmatrix} 1 & c_1 & c_3 \\ c_{-1} & 1 & c_2 \\ c_{-2} & c_{-1} & c_1 \end{vmatrix}.$$

To the first row add $\bar{z}_1 z_2-z_1$ times the second row and $-z_2$ times the third. We get

$$\gamma_2=-(1-\rho_1^2)(1-\rho_2^2)\begin{vmatrix} 1 & 0 & z_3 \\ c_{-1} & 1 & c_2 \\ c_{-2} & c_{-1} & c_1 \end{vmatrix}$$

$$=(z_3\bar{z}_2-z_1+\bar{z}_1 z_2)(1-\rho_1^2)^2(1-\rho_2^2).$$

Similarly, $$\gamma_1=(z_3\bar{z}_1-z_3 z_1\bar{z}_2-z_2)(1-\rho_1^2)^2(1-\rho_2^2),$$

$$\gamma_0=-z_3(1-\rho_1^2)^2(1-\rho_2^2).$$

The recurrence formula is therefore

$$c_n+(z_3\bar{z}_2-z_1+\bar{z}_1 z_2)c_{n-1}+(z_3\bar{z}_1-z_3 z_1\bar{z}_2-z_2)c_{n-2}-z_3 c_{n-3}=0 \quad (n>3). \quad (48)$$

If we choose A, B, C so that

$$\frac{A+B_x+C_x^2}{1+(z_3\bar{z}_2-z_1+\bar{z}_1 z_2)x+(z_3\bar{z}_1-z_3\bar{z}_1 z_2-z_2)x^2-z_3 x^3}$$

$$=c_1+c_2 x+c_3 x^2+\sum_{n=4}^{\infty} u_n x^{n-1},$$

we have $c_n=u_n$. We find that $A=z_1$, $B=z_2+z_3 z_1\bar{z}_2$, $C=z_3$. Thus

$$\frac{z_1+(z_2+z_3 z_1\bar{z}_2)x+z_3 x^2}{1+(z_3\bar{z}_2-z_1+\bar{z}_1 z_2)x+(z_3\bar{z}_1-z_3 z_1\bar{z}_2-z_2)x^2-z_3 x^3}=\sum_{n=1}^{\infty} c_n x^{n-1}.$$

We have

$$c_n = z_1{}^n + (1-\rho_1{}^2)\,p_n^{(2)}(z_1, \bar{z}_1, z_2) + (1-\rho_1{}^2)(1-\rho_2{}^2)\,p_n^{(3)}(z_1, \bar{z}_1, z_2, \bar{z}_2, z_3). \quad (49)$$

We write $p_2^{(2)} = z_2$, and $p_1^{(2)} = p_0^{(2)} = 0$. We write $p_n^{(3)}$ for $p_n^{(3)}(z_1, \bar{z}_1, z_2, \bar{z}_2, z_3)$ and $p_3^{(3)} = z_3$, $p_2^{(3)} = p_1^{(3)} = p_0^{(3)} = 0$. We substitute (49) in (48). The terms which involve z_3 must add up to zero. We get

$$\begin{aligned}(1-\rho_1{}^2)(1-\rho_2{}^2)\,p_n^{(3)} &+ z_3\bar{z}_2\{z_1^{n-1} + (1-\rho_1{}^2)\,p_{n-1}^{(2)} + (1-\rho_1{}^2)(1-\rho_2{}^2)\,p_{n-1}^{(3)}\} \\ &+ (\bar{z}_1 z_2 - z_1)(1-\rho_1{}^2)(1-\rho_2{}^2)\,p_{n-1}^{(3)} + z_3(\bar{z}_1 - z_1\bar{z}_2) \\ &\times\{z_1^{n-2} + (1-\rho_1{}^2)\,p_{n-2}^{(2)} + (1-\rho_1{}^2)(1-\rho_2{}^2)\,p_{n-2}^{(3)}\} - z_2(1-\rho_1{}^2)(1-\rho_2{}^2)\,p_{n-2}^{(3)} \\ &- z_3\{z_1^{n-3} + (1-\rho_1{}^2)\,p_{n-3}^{(2)} + (1-\rho_1{}^2)(1-\rho_2{}^2)\,p_{n-3}^{(3)}\} = 0. \quad (50)\end{aligned}$$

In (50), the terms which do not contain the factor $(1-\rho_1{}^2)(1-\rho_2{}^2)$ are

$$\begin{aligned}z_3\bar{z}_2 z_1^{n-1} + z_3(\bar{z}_1 - z_1\bar{z}_2)\,z_1^{n-2} - z_3 z_1^{n-3} &\\ + (1-\rho_1{}^2)\{z_3\bar{z}_2\,p_{n-1}^{(2)} + z_3(z_1 - z_1 z_2)\,p_{n-2}^{(2)} - z_3 p_{n-3}^{(2)}\} &\\ = z_3(1-\rho_1{}^2)[-z_1^{n-3} + \bar{z}_2\,p_{n-1}^{(2)} + (\bar{z}_1 - z_1\bar{z}_2)\,p_{n-2}^{(2)} - p_{n-3}^{(2)}]. \quad (51)\end{aligned}$$

By (47), $$p_{n-1}^{(2)} = (z_1 - \bar{z}_1 z_2)\,p_{n-2}^{(2)} + z_2 p_{n-3}^{(2)} + z_2 z_1^{n-3}.$$

Hence, in (51), the expression in square brackets is

$$p_{n-2}^{(2)}\{\bar{z}_1 - z_1\bar{z}_2 + z_1\bar{z}_2 - \bar{z}_1 z_2\bar{z}_2\} + p_{n-3}^{(2)}(z_2\bar{z}_2 - 1) - z_1^{n-3}(1 - z_2\bar{z}_2),$$

and so (51) is

$$z_3(1-\rho_1{}^2)(1-\rho_2{}^2)\,[\bar{z}_1\,p_{n-2}^{(2)} - p_{n-3}^{(2)} - z_1^{n-3}].$$

We can therefore divide (50) by $(1-\rho_1{}^2)(1-\rho_2{}^2)$, and obtain

$$\begin{aligned}p_n^{(3)} + z_3[\bar{z}_1 p_{n-2}^{(2)} - p_{n-3}^{(2)} - z_1^{n-3}] + z_3\bar{z}_2 p_{n-1}^{(3)} &\\ + (\bar{z}_1 z_2 - z_1)\,p_{n-1}^{(3)} + z_3(\bar{z}_1 - z_1\bar{z}_2)\,p_{n-2}^{(3)} - z_2 p_{n-2}^{(3)} - z_3 p_{n-3}^{(3)} = 0.\end{aligned}$$

We further observe that, by (47),

$$\frac{p_{n-1}^{(2)} - z_1 p_{n-2}^{(2)}}{z_2} = -[\bar{z}_1\,p_{n-2}^{(2)} - p_{n-3}^{(2)} - z_1^{n-3}].$$

Hence

$$\begin{aligned}p_n^{(3)} = z_3\left[\frac{p_{n-1}^{(2)} - z_1 p_{n-2}^{(2)}}{z_2}\right] + z_3\,[p_{n-3}^{(3)} + (z_1\bar{z}_2 - \bar{z}_1)\,p_{n-2}^{(3)} - \bar{z}_2 p_{n-1}^{(3)}] &\\ + z_2 p_{n-2}^{(3)} + (z_1 - \bar{z}_1 z_2)\,p_{n-1}^{(3)}. \quad (52)\end{aligned}$$

This relation enables us to calculate the first few values of $p_n^{(3)}$. Since $p_n^{(3)}$ is of degree $n-2$ in z_3, we can write

$$p_n^{(3)} = z_3 p_n^{(3)}(1) + z_3^2 p_n^{(3)}(2) + \ldots + z_3^{n-2} p_n^{(3)}(n-2). \tag{53}$$

It is best to calculate separately the coefficients in (53). By (52) and (53), the formula for $p_n^{(3)}(1)$ is

$$p_n^{(3)}(1) = \frac{p_{n-1}^{(2)} - z_1 p_{n-2}^{(2)}}{z_2} + z_2 p_{n-2}^{(3)}(1) + (z_1 - \bar{z}_1 z_2) p_{n-1}^{(3)}(1).$$

The formula for $p_n^{(3)}(2)$ is

$$p_n^{(3)}(2) = p_{n-3}^{(3)}(1) + (z_1 \bar{z}_2 - \bar{z}_1) p_{n-2}^{(3)}(1) - \bar{z}_2 p_{n-1}^{(3)}(1)$$
$$+ z_2 p_{n-2}^{(3)}(2) + (z_1 - \bar{z}_1 z_2) p_{n-1}^{(3)}(2),$$

and so on.

The values of $p_n^{(3)}(1)$ for $n \leqslant 8$ are given in Table II, arranged in powers of z_2. The functions $p_n^{(3)}(2)$ are linear functions of $\bar{z}_2$. We write

$$p_n^{(3)}(2) = P_n^{(3)}(2) + \bar{z}_2 Q_n^{(3)}(2).$$

The values of $P_n^{(3)}(2)$, $Q_n^{(3)}(2)$ for $n \leqslant 8$ are given in Tables III, IV, arranged in powers of z_2. In Table V, the values of $p_n^{(3)}(3)$ are given, arranged in powers of z_2, $\bar{z}_2$. In Table VI the values of $p_n^{(3)}(4) + z_3 p_n^{(3)}(5) + z_3^2 p_n^{(3)}(6)$ are given, arranged in powers of $\bar{z}_2$, z_2, z_3.

The class P_4.

In P_3, the value of c_4 is $z_1^4 + (1-\rho_1^2) p_4^{(2)} + (1-\rho_1^2)(1-\rho_2^2) p_4^{(3)}$. We call this value $c_4^{(3)}$. Then by Tables I–IV, we have

$$c_4^{(3)} = z_1^4 + (1-\rho_1^2)[3 z_1^2 \bar{z}_2 + (1 - 3 z_1 \bar{z}_1) z_2^2 + \bar{z}_1^2 z_2^3]$$
$$+ (1-\rho_1^2)(1-\rho_2^2)[2 z_1 z_3 - 2 \bar{z}_1 z_2 z_3 - \bar{z}_2 z_3^2],$$

and the general value of c_4 is given by

$$c_4 = c_4^{(3)} + z_4 (1-\rho_1^2)(1-\rho_2^2)(1-\rho_3^2).$$

We shall find the recurrence formula for the class P_4. It will be seen that, given the recurrence formula for P_3, that for P_4 can be obtained without trouble. It will be manifest that this remark applies for $n > 4$.

To find the recurrence relation for the class P_4, we must evaluate the cofactors of the elements in the last row of D_4. We call these cofactors $G_0, \ldots, G_4$ respectively. Then

$$G_4 = D_3 = (1-\rho_1^2)^3 (1-\rho_2^2)^2 (1-\rho_3^2).$$

We give the same meaning to $\gamma_0, \ldots, \gamma_3$ as in the preceding analysis of P_3. Then we have the relations

$$c_4^{(3)}\gamma_3+c_3\gamma_2+c_2\gamma_1+c_1\gamma_0=0,$$

$$c_3\gamma_3+c_2\gamma_2+c_1\gamma_1+c_0\gamma_0=0,$$

$$c_2\gamma_3+c_1\gamma_2+c_0\gamma_1+c_{-1}\gamma_0=0,$$

$$c_1\gamma_3+c_0\gamma_2+c_{-1}\gamma_1+c_{-2}\gamma_0=0,$$

$$c_0\gamma_3+c_{-1}\gamma_2+c_{-2}\gamma_1+c_{-3}\gamma_0=D_3.$$

We use these relations to evaluate $G_0, \ldots, G_3$. We have

$$G_3=-\begin{vmatrix} 1 & c_1 & c_2 & c_4 \\ c_{-1} & 1 & c_1 & c_3 \\ c_{-2} & c_{-1} & 1 & c_2 \\ c_{-3} & c_{-2} & c_{-1} & 1 \end{vmatrix}.$$

Multiply the second row by γ_2/γ_3 and add to the first, the third row by γ_1/γ_3 and add to the first, the fourth row by γ_0/γ_3 and add to the first. We get

$$G_3=-(1-\rho_1^2)(1-\rho_2^2)(1-\rho_3^2)\begin{vmatrix} 1 & 0 & 0 & z_4 \\ c_{-1} & 1 & c_1 & c_3 \\ c_{-2} & c_{-1} & 1 & c_2 \\ c_{-3} & c_{-2} & c_{-1} & c_1 \end{vmatrix}$$

$$=(1-\rho_1^2)(1-\rho_2^2)(1-\rho_3^2)\,[\gamma_2-z_4\bar{\gamma}_0].$$

Similarly

$$G_2=(1-\rho_1^2)(1-\rho_2^2)(1-\rho_3^2)\,[\gamma_1-z_4\bar{\gamma}_1],$$

$$G_1=(1-\rho_1^2)(1-\rho_2^2)(1-\rho_3^2)\,[\gamma_0-z_4\bar{\gamma}_2],$$

$$G_0=-(1-\rho_1^2)(1-\rho_2^2)(1-\rho_3^2)\,z_4\gamma_3.$$

The required recurrence formula is therefore

$$c_n+(z_3\bar{z}_2-z_1+\bar{z}_1z_2+z_4\bar{z}_3)\,c_{n-1}$$

$$+(z_3\bar{z}_1-z_3z_1\bar{z}_2-z_2-z_4\bar{z}_3z_1+z_4\bar{z}_3\bar{z}_1z_2+z_4\bar{z}_2)\,c_{n-2}$$

$$+(\bar{z}_1z_4-z_3-z_4\bar{z}_3z_2-z_4z_1\bar{z}_2)\,c_{n-3}-z_4c_{n-4}=0 \quad (n>5).$$

TABLE I.

$p_n^{(2)}(z_1, \bar{z}_1, z_2)$. (In this table, $x = z_1$, $y = \bar{z}_1$.).

n	z_2	z_2^2	z_2^3	z_2^4	z_2^5	z_2^6	z_2^7	z_2^8	z_2^9
2	1								
3	$2x$	$-y$							
4	$3x^2$	$1-3xy$	y^2						
5	$4x^3$	$3x-6x^2y$	$-2y+4xy^2$	$-y^3$					
6	$5x^4$	$6x^2-10x^3y$	$1-8xy+10x^2y^2$	$3y^2-5xy^3$	y^4				
7	$6x^5$	$10x^3-15x^4y$	$4x-20x^2y+20x^3y^2$	$-3y+15xy^2-15x^2y^3$	$-4y^3+6xy^4$	$-y^5$			
8	$7x^6$	$15x^4-21x^5y$	$10x^2-40x^3y+35x^4y^2$	$1-15xy+45x^2y^2-35x^3y^3$	$6y^2-24xy^3+21x^2y^4$	$5y^4-7xy^5$	y^6		
9	$8x^7$	$21x^5-28x^6y$	$20x^3-70x^4y+56x^5y^2$	$5x-45x^2y+105x^3y^2-70x^4y^3$	$-4y+36xy^2-84x^2y^3+56x^3y^4$	$-10y^3+35xy^4-28x^2y^5$	$-6y^5+8xy^6$	$-y^7$	
10	$9x^8$	$28x^6-36x^7y$	$35x^4-112x^5y+84x^6y^2$	$15x^2-105x^3y+210x^4y^2-126x^5y^3$	$1-24xy+126x^2y^2-224x^3y^3+126x^4y^4$	$10y^2-70xy^3+140x^2y^4-84x^3y^5$	$15y^4-48xy^5+36x^2y^6$	$7y^6-9xy^7$	y^8

TABLE II.

$p_n^{(3)}(1)$.

n	z_2^0	z_2^1	z_2^2	z_2^3	z_2^4	z_2^5
3	1					
4	$2z_1$	$-2\bar{z}_1$				
5	$3z_1^2$	$2-6z_1\bar{z}_1$	$3\bar{z}_1^2$			
6	$4z_1^3$	$6z_1-12z_1^2\bar{z}_1$	$-6\bar{z}_1+12z_1\bar{z}_1^3$	$-4\bar{z}_1^3$		
7	$5z_1^4$	$12z_1^2-20z_1^3\bar{z}_1$	$3-24z_1\bar{z}_1+30z_1^2\bar{z}_1^2$	$12\bar{z}_1^2-20z_1\bar{z}_1^3$	$5\bar{z}_1^4$	
8	$6z_1^5$	$20z_1^3-30z_1^4\bar{z}_1$	$12z_1-60z_1^2\bar{z}_1+60z_1^3\bar{z}_1^2$	$-12\bar{z}_1+60z_1\bar{z}_1^2-60z_1^2\bar{z}_1^3$	$-20z_1^3+30z_1\bar{z}_1^4$	$-6\bar{z}_1^5$

TABLE III.

$P_n^{(3)}(2)$.

n	z_2^0	z_2^1	z_2^2	z_2^3
4	0			
5	$-\bar{z}_1$			
6	$1-3z_1\bar{z}_1$	$3\bar{z}_1^2$		
7	$3z_1-6z_1^2\bar{z}_1$	$-6\bar{z}_1+12z_1\bar{z}_1^2$	$-6\bar{z}_1^3$	
8	$6z_1^2-10z_1^3\bar{z}_1$	$3-24z_1\bar{z}_1+30z_1^2\bar{z}_1^2$	$18\bar{z}_1^2-30z_1\bar{z}_1^3$	$10\bar{z}_1^4$

TABLE IV.

$Q_n^{(3)}(2)$.

n	z_2^0	z_2^1	z_2^2	z_2^3	z_2^4
4	-1				
5	$-2z_1$	$3\bar{z}_1$			
6	$-3z_1^2$	$-3+9z_1\bar{z}_1$	$-6\bar{z}_1^2$		
7	$-4z_1^3$	$-9z_1+18z_1^2\bar{z}_1$	$12\bar{z}_1-24z_1\bar{z}_1^2$	$10\bar{z}_1^3$	
8	$-5z_1^4$	$-18z_1^2+30z_1^3\bar{z}_1$	$-6+48z_1\bar{z}_1-60z_1^2\bar{z}_1^2$	$-30z_1^2+50z_1\bar{z}_1^3$	$-15\bar{z}_1^4$

TABLE V.

$p_n^{(3)}(3)$.

n	z_2^0	z_2^1	$\bar{z}_2$	$z_2\bar{z}_2$	$z_2^2\bar{z}_2$	$\bar{z}_2^2$	$z_2\bar{z}_2^2$	$z_2^2\bar{z}_2^2$	$z_2^3\bar{z}_2^2$
5						1			
6			$2\bar{z}_1$			$2z_1$	$-4\bar{z}_1$		
7	$\bar{z}_1^2$		$-2+6z_1\bar{z}_1$	$-8\bar{z}_1^2$		$3\bar{z}_1^2$	$4-12z_1\bar{z}_1$	$10\bar{z}_1^2$	
8	$-2\bar{z}_1+4z_1\bar{z}_1^2$	$-4\bar{z}_1^3$	$-6z_1+12z_1^2\bar{z}_1$	$16\bar{z}_1-32z_1\bar{z}_1^2$	$20\bar{z}_1^3$	$4z_1^3$	$12z_1-24z_1^2\bar{z}_1$	$-20\bar{z}_1+40z_1\bar{z}_1^2$	$-20\bar{z}_1^3$

TABLE VI.

$p_n^{(3)}(4)+z_3p_n^{(3)}(5)+z_3^2p_n^{(3)}(6)$.

n	$\bar{z}_2$	$\bar{z}_2^2$	$z_2\bar{z}_2^2$	$\bar{z}_2^3$	$z_2\bar{z}_2^3$	$z_2^2\bar{z}_2^3$	$z_3\bar{z}_2^3$	$z_3\bar{z}_2^4$	$z_3z_2\bar{z}_2^4$	$z_3^2\bar{z}_2^5$
6				-1						
7		$-3\bar{z}_1$		$-2z_1$	$5\bar{z}_1$			1		
8	$-3\bar{z}_1^2$	$3-9z_1\bar{z}_1$	$15\bar{z}_1^2$	$-3z^2$	$-5+15z_1\bar{z}_1$	$-15\bar{z}_1^2$	$4\bar{z}_1$	$2z_1$	$-6\bar{z}_1$	-1

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ON FUNCTIONS WHICH ARE REGULAR AND OF FINITE ORDER IN AN ANGLE

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Introduction.

1.1. Suppose that $f(z)$ is regular and of finite positive order ρ in the angle $\alpha \leqslant \arg z \leqslant \beta$ for $|z| \geqslant l > 0$. Let $\rho(r)$ be the Lindelöf proximate order* of any integral function of order ρ, and let

$$h(\theta) = \overline{\lim_{r\to\infty}} \frac{\log |f(re^{i\theta})|}{r^{\rho(r)}}.$$

The object of this paper is to deduce something about the zeros, and the behaviour of $f(z)$ in general, from the form of $h(\theta)$ when

$$h(\theta) = A \cos\theta\rho + B \sin\theta\rho, \tag{1.11}$$

where A and B are finite. When $h(\theta)$ satisfies (1.11) for $\alpha \leqslant \theta \leqslant \beta$, it is said to be a sinusoid in that interval. This form of $h(\theta)$ is a very special one; but it is, in a sense, fundamental. For Phragmén and Lindelöf† have proved the following theorem.

THEOREM A. *If $h(\theta)$ satisfies (1.11) for $\theta = \alpha$ and $\theta = \beta$, where*

$$0 < \beta - \alpha < \pi/\rho,$$

then

$$h(\theta) \leqslant A \cos\theta\rho + B \sin\theta\rho \tag{1.12}$$

for $\alpha \leqslant \theta \leqslant \beta$.

From this theorem Phragmén and Lindelöf have deduced many other results which further emphasize the importance of the sinusoid form of $h(\theta)$.

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* See G. Valiron, *Lectures on the general theory of integral functions* (1923), 64, and § 2.4 of this paper.

† E. Phragmén and E. Lindelöf, *Acta Math.*, 31 (1908), 381–406.

In a former paper* I showed that (1.12) could be improved if $f(z)$ had a sufficient number of zeros inside the angle; the following theorem is typical of my results.

THEOREM B. *Suppose that $h(\theta)$ satisfies the hypotheses of Theorem* A, *and that the zeros of $f(z)$ include a sequence of numbers,*

$$z_1, z_2, \ldots, z_n = r_n e^{i\theta_n}, \ldots,$$

where $1 \leqslant r_1 \leqslant r_2 \leqslant \ldots$, and $\lim_{n\to\infty} \theta_n = \frac{1}{2}(\alpha+\beta)$. Suppose further that, as $r \to \infty$,

$$n(r) \sim \lambda r^{\rho(r)},$$

where $n(r)$ is the number of numbers $r_n \leqslant r$. Then

$$(1.13) \quad h(\theta) \leqslant A\cos\theta\rho + B\sin\theta\rho - \frac{\pi\lambda \sin\{\frac{1}{2}(\beta-\alpha) - |\theta - \frac{1}{2}(\alpha+\beta)|\}\rho}{\cos\frac{1}{2}(\beta-\alpha)\rho}$$

for $\alpha \leqslant \theta \leqslant \beta$.

In the light of this result we may hope to show that, if $h(\theta)$ is a sinusoid for $\alpha \leqslant \theta \leqslant \beta$, then $f(z)$ has comparatively few zeros inside the angle, at any rate in some intervals of $|z|$; and this we shall proceed to do. First, however, it is as well to mention some of the classes of integral functions for which $h(\theta)$ is of the sinusoid form in some angle, or angles. These include exponential polynomials, Bessel functions, $1/\Gamma(z+1)$, the *fonctions orientées* of Valiron†, Mittag-Leffler's function, $E_a(z) = \Sigma z^n/\Gamma(1+an)$, and functions of finite order $\rho > 1$ whose directions of Borel lie in an angle not greater than π/ρ. Also, if $f(z) = \Sigma a_n z^n$ is an integral function of order 1 and mean type such that $h(\theta)$ is a sinusoid for $\alpha \leqslant \theta \leqslant \beta$, then the Borel polygon of $\phi(z) = \Sigma n! a_n z^n$ has no vertex for which $\alpha < \arg z < \beta$, and conversely‡. The chief types of function for which $h(\theta)$ is not a sinusoid in any angle are functions whose zeros are very irregularly distributed, and "ring functions", *i.e.* functions whose zeros lie in rings round the origin, such as§

$$\sum_{n=1}^{\infty} \frac{z^{n^3}}{\Gamma(1+n^3)}.$$

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* M. L. Cartwright, *Proc. London Math. Soc.* (2), 32 (1930), 51–71. Compare V. Bernstein, *Rendiconti d. R. Accad. d. Lincei, Sc. fis. mat. e nat.*, 15 (1932), 30. In Bernstein's theorem the points are not zeros, but $|f(z)|$ is so small that they have the same effect as if they were zeros.

† G. Valiron, *Annales de la Faculté des Sciences, Toulouse* (3), 5 (1913), 117–257 (230).

‡ See E. Borel, *Leçons sur les séries divergentes* (Paris, 1928), 171.

§ See G. H. Hardy, *Proc. London Math. Soc.* (2), 2 (1905), 332–339.

1.2. Theorem A shows that $h(\theta)$ cannot increase very rapidly with θ inside an angle less than π/ρ, not much more rapidly than the $h(\theta)$ formed from the exponential, $\exp(A-iB)z^\rho$, say, which has the same $h(\theta)$ on the boundaries of the angle. It is natural to suppose that this is because $f(z)$ behaves like $\exp\{(A-iB)z^\rho\}$ in those regions where $|f(z)|$ is large. Theorem B supports this conjecture, and I now show that it is in fact true. We may express the result briefly by saying that, *if* $h(\theta)$ *satisfies* (1.11) *for* $\alpha \leqslant \theta \leqslant \beta$, *then, given any* $\delta > 0$, *we have*

$$\log|f(re^{i\theta})| \sim (A\cos\theta\rho + B\sin\theta\rho)\, r^{\rho(r)} \tag{1.21}$$

for $\alpha+\delta \leqslant \theta \leqslant \beta+\delta$, *as* $r\to\infty$ *in a set of upper linear density* 1*. The number of zeros of $f(z)$ is, of course, small in those regions where (1.21) holds. If $\beta-\alpha < \pi/\rho$, there may be a set of values of r for which (1.21) does not hold; for these values of r, $|f(re^{i\theta})|$ is comparatively small†. But if, as in Theorem II, $\beta-\alpha > \pi/\rho$, then (1.21) holds for all r except a set of density 0. Between these two cases we have Theorem III in which $\beta-\alpha = \pi/\rho$, and we use an extra condition to obtain a result as precise as that for $\beta-\alpha > \pi/\rho$.

1.3. From these theorems we deduce a few results for three special types of integral function. The first (see Theorem VI) are like Valiron's *fonctions orientées*, and have nearly all their zeros in one direction. The second (see Theorem VII) are like Mittag-Leffler's function $E_\alpha(z)$, and are closely related to functions whose directions of Borel all lie in an angle less than π/ρ. The third type of function is the origin of the whole paper. For Prof. Titchmarsh has suggested that the methods which he used for‡

$$f(z) = \int_a^b e^{zt}\psi(t)\,dt \tag{1.31}$$

might be applied to any function of order 1 and mean type which merely has the same $h(\theta)$ as (1.31), viz.

$$h(\theta) = \max(a\cos\theta;\ b\cos\theta). \tag{1.32}$$

This is not so, and I show here that his results do not hold for some integral functions satisfying (1.32). If, however, we use a slight

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* See § 3.1, Theorem I. The terms upper and lower density have been defined by Besicovitch, *Math. Annalen*, 97 (1927), 677–695, but it is preferable to regard the above statement as a contracted form of the more precise statement given in Theorem I.

† See § 4.1.

‡ See E. C. Titchmarsh, *Proc. London Math. Soc.* (2), 25 (1926), 283–302.

additional hypothesis, we can obtain all his results* from one of the general theorems for $h(\theta)$.

A statement of some of the results of this paper has appeared in the *Comptes rendus*†, but Theorem III and the various theorems depending on it are new. I should like to express my thanks to Prof. Titchmarsh for the suggestion which has been so fruitful, and to Dr. E. F. Collingwood for much valuable advice.

1.4. I suppose throughout that $f(z)$ is regular in and on the boundary of the angle considered for $|z| \geqslant l > 0$, and that

$$\log|f(re^{i\theta})| < r^{\rho+\epsilon} \tag{1.41}$$

uniformly in that angle for every $\epsilon > 0$ and every $r > r_0(\epsilon)$. We denote by $n(r, \alpha, \beta)$ the number of zeros of $f(z)$ for which $l < |z| \leqslant r$, $\alpha \leqslant \arg z \leqslant \beta$, and we write $n(r) = n(r, 0, 2\pi)$ unless otherwise stated. We also denote by $z_1, z_2, \ldots, z_n = r_n e^{i\theta_n}$, where $l < r_1 \leqslant r_2 \leqslant \ldots$, the zeros of $f(z)$ lying in the angle of the theorem unless otherwise stated. If, as in Theorems VI–X, $f(z)$ is an integral function, then the sequence includes all the zeros of $f(z)$ in their order of multiplicity except perhaps for one at the origin. As usual, $\{u\}^+ = u$ if $u > 0$ and 0 otherwise. If $\rho(r) \equiv \rho$, and $C = \max_{\alpha \leqslant \theta \leqslant \beta} h(\theta)$, where C is finite and positive, then $f(z)$ is said to be of *order* ρ *and mean type* for $\alpha \leqslant \arg z \leqslant \beta$. The symbols $\mathfrak{R}(z)$ and $\mathfrak{I}(z)$ are used to denote the real and imaginary parts respectively of z; and ϵ, δ, η denote positive numbers which are always less than 1. The letter H is used to denote a number independent of z, R, r, N, but depending sometimes on a parameter δ, η, or k which will be indicated. It is not necessarily the same in each place, but H' is different from H'', and so on.

Preliminary lemmas.

2.1. We shall make considerable use of the following lemmas.

LEMMA 1. *Suppose that* $F(z)$ *is regular for* $|z| \leqslant R$, $F(0) = 1$, *and that* $M(r) = \max_{0 \leqslant \theta \leqslant 2\pi} |F(re^{i\theta})|$. *Then* (1)

$$n(r) \leqslant \frac{\log M(R)}{\log (R/r)},$$

* Titchmarsh's method will give the results when the extra condition is used, but it requires several adjustments. It is quite different from the method used here.

† M. L. Cartwright, *Comptes rendus*, 194 (1932), 1718.

where $n(r)$ is the number of zeros of $F(z)$ for which $|z| \leqslant r$; (2) given any $\eta > 0$, $k > 1$, we can choose $H = H(\eta, k)$ independently of z and R so that

$$\log|F(z)| > -H \log M(R)$$

for $0 \leqslant |z| \leqslant R/k$, except for values of z which can be enclosed in circles the sum of whose radii is at most ηR.

The first part is a well-known deduction from Jensen's formula. The second part, which is more important for our purposes here, is a sharpened form of Valiron's theorem*, and was obtained by V. Bernstein† by means of a theorem of H. Cartan‡.

2.2. Jensen's formula is too well known to need stating here, but we also require the corresponding result for a sector.

LEMMA 2. *Suppose that $F(z)$ is regular for $|\arg z| \leqslant \frac{1}{2}\pi/\rho$, $l \leqslant |z| \leqslant R$, and let $z_1, z_2, \ldots, z_n = r_n e^{i\theta_n}, \ldots$ be the zeros of $F(z)$ for which $|\theta_n| < \frac{1}{2}\pi/\rho$, $l \leqslant r_n < R$. Then*

$$\sum_{l \leqslant r_n < R} \frac{\cos \rho\theta_n}{r_n^{\rho}} \left(1 - \frac{r_n^{2\rho}}{R^{2\rho}}\right) = \frac{1}{\pi R^{\rho}} \int_{-\frac{1}{2}\pi/\rho}^{\frac{1}{2}\pi/\rho} \log|F(re^{i\theta})| \cos\theta\rho \, d\theta$$

$$+ \frac{1}{2\pi} \int_l^R \{\log|F(ue^{i\frac{1}{2}\pi/\rho})| + \log|F(ue^{-i\frac{1}{2}\pi/\rho})|\} \left(1 - \frac{u^{2\rho}}{R^{2\rho}}\right) \frac{du}{u^{\rho+1}} + \psi(R),$$

where $\psi(R)$ is bounded as $R \to \infty$.

This is a generalization of Carleman's formula, and may be obtained in the same way§. The next lemma‖ is the result of combining the previous formula with the corresponding formulae for other sectors.

LEMMA 3. *Suppose that $F(z)$ is regular for $|z| \leqslant R$, and that $F(0) = 1$. Let $z_1, z_2, \ldots, z_n = r_n e^{i\theta_n}$ be the zeros of $F(z)$ for which $r_n \leqslant R$. Then, for every positive integer m, we have*

$$\left[\frac{d^m \log F(z)}{dz^m}\right]_{z=0} + \frac{1}{m} \sum_{r_n < R} \frac{e^{-mi\theta_n}}{r_n^m} \left(1 - \frac{r_n^{2m}}{R^{2m}}\right) = \frac{1}{\pi R^m} \int_{-\pi}^{\pi} \log|F(Re^{i\theta})| e^{-mi\theta} \, d\theta.$$

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* See G. Valiron, *Lectures on the general theory of integral functions* (1923), 89.

† V. Bernstein, *Annali di Mat.* (4), 12 (1934), 179–181. In the theorems which I stated in the *Comptes rendus*, I used a similar, but weaker, result obtained without Cartan's theorem.

‡ H. Cartan, *Annales Sci. de l'Ecole Normale Sup.* (3), 5 (1928), 273–276.

§ See G. Pólya and G. Szegö, *Aufgaben und Lehrsätze*, 1 (1925), 120, Abschn. III, No. 178.

‖ See M. L. Cartwright, *Proc. London Math. Soc.* (2), 33 (1931), 209–224 (220), Lemma C. The reference to Pólya and Szegö on page 220 should be to 120, Abschn. III, No. 178, as above, and not to page 310.

2.3. In virtue of the definition of Lindelöf proximate order*, we know that $\rho(r)$ is a positive continuous function, defined for $r > l > 0$, and differentiable in adjacent intervals, such that

$$\lim_{r\to\infty} \rho(r) = \rho, \quad \lim_{r\to\infty} r\rho'(r)\log r = 0.$$

It is also known that $\rho(r)$ is differentiable on the right and on the left at every point $r > l$, and that $\lim r\rho'(r)\log r = 0$ when $\rho'(r)$ is interpreted as either the right-hand or the left-hand derivative. Inside the intervals in which $\rho(r)$ is differentiable it has a continuous differential coefficient. *All the general results proved here for $\rho(r)$ hold for any $\rho(r)$ satisfying the above conditions.* Of course, if $f(z)$ is an integral function of proximate order $\rho(r)$, we have, as well,

$$\overline{\lim_{r\to\infty}} \frac{\log M(r)}{r^{\rho(r)}} = 1, \quad \text{where } M(r) = \max_{0\leqslant\theta\leqslant 2\pi} |f(re^{i\theta})|.$$

If $\rho(r) \neq \rho$, we need a function corresponding to $\exp(z^\rho)$ in order to be able to apply the methods of Phragmén and Lindelöf to $h(\theta)$.

LEMMA 4. *If $\rho(r)$ satisfies the conditions stated above, then there is a function $V(z)$ regular for $|z| \geqslant r_0 \geqslant l$, $|\arg z| \leqslant \alpha$, and real and positive for real positive values of z, such that, as $r\to\infty$,*

$$V(re^{i\theta}) \sim r^{\rho(r)}e^{i\theta\rho}$$

uniformly for $|\theta| \leqslant \alpha$.

This result is substantially due to Valiron. We choose a positive number $m < \pi/\alpha$ so that $m\rho$ is not an integer. Let $P(z)$ be a canonical product whose zeros all lie on the negative real axis such that

$$n(r) \sim r^{m\rho(r)} \frac{\sin \pi m\rho}{\pi},$$

where $n(r)$ is the number of zeros of $P(z)$ for which $|z| \leqslant r$.

Valiron† has shown that, given any $\delta > 0$, we have

$$\log P(re^{i\theta}) \sim r^{m\rho(r)}e^{im\rho}$$

for $|\theta| \leqslant \pi - \delta$. Putting $V(z) = \log P(z^{1/m})$, we have the required result.

2.4. Besides Theorem A we shall use various properties of $h(\theta)$ which have been proved by Phragmén and Lindelöf‡. We assume, as we said in §1.4, that (1.41) is satisfied uniformly for the values of θ concerned.

* See G. Valiron, *Lectures on the general theory of integral functions* (1923), 64.

† See G. Valiron, *Ann. de la Faculté des Sciences, Toulouse* (3), 5 (1913), 230.

‡ E. Phragmén and E. Lindelöf, *Acta Math.*, 31 (1908), 381–406.

LEMMA 5. *If $h(\theta) \leqslant C$ for $\alpha \leqslant \theta \leqslant \beta$, then, given any $\epsilon > 0$, we have*

$$\log|f(re^{i\theta})| < (C+\epsilon)\, r^{\rho(r)}$$

for $\alpha \leqslant \theta \leqslant \beta$, provided that $r > r_0(\epsilon)$.

LEMMA 6. *If $h(\theta)$ is finite for $\alpha \leqslant \theta \leqslant \beta$, then $h(\theta)$ is continuous.*

From these two it follows that

LEMMA 7. *If $h(\theta)$ is finite for $\alpha \leqslant \theta \leqslant \beta$, then, given any $\epsilon > 0$, we have*

$$\log|f(re^{i\theta})| < \{h(\theta)+\epsilon\}\, r^{\rho(r)}$$

for $\alpha \leqslant \theta \leqslant \beta$, $r > r_0(\epsilon)$.

General theorems for $h(\theta) = A\cos\theta\rho + B\sin\theta\rho$.

3.1. The first theorem, or rather the method used to prove it, is the foundation of all the subsequent work.

THEOREM I. *If (1.11) is satisfied for $|\theta| \leqslant \alpha$, then*

$$\lim_{r\to\infty} \{n(r, -\alpha+\delta, \alpha-\delta) - n(\eta r, -\alpha+\delta, \alpha-\delta)\}/r^{\rho(r)} = 0 \tag{3.11}$$

for every $\delta > 0$ and every $\eta > 0$. Also, given any $\epsilon > 0$, $\delta > 0$, $\eta > 0$, $\zeta > 0$, we can choose $R_1, R_2, \ldots, R_m \to \infty$ so that

$$\log|f(re^{i\theta})| > (A\cos\theta\rho + B\sin\theta\rho - \epsilon)\, r^{\rho(r)} \tag{3.12}$$

for $|\theta| \leqslant \alpha-\delta$, $\eta R_m \leqslant r \leqslant R_m$, except for values of $re^{i\theta}$ which can be enclosed in circles the sum of whose radii is less than ζR_m.

It follows from Lemma 7 that, given any $\epsilon > 0$, we have

$$\log|f(re^{i\theta})| < (A\cos\theta\rho + B\sin\theta\rho + \epsilon)\, r^{\rho(r)}, \tag{3.13}$$

for $|\theta| \leqslant \alpha$, $r > r_0(\epsilon)$. It is sometimes convenient to combine the results (3.12) and (3.13) and say, briefly, that

$$\log|f(re^{i\theta})| \sim (A\cos\theta\rho + B\sin\theta\rho)\, r^{\rho(r)} \tag{3.131}$$

for $|\theta| \leqslant \alpha-\delta$ in *a set of r of upper linear density* 1. This expression will be used here simply as a contracted form of the two statements made above. It should be observed that the size of the angle α is not in any way restricted.

Proof. Suppose, first, that $\rho(r) = \rho$ and $\alpha = \frac{1}{2}\pi$. Let

$$F(z) = f(z)\, e^{-(A-iB)z^\rho}.$$

In virtue of (1.11), given any $\epsilon_1 > 0$, we can choose a sequence of real positive numbers $X_1, X_2, \ldots, X_m \to \infty$, such that

$$(3.14) \quad \log|F(X_m)| = \log|f(X_m)\exp\{-(A-iB)X_m^{\rho}\}| > -\epsilon_1 X_m^{\rho}.$$

But it follows from (3.13) that

$$(3.15) \quad \log|F(re^{i\theta})| < \epsilon_1 r^{\rho}$$

for $|\theta| \leqslant \alpha$, $r > r_0(\epsilon_1)$. Hence, applying Lemma 1 to $F(z)/F(X_m)$ in the circle $|z-X_m| \leqslant X_m - r_0$, we have

$$(3.16) \quad n(|z-X_m| = X_m/k) < \epsilon_1 H X_m^{\rho},$$

where $H = H(k)$, and $n(|z-X_m| = X_m/k)$ denotes the number of zeros of $f(z)$ for which $|z-X_m| \leqslant X_m/k$, for every $k > 1$, provided that X_m is sufficiently large.

The first result required, viz. (3.11), follows almost immediately from (3.16). For, given any $\delta > 0$, $\eta > 0$, we can choose k so near 1 that the circle $|z-X_m| \leqslant X_m/k$ includes the whole region $|\arg z| \leqslant \frac{1}{2}\pi - \delta$, $\eta X_m \sin\delta \leqslant |z| \leqslant X_m \sin\delta$, provided that $m > m_0(k) = m_0(\eta, \delta)$. Hence, putting $R_m = X_{m+m_0}\sin\delta$, we have

$$n(R_m, -\tfrac{1}{2}\pi+\delta, \tfrac{1}{2}\pi-\delta) - n(\eta R_m, -\tfrac{1}{2}\pi+\delta, \tfrac{1}{2}\pi-\delta)$$
$$\leqslant n(|z-X_{m+m_0}| = X_{m+m_0}/k) < \epsilon_1 H R_m^{\rho}(\operatorname{cosec}\delta)^{\rho}.$$

Now, given any $\eta > 0$, $\delta > 0$, $\epsilon > 0$, we can choose ϵ_1 so that

$$\epsilon_1 H(\operatorname{cosec}\delta)^{\rho} < \epsilon,$$

and we have

$$n(R_m, -\tfrac{1}{2}\pi+\delta, \tfrac{1}{2}\pi-\delta) - n(\eta R_m, -\tfrac{1}{2}\pi+\delta, \tfrac{1}{2}\pi-\delta) < \epsilon R_m^{\rho}$$

for a sequence $R_1, R_2, \ldots, R_m \to \infty$, which is the required result.

The second part of Lemma 1 applied to $F(z)/F(X_m)$ shows that given any $k > 1$, $\zeta_1 > 0$, we have

$$(3.17) \quad \log|F(z)| > -\epsilon_1 H X_m^{\rho}$$

for $|z-X_m| \leqslant X_m/k$, except in circles the sum of whose radii is less than $\zeta_1 X_m$, provided that $X_m > X_0(k_1, \zeta_1)$. Choosing k so near 1 that the circle $|z-X_{m+m_0}| \leqslant X_{m+m_0}/k$ covers the region

$$|\arg z| \leqslant \tfrac{1}{2}\pi - \delta, \ \eta R_m \leqslant |z| \leqslant R_m = X_{m+m_0}\sin\delta,$$

and choosing $\zeta_1 < \zeta \operatorname{cosec}\delta$, we have the second part of the result. If $\alpha \neq \frac{1}{2}\pi$, we put $z = w^{2\alpha/\pi}$, apply the result just obtained, and then transform

back again. If $\rho(r) \neq \rho$, we put

$$F(z) = f(z) \exp\{-(A-iB)V(z)\},$$

where $V(z)$ is the function described in Lemma 4, and use the same method.

3.2. The upper limit of $n(r, -\alpha+\delta, \alpha-\delta)/r^{\rho(r)}$, and the behaviour of $|f(z)|$ in general, depend very much on the size of $|f(z)|$ on the arms of the angle. The two theorems which follow depend on the hypothesis that $|f(z)|$, or the modulus of $f(z)$ multiplied by some exponential, is small on the arms of the angle compared with its value inside the angle. If the angle of the theorem is equal to π/ρ_1, where $\rho_1 > \rho$, we can find functions of order ρ which tend to 0 like $\exp(-r^{\rho_2})$, where $\rho < \rho_2 < \rho_1$, through a certain sequence of regions inside the angle. In these regions $f(z)$ may have a number of zeros approximately equal to r^{ρ_2}, and so

$$\overline{\lim_{r\to\infty}} \frac{n(r, \alpha+\delta, \beta-\delta)}{r^{\rho}} = \infty.$$

If the angle of the theorem is greater than π/ρ, then $f(z)$ cannot tend to 0 so rapidly in any sequence of regions unless it is identically equal to 0, a possibility which is obviously excluded by (1.11).

THEOREM II. *If* (1.11) *is satisfied for* $|\theta| \leqslant \alpha$, *where* $2\alpha > \pi/\rho$, *then*

$$\lim_{r\to\infty} n(r, -\alpha+\delta, \alpha-\delta)/r^{\rho(r)} = 0 \tag{3.21}$$

for every $\delta > 0$. *Also, given any* $\epsilon > 0$, $\delta > 0$, $\eta > 0$, $\zeta > 0$, *we can choose* R_0 *so that* (3.12) *is satisfied for* $|\theta| \leqslant \alpha-\delta$, $\eta R \leqslant r \leqslant R$, $R > R_0$, *except for values of* $re^{i\theta}$ *lying in circles, the sum of whose radii is less than* ζR.

Again (3.13) holds for $|\theta| \leqslant \alpha$, $r > r_0(\epsilon)$, and we say briefly that (3.131) holds in *a set of r of linear density* 1.

Proof. Suppose for convenience that $\rho(r) = \rho$, and let β be a number such that $0 < 2\beta < \min\{\frac{1}{2}\pi/\rho;\ \alpha-\frac{1}{2}\pi/\rho\}$. Consider*

$$F(z) = f(z) \exp\{-(A-iB)z^{\rho} + \xi z^{\rho} - \xi' r_0^{\rho}\},$$

where $0 < \xi < \xi'$. Then

$$|F(re^{i\theta})| < 1$$

* The following method is similar to that used by H. Milloux, *Acta Math.*, 52 (1928), 243.

for $\theta = \pm(\frac{1}{2}\pi/\rho+\beta)$, $r \geqslant r_0$, and also for $r = r_0$, $|\theta| \leqslant \frac{1}{2}\pi/\rho+\beta$, provided that $r_0 = r_0(\xi, \xi'-\xi)$ is sufficiently large. But, by our hypothesis, for every $\epsilon > 0$, there is a sequence of values of r tending to infinity such that

$$\log|f(r)| > (A-\epsilon)r^\rho;$$

and so, choosing $\epsilon < \xi$, we have a sequence of values of r for which

$$|F(r)| > 1.$$

Hence, by the maximum modulus principle, there is a point on every arc. $|z| = r$, $|\arg z| \leqslant \frac{1}{2}\pi/\rho+\beta$, at which $|F(z)| > 1$, provided that r is sufficiently large. But we can choose ξ as small as we please; and so, given any $\epsilon_1 > 0$, we can choose R_0' so that

$$\log|f(re^{i\theta})| > (A\cos\theta\rho + B\sin\theta\rho - \epsilon_1)r^\rho \tag{3.22}$$

for some $\theta = \theta(r)$ such that $|\theta(r)| \leqslant \frac{1}{2}\pi/\rho+\beta$, and every $r > R_0'$.

Suppose, first, that there is a number $K > 1$, and a sequence $R_1, R_2, \ldots, R_m < KR_{m-1}$, $(R_m \to \infty)$ such that (3.22) holds for $r = R_m$, $\theta = 0$. Suppose also that $\alpha = \frac{1}{2}\pi$, and apply Lemma 1 to $F_1(z)/F_1(R_m)$, where $F_1(z) = f(z)\exp\{-(A-iB)z^\rho\}$ in the circle $|z-R_m| \leqslant R_m - r_0$. By a suitable choice of k depending on K, δ, and η, we can cover the region $|\arg z| \leqslant \frac{1}{2}\pi-\delta$, $\eta R \leqslant |z| \leqslant R$ by one of the circles $|z-R_m| \leqslant R_m/k$, and the second part of the result follows as in Theorem I. We also have

$$n(r, -\alpha+\delta, \alpha-\delta) - n(\eta r, -\alpha+\delta, \alpha-\delta) < \epsilon_1 H r^\rho \tag{3.23}$$

for every $\eta > 0$, $\delta > 0$, provided that $r > r_0(\epsilon_1, \delta, \eta)$. Suppose that $R_0 \geqslant r_0(\epsilon_1, \delta, \frac{1}{2})$, and that $2^{m-1}R_0 \leqslant r \leqslant 2^m R_0$, where m is a positive integer. Then

$$\begin{aligned} n(r, -\alpha+\delta, \alpha-\delta) &< n(2^m R_0, -\alpha+\delta, \alpha-\delta) \\ &< n(2^m R_0, -\alpha+\delta, \alpha-\delta) - n(R_0, -\alpha+\delta, \alpha-\delta) + H', \end{aligned}$$

where H' is independent of m. Putting $r = R_0, 2R_0, \ldots, 2^m R_0$ in (3.23) with $\eta = \frac{1}{2}$ and adding, we have

$$\begin{aligned} n(r, -\alpha+\delta, \alpha-\delta) &< \epsilon_1 H R_0^\rho (1+2^\rho+\ldots+2^{m\rho}) + H' \\ &< \frac{\epsilon_1 H R_0^\rho 2^{(m+1)\rho}}{2^\rho - 1} + H' < \epsilon r^\rho \end{aligned}$$

for every $r > r_0(\epsilon)$, which is the required result.

If $\alpha \neq \frac{1}{2}\pi$, we put $z = w^{2\alpha/\pi}$ before applying Lemma 1, and then transform back again after applying it.

Even if there is no sequence of the kind just considered, we may suppose that $\beta < \frac{1}{2}\pi$ and we can apply Lemma 1 to $F_1(z)/F_1(re^{i\theta(r)})$ with $k=2$, $\zeta = \frac{1}{4}\sin\beta$, in the circle $|z-re^{i\theta(r)}| \leqslant r\sin\beta$; and we obtain a new point $re^{i\theta_1(r)}$ such that $|\theta_1(r)| \leqslant |\theta(r)| - \frac{1}{4}\beta < \frac{1}{2}\pi/\rho + \frac{3}{4}\beta$, at which

$$\log|f(re^{i\theta})| > (A\cos\theta\rho + B\sin\theta\rho - \epsilon_1 H)r^\rho. \tag{3.24}$$

Repeating the process $[2\pi/(\rho\beta)+4]$ times, we get a point $z = R_m$, such that $r(1-\frac{1}{4}\sin\beta) < R_m < r(1+\frac{1}{4}\sin\beta)$, at which (3.24) holds, and H depends only on β. Hence we may take $K = \frac{1}{2}\sin\beta$, and the argument given above now yields the required results.

3.3. We now come to the more difficult theorems for the critical angle π/ρ; and we consider first a very precise result for functions of order ρ and mean type.

THEOREM III. *Suppose that $f(z)$ is of order ρ and mean type, and that* (1.11) *is satisfied for $|\theta| \leqslant \frac{1}{2}\pi/\rho$. Suppose further that*

$$\int_1^\infty \{\log|f(re^{i\frac{1}{2}\pi/\rho})| + \log|f(re^{-i\frac{1}{2}\pi/\rho})|\}^+ \frac{dr}{r^{\rho+1}} \tag{3.31}$$

is convergent. Then (i)

$$\sum_{n=1}^{\infty} \frac{\cos\rho\theta_n}{r_n{}^\rho} \tag{3.32}$$

is convergent, (ii)

$$\lim_{r\to\infty} \frac{2\rho}{\pi r^\rho} \int_{-\frac{1}{2}\pi/\rho}^{\frac{1}{2}\pi/\rho} \log|f(re^{i\theta})| \cos\theta\rho\, d\theta = A, \tag{3.33}$$

(iii) *the conclusions of Theorem II hold with $\alpha = \frac{1}{2}\pi/\rho$.*

Proof. We have $\rho(r) \equiv \rho$. The convergence of (3.32) is the consequence of a theorem due to F. and R. Nevanlinna*; it can also be deduced from Lemma 2 by a classical Tauberian argument. Parts (i) and (ii) are in some respects stronger than (iii), and, in fact, we deduce (iii) from (i) and (ii) as follows: An elementary argument shows that (3.21) follows from the convergence of (3.32). Since, given any $\epsilon > 0$, we have

$$\log|f(re^{i\theta}| < (A\cos\theta\rho + B\sin\theta\rho + \epsilon)r^\rho \tag{3.34}$$

for $|\theta| \leqslant \frac{1}{2}\pi/\rho$, $r > r_0(\epsilon)$, it follows from (3.33) that, given any $\epsilon > 0$, $\delta > 0$, we can choose $r_0(\epsilon, \delta)$ so that

$$\log|f(re^{i\theta})| > (A\cos\theta\rho + B\sin\theta\rho - \epsilon)r^\rho \tag{3.35}$$

* F. and R. Nevanlinna, *Acta Soc. Sci. Fennicae*, 50 (1922), No. 5, 45.

for $|\theta| \leqslant \frac{1}{2}\pi/\rho$, except, perhaps, in a set of θ of measure less than δ, provided that $r > r_0(\epsilon, \delta)$. Hence there is a point at which (3.35) holds in the angle $|\theta| < \delta$ for every $\delta > 0$ and every $r > r_0(\epsilon, \delta)$. Taking one of these points as centre, and applying Lemma 1 as before, we have the required result.

It follows immediately from the Nevanlinnas' work* that (3.33) holds, provided that (1.11) holds and also

$$\int_1^\infty \{\log^+|f(re^{i\frac{1}{2}\pi/\rho})|+\log^+|f(re^{-i\frac{1}{2}\pi/\rho})|\}\frac{dr}{r^{\rho+1}} \tag{3.351}$$

is convergent, and the same result follows from the convergence of (3.31) by a slight adjustment of their method. For completeness we give the proof more or less in full. Suppose, first, that $\rho = 1$, and let $z = x+iy$, where x and y are real. Then there is a real continuous function $C(y)$ such that $C(y) = \max\{\log|f(iy)|\,;\ -\log|f(-iy)|\}$ if either $\log|f(iy)| > 0$ or $\log|f(-iy)| > 0$, and $C(y) = 0$ otherwise. We have

$$C(y)+C(-y) \equiv \{\log|f(iy)|+\log|f(-iy)|\}^+;$$

and so

$$\int_1^\infty \{C(y)+C(-y)\}\frac{dy}{y^2} \tag{3.36}$$

is convergent. It follows from the Nevanlinnas' work that if $|z| < R$, $|\arg z| < \frac{1}{2}\pi$, we have

$$\log|f(z)| \leqslant U_R(z)+V_R(z),$$

where

$$U_R(z) = \frac{1}{2\pi}\int_{\phi=-\frac{1}{2}\pi}^{\phi=\frac{1}{2}\pi} \log|f(Re^{i\phi})|\,dg_R(Re^{i\phi}, z),$$

$$V_R(z) = \frac{1}{2\pi}\int_{t=r}^{t=-r} C(t)\,dg_R(it, z),$$

and $-g_R(w, z)$ is a function conjugate to the Green's function for the semicircle $|w| \leqslant R$, $|\arg w| \leqslant \frac{1}{2}\pi$ with a pole at z. We also know that

$$U_R(x) = \frac{2x}{\pi R}(1+\epsilon_R)\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \log|f(Re^{i\phi})|\cos\phi\,d\phi;$$

where ϵ_R denotes generally a number which tends to 0 as $R\to\infty$, not necessarily the same in each place.

We observe that $V_R(z)$ takes the value $C(y)$ on $z = iy$, $|y| \leqslant R$, and vanishes on $|z| = R$, $|\arg z| \leqslant \frac{1}{2}\pi$. It follows from the reflection principle that $V_R(z)$ exists in the whole half plane $x \geqslant 0$, and takes the value

* *Loc. cit.*

$-C(R^2/y)$ on the imaginary axis for $|y| > R$. Hence Green's formula for the half plane gives

$$V_R(x) = \frac{x}{\pi}\int_{-\infty}^{\infty}\frac{V_R(it)}{t^2+x^2}\,dt = \frac{x}{\pi}\int_0^{\infty}\{V_R(it)+V_R(-it)\}\frac{dt}{t^2+x^2}$$

$$= \frac{x}{\pi}\int_0^{R}\{C(t)+C(-t)\}\frac{dt}{t^2+x^2} - \frac{x}{\pi}\int_0^{R}\{C(t)+C(-t)\}\frac{dt}{(tx/R)^2+R^2}.$$

Since (3.36) is convergent, the first term tends to

$$I(x) = \frac{x}{\pi}\int_0^{\infty}\{C(t)+C(-t)\}\frac{dt}{t^2+x^2},$$

as $R\to\infty$, while the last term tends to 0. Further, since $h(\frac{1}{2}\pi)+h(-\frac{1}{2}\pi) = 0$, we have $C(t)+C(-t) = \epsilon_t t$; and since (3.36) converges, it is easy to show that

$$I(x) = \epsilon_x x.$$

Hence

$$\frac{\log|f(x)|}{x} < (1+\epsilon_R)\frac{2}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{\log|f(Re^{i\phi})|}{R}\cos\phi\,d\phi+\epsilon_x;$$

and so

$$\frac{\log|f(x)|}{x} \leqslant \lim_{R\to\infty}\frac{2}{\pi R}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{\log|f(Re^{i\phi})|}{R}\cos\phi\,d\phi+\epsilon_x.$$

But $h(0) = A$ so that $\log|f(x)| > (A-\epsilon)x$ for every $\epsilon > 0$ and a sequence of values of x tending to infinity. Hence

$$\lim_{R\to\infty}\frac{2}{\pi R}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{\log|f(Re^{i\phi})|}{R}\cos\phi\,d\phi \geqslant A.$$

It follows from (3.34) that the upper limit of the expression is less than or equal to A; and so we have the required result for $\rho = 1$. The general result follows by putting $z = w^{\rho}$.

3.4. It is interesting to compare the last theorem with certain results about the convergence or divergence of

$$\Sigma\frac{\cos\rho\theta_n(a)}{r_n(a)},$$

where $z_1(a), z_2(a), \ldots, z_n(a) = r_n(a)e^{i\theta_n(a)}, \ldots$ $[l < r_1(a) \leqslant r_2(a) \leqslant \ldots]$ are the zeros of $f(z)-a$ for which $|\theta_n(a)| \leqslant \frac{1}{2}\pi/\rho$. R. Nevanlinna* has proved a

* R. Nevanlinna, *Acta Soc. Sci. Fennicae*, 50 (1922), No. 12, p. 18, Satz II, and p. 20, Satz III.

number of beautiful theorems from which it is easy to deduce the following:

THEOREM IV. *If $f(z)$ is regular and of finite order for $|\arg z| \leqslant \frac{1}{2}\pi/\rho$, $|z| \geqslant l$, and if*

$$(3.41)\quad \overline{\lim_{r\to\infty}} \int_l^r \{\log^+|f(ue^{i\frac{1}{2}\pi/\rho})|+\log^+|f(ue^{-i\frac{1}{2}\pi/\rho})|\}\frac{du}{u^{\rho+1}} + \overline{\lim_{r\to\infty}} \int_{-\frac{1}{2}\pi/\rho}^{\frac{1}{2}\pi/\rho} \frac{\log^+|f(re^{i\theta})|}{r^\rho}\cos\theta\rho\, d\theta = \infty,$$

then

$$(3.42)\qquad \Sigma \frac{\cos\rho\theta_n(a)}{\{r_n(a)\}^\rho}$$

is divergent except possibly for a single exceptional value of a.

THEOREM V. *If* (3.42) *converges for two different values of a, then the upper limits of both the integrals in* (3.41) *are finite.*

It follows that *if $f(z)$ is of order ρ and mean type, and if* (1.11) *holds, and* (3.42) *converges for two different values of a, then the conclusions of Theorem III hold.* For in this case (3.31) converges.

Negative results.

4.1. The results given are in most respects the best possible, and we now proceed to construct examples to show that this is so. In (3.11) and (3.21) it is not possible to replace $n(r, -\alpha+\delta, \alpha-\delta)$ by $n(r, -\alpha, \alpha)$, as may easily be seen by considering the function $e^{V(z)}+a$, where $|a|>1$. Also, since multiplication by an integral function $\phi(z)$ such that

$$\log|\phi(re^{i\theta})| < \epsilon r^{\rho(r)}$$

for every $\epsilon > 0$ and every $r > r_0(\epsilon)$ will not alter the hypotheses in Theorems I and II, we cannot expect anything better than (3.21). Similarly we cannot expect to improve (3.32) substantially.

It is more difficult to show that under the conditions of Theorem I the behaviour of $f(z)$ may vary considerably with r, and that Theorem II need not hold if $\alpha = \frac{1}{2}\pi/\rho$. We use certain functions of integral order which I considered in a former paper*.

LEMMA 8. *Suppose that $f(z)$ is a canonical product of order ρ and mean type such that $n(r) = O(r^\rho)$. Then* (1)

$$\log|f(re^{i\theta})| < [\Re\{S(r)e^{i\theta\rho}\}+\epsilon]r^\rho,$$

* M. L. Cartwright, *Proc. London Math. Soc.* (2), 33 (1931), 209–224, in particular 212, 218, 219.

where $$S(r) = \frac{1}{\rho} \sum_{r_n \leqslant r} \frac{1}{z_n^{\rho}},$$

for every $\epsilon > 0$ *and every* $r > r_0(\epsilon)$; (2) *given any* $\epsilon > 0$, $\eta > 0$, *we can choose* $R_0(\epsilon, \eta)$ *so that*

$$\log|f(re^{i\theta})| > [\mathfrak{R}\{S(r)e^{i\theta\rho}\} - \epsilon]r^{\rho}$$

for $0 < r < R$, $R > R_0$, *except in a set of measure less than* ηR.

Let $r_n^{\rho} = n \log n$ for $n \geqslant 2$; and let

$$A \cos\theta\rho + B \sin\theta\rho = C \cos(\theta - \gamma)\rho.$$

Then $\Sigma 1/r_n^{\rho}$ is divergent, and we can choose θ_n so that $S(r)$ oscillates between any desired limits; in particular, we can choose θ_n so that

$$S(r) \sim Ce^{-i\gamma\rho} - D(r),$$

where $D(r)$ oscillates between 0 and 1. Hence, if $|\theta| \leqslant \alpha < \frac{1}{2}\pi/\rho$, we have

$$\mathfrak{R}\{S(r)e^{i\theta\rho}\} \sim C \cos(\theta - \gamma)\rho - D(r) \leqslant C \cos(\theta - \gamma)\rho, \tag{4.11}$$

and $$\overline{\lim_{r\to\infty}}\, \mathfrak{R}\{S(r)\, e^{i\theta\rho}\} = C \cos(\theta - \gamma)\rho.$$

It follows that if $f(z)$ satisfies the conditions of Lemma 8 and has its zeros arranged in this way, it also satisfies the conditions of Theorem I for $|\theta| \leqslant \alpha \leqslant \frac{1}{2}\pi/\rho$. But, since $\overline{\lim}\, D(r) = 1$, it follows from (4.11) that (3.12) is not satisfied for certain large intervals of r.

In order to show that $\underline{\lim}$ in (3.11) cannot be replaced by lim if $\alpha < \frac{1}{2}\pi/\rho$, we have to combine two canonical products f_1 and f_2 of the type just considered. Suppose, for simplicity, that $\rho = 1$, and that $h(\theta) = \cos\theta$ for $|\theta| \leqslant \frac{1}{4}\pi$; and let $r_n' = n \log n$, $r_n'' = n \log n$, where r_n' and r_n'' are the moduli of zeros of f_1 and f_2 respectively. Further, let the zeros of f_1 be the conjugates of those of f_2. We arrange the zeros of f_1 so that $S_1(r) + \epsilon_r$ oscillates between 1 and i, taking only values between 0 and 1 on the real axis and values between 0 and i on the imaginary axis. Then $S_2(r) + \epsilon_r$ oscillates between 1 and $-i$, passing from 1 to 0 along the real axis and from 0 to $-i$ along the imaginary axis, and back again. We have

$$\overline{\lim_{r\to\infty}}\, \mathfrak{R}\{S_1(r)\, e^{i\theta}\} = \cos\theta \tag{4.12}$$

for $|\theta| \leqslant \frac{1}{4}\pi$, and also

$$\underline{\lim_{r\to\infty}}\, \mathfrak{R}\{S_1(r)\, e^{i\theta}\} = -\sin\theta, \tag{4.13}$$

while $S_2(r)$ satisfies (4.12), and (4.13) with the sign changed. Let $f(z)=f_1(z)+f_2(z)$; then applying Lemma 8 to $f_1(z)$ and $f_2(z)$ and using (4.12) we have $h(\theta)=\cos\theta$ for $|\theta|\leqslant\frac{1}{4}\pi$, as required. Also, using (4.13) and the corresponding result for $S_2(r)$ together with Lemma 8, we have

$$\log|f(re^{i\theta})|\sim|\sin\theta|r$$

for $\delta\leqslant(\theta|\leqslant\frac{1}{2}\pi-\delta$, where $\delta>0$, in a set of upper linear density 1*. It is easy to show by means of Jensen's formula and Lemma 1 that

$$\overline{\lim_{r\to\infty}}\frac{n(r,\ -\delta,\ \delta)}{r}>0.$$

The exact value of this upper limit is $1/\pi$, and it can be obtained by a method which I shall describe in another paper.

Similar examples can be constructed for the general form

$$h(\theta)=A\cos\theta\rho+B\sin\theta\rho,$$

provided that the angle considered is less than π/ρ.

All the examples derived from Lemma 8 have been of integral order and mean type. It is easy to prove a corresponding lemma for functions of integral order and maximum type, and functions of any proximate order can be constructed from those of Lemma 8 by conformal representation. The latter will not, however, be integral functions as a rule.

Special functions.

5.1. We now suppose that $f(z)$ is an integral function of proximate order $\rho(r)$. Then $f(z)$ can be written in the form

$$f(z)=z^m\exp(c_0+c_1z+\ldots+c_pz^p)\prod_{n=1}^{\infty}\left(1-\frac{z}{z_n}\right)\exp\left(\frac{z}{z_n}+\frac{z^2}{2z_n^2}+\ldots+\frac{z_p}{pz_n^p}\right),$$

where $z_n=r_ne^{i\theta_n}$, m is a positive integer or zero, and p is the integral part of ρ.

THEOREM VI. *Suppose that* $\rho>\frac{1}{2}$, *and that*

$$h(\theta)=(-1)^p\cos\theta\rho \tag{5.11}$$

for $|\theta|\leqslant\pi$. *Then, for every* $\delta>0$, *we have*

$$n(r,\ -\pi+\delta,\ \pi-\delta)=o(r^{\rho(r)}), \tag{5.12}$$

$$n(r,\ \pi-\delta,\ \pi+\delta)\sim n(r)\sim|\sin\pi\rho|r^{\rho(r)}/\pi \tag{5.13}$$

* See § 3.1.

as $r \to \infty$. *Further, if* ρ *is an integer,*

$$c_\rho + \frac{1}{\rho} \sum_{r_n \leqslant r} \frac{1}{z_n^{\rho}} \sim (-1)^\rho r^{\rho(r)-\rho} \tag{5.14}$$

as $r \to \infty$.

If ρ is not an integer, it follows from (5.12) that any function of order $\rho > \frac{1}{2}$ satisfying (5.11) is equal to $F(z) \times \phi(z)$, where $F(z)$ is one of the *fonctions orientées* considered by Valiron, and

$$\log|\phi(re^{i\theta})| < \epsilon r^{\rho(r)}$$

for every $\epsilon > 0$ and every $r > r_0(\epsilon)$.

Proof. We may suppose, without real loss of generality, that $f(0) = 1$. It follows at once from Theorem II that (5.12) holds, and also that

$$\log|f(re^{i\theta})| \sim h(\theta) r^{\rho(r)}$$

for $|\theta| \leqslant \pi - \delta$ in a set of r of linear density 1. By Lemma 1, given any $\eta > 0$, we can choose $H = H(\eta)$ so that

$$\log|f(re^{i\theta})| > -H R^{\rho(R)}$$

for $0 < r < R$, except in a set of measure less than ηR. Combining these results with Jensen's formula, we have

$$N(R) = \int_0^R \frac{n(u)\,du}{u} > \frac{r^{\rho(r)}}{2\pi} \int_{-\pi}^{\pi} h(\theta)\,d\theta - \epsilon_r r^{\rho(r)} - H\delta\, r^{\rho(r)},$$

where $(1-\eta) R < r < R$, and so

$$N(R) = \frac{|\sin \pi\rho|}{\pi\rho} R^{\rho(R)} - \epsilon_R R^{\rho(R)} - H\delta R^{\rho(R)} - \epsilon(\eta) R^{\rho(R)},$$

where $\epsilon(\eta) \to 0$ as $\eta \to 0$. The "less than" inequality for $N(r)$ follows immediately from the hypothesis and Jensen's formula; and so, making $\delta \to 0$, $\eta \to 0$, we have

$$N(r) \sim \frac{|\sin \pi\rho|}{\pi\rho} r^{\rho(r)}$$

as $r \to \infty$. A well-known argument now gives (5.13), and (5.14) follows in a similar manner from Lemma 3.

5.2. The functions considered in the next theorem behave rather like Mittag-Leffler's function, $E_\alpha(z) = \Sigma z^n / \Gamma(1+\alpha n)$, where $\alpha = 1/\rho < 1$.

THEOREM VII. *Suppose that $f(z)$ is of order $\rho > 1$ and mean type. Suppose also that*

$$\begin{cases} h(\theta) = A\cos\theta\rho & \text{for } |\theta| \leqslant \tfrac{1}{2}\pi/\rho \\ h(\theta) = 0 & \text{for } \tfrac{1}{2}\pi/\rho \leqslant |\theta| \leqslant \pi, \end{cases} \tag{5.21}$$

and that

$$\int_1^\infty \{\log|f(re^{i\frac{1}{2}\pi/\rho})| + \log|f(re^{-i\frac{1}{2}\pi/\rho})|\}^+ \frac{dr}{r^{\rho+1}} \tag{5.22}$$

is convergent. Then (1)

$$\Sigma' \frac{\cos\rho\theta_n}{r_n^{\rho}}, \tag{5.13}$$

where Σ' denotes summation over zeros for which $|\theta_n| \leqslant \frac{1}{2}\pi/\rho$, is absolutely convergent,

(2) *for every $\delta > 0$,*

$$n(r,\ \tfrac{1}{2}\pi/\rho+\delta,\ 2\pi-\tfrac{1}{2}\pi/\rho-\delta) = o(r^\rho), \tag{5.24}$$

$$n(r) \sim Ar^\rho/\pi \tag{5.25}$$

as $r \to \infty$.

We apply Theorem III to $f(z)$ in the angle $|\theta| \leqslant \frac{1}{2}\pi/\rho$, and Theorem II to $f(z)$ in the angle $\frac{1}{2}\pi/\rho \leqslant \theta \leqslant 2\pi-\frac{1}{2}\pi/\rho$. Part (1) and (5.24) follow immediately, and we also have

$$\log|f(re^{i\theta})| \sim h(\theta)r^\rho$$

for $|\theta| \leqslant \frac{1}{2}\pi/\rho-\delta$, $\frac{1}{2}\pi/\rho+\delta \leqslant |\theta| \leqslant \pi$ in a set of r of linear density 1. Hence, as in the previous theorem,

$$N(r) \sim \frac{r^\rho}{2\pi}\int_{-\pi}^{\pi} h(\theta)\,d\theta = \frac{Ar^\rho}{\pi\rho};$$

and so we have (5.25).

It is natural to suppose that the zeros are more or less equally divided between the angle $|\arg z-\frac{1}{2}\pi/\rho| \leqslant \delta$ and $|\arg z+\frac{1}{2}\pi/\rho| \leqslant \delta$; and we can obtain confirmation of this conjecture by putting $m = 1$ in Lemma 3 and taking the imaginary part of both sides. We have

$$-c_1 + \sum_{r_n \leqslant r} \frac{\sin\theta_n}{r_n}\left(1-\frac{r_n^2}{r^2}\right) = o(r^{\rho-1});$$

and, comparing this with (5.25), we see that the zeros are more or less

evenly divided between the upper and lower half planes. By considering $\int d \arg f(z)$, taken round a contour enclosing the region $\eta r \leqslant |z| \leqslant r$, $|\arg z - \frac{1}{2}\pi/\rho| \leqslant \delta$, and using a lemma from one of my former papers*, we can show that

$$n(r,\ \tfrac{1}{2}\pi/\rho - \delta,\ \tfrac{1}{2}\pi/\rho + \delta) \sim Ar^\rho/(2\pi).$$

The details of this method are somewhat cumbersome, and so I leave them for another occasion.

5.3. Theorem VII throws some light on functions whose directions of Borel all lie in the angle $|\arg z| \leqslant \frac{1}{2}\pi/\rho$. Let

$$z_1(a),\ z_2(a),\ \ldots, z_n(a) = r_n(a) \exp\{i\theta_n(a)\} \ldots \quad [0 \leqslant r_1(a) \leqslant r_2(a) \leqslant \ldots]$$

be the zeros of $f(z) - a$; and let Σ' and Σ'' denote summation over zeros for which $|\theta_n(a)| \leqslant \frac{1}{2}\pi/\rho$, and $\frac{1}{2}\pi/\rho \leqslant |\theta_n(a)| \leqslant \pi$ respectively. More generally let $\sum_{\mathfrak{A}}$ denote summation over zeros lying in the angle $\mathfrak{A}$. If

$$\sum_{\mathfrak{A}} \frac{1}{\{r_n(a)\}^\rho}$$

is divergent for every angle $\mathfrak{A}$ bisected by $\arg z = \alpha$, and for every value of a except perhaps one, then $\arg z = \alpha$ is said to be a *direction of Borel of divergent type*†.

THEOREM VIII. *Suppose that $f(z)$ satisfies the hypotheses of Theorem VII, and that*

$$\int_1^\infty \{\log^+|f(re^{i\frac{1}{2}\pi/\rho})| + \log^+|f(re^{-i\frac{1}{2}\pi/\rho})|\} \frac{dr}{r^{\rho+1}} \tag{5.31}$$

is convergent. Then

$$\Sigma' \frac{\cos \rho\theta_n(a)}{\{r_n(a)\}^\rho}$$

is convergent for every value of a, and $f(z)$ has no directions of Borel of divergent type for which $|\arg z| < \frac{1}{2}\pi/\rho$.

It follows from (5.31) that $f(z) - a$ satisfies the hypotheses of Theorem VII for every value of a, and the first part follows at once. The second part is a trivial corollary of the first.

* M. L. Cartwright, *Quart. J. of Math.* (Oxford series), 1 (1930), 38–59.

† See G. Valiron, *Journal de Math.* (9), 10 (1931), 457–480.

Let $n(r, \alpha, \beta, a)$ denote the number of zeros of $f(z)-a$ for which $|z| \leqslant r$, $\alpha \leqslant \arg z \leqslant \beta$; and suppose that

$$\overline{\lim_{r\to\infty}} \frac{n(r, \alpha-\delta, \alpha+\delta, a)}{r^{\rho(r)}} = \zeta > 0$$

for every $\delta > 0$ and every a except perhaps one. Then $\arg z = \alpha$ is said to be *a direction of Borel of the maximum kind* of $f(z)$.

THEOREM IX. *Suppose that $f(z)$ is of order $\rho > 1$ and mean type, and that its directions of Borel of the maximum kind all lie in the angle*

$$|\arg z| \leqslant \tfrac{1}{2}\pi/\rho.$$

Suppose further that

(5.32) *either* $$\Sigma' \frac{\cos \rho\theta_n(a)}{\{r_n(a)\}^\rho} \quad \textit{or} \quad \Sigma'' \frac{\cos \rho\theta_n(a)}{\{r_n(a)\}^\rho}$$

is absolutely convergent for two values of a. Then $f(z)$ satisfies the hypotheses of Theorem VIII and therefore of Theorem VII. In particular, $\arg z = \pm\frac{1}{2}\pi/\rho$ are directions of Borel of the maximum kind without exceptional values.

Since $f(z)$ has no directions of Borel of the maximum kind for which $\frac{1}{2}\pi/\rho < |\arg z| \leqslant \pi$, it follows from a theorem which I have proved elsewhere* that $h(\theta) = 0$ for $\frac{1}{2}\pi/\rho \leqslant |\theta| \leqslant \pi$; and so, by a theorem of Phragmén and Lindelöf, we have $h(\theta) = A\cos\theta\rho$, where A is the type of $f(z)$, for $|\theta| \leqslant \frac{1}{2}\pi/\rho$. Applying Theorem V to $f(z)$ in the angle $|\arg z| \leqslant \frac{1}{2}\pi/\rho$, we obtain the required result immediately from the hypothesis on Σ'. It follows from the hypothesis on Σ'' that

$$\sum_{\mathfrak{A}} \frac{\cos \rho\{\theta_n(a)-\pi/\rho\}}{\{r_n(a)\}^\rho} = -\sum_{\mathfrak{A}} \frac{\cos \rho\theta_n(a)}{\{r_n(a)\}^\rho},$$

where $\mathfrak{A}$ is the angle $|\arg z-\pi/\rho| \leqslant \frac{1}{2}\pi/\rho$, is convergent for two values of a. Hence, applying Theorem V to $f(z)$ in the angle $\mathfrak{A}$, we see that

$$\int_1^\infty \{\log^+|f(re^{i\frac{1}{2}\pi/\rho})|+\log^+|f(re^{i\frac{3}{2}\pi/\rho})|\} \frac{dr}{r^{\rho+1}}$$

is convergent. Combining this with a similar result for the angle $|\arg z+\pi/\rho| \leqslant \frac{1}{2}\pi/\rho$, we have (5.31), which is the required result.

* M. L. Cartwright, *Comptes rendus*, 194 (1932), 1889–1891.

5.4. Theorem III gives a solution of the problem suggested by Prof. Titchmarsh from which these investigations began. Suppose that

$$f(z) = \int_a^b e^{zt}\psi(t)\,dt \quad (a < b),$$

where $\psi(t)$ is integrable in the sense of Lebesgue, and a and b are the effective limits of integration. Titchmarsh* showed that

(A) $\sum_{n=1}^{\infty} \frac{|\cos\theta_n|}{r_n}$ *converges,*

(B) *as* $r\to\infty$, $n(r)\sim(b-a)r/\pi$,

(C) $c_1+\sum_{n=1}^{\infty}\frac{1}{z_n}=\frac{1}{2}(a+b)$.

It is easy to see that $f(z)$ is of order 1 and mean type, and he showed that

$$h(\theta) = \max(a\cos\theta;\ b\cos\theta). \tag{5.41}$$

The problem is whether these results follow simply from (5.41), or, if not, with some simple additional condition.

It should be observed first that (B) implies (C). For I have shown elsewhere† that, *if* $f(z)$ *is an integral function of order* ρ *and mean type and if*

$$\lim_{r\to\infty}\frac{n(r)}{\rho r^{\rho}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\theta)\,d\theta,$$

then

$$c^{\rho}+\frac{1}{\rho}\sum_{n=1}^{\infty}\frac{1}{z_n{}^{\rho}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\theta)\,e^{-i\theta\rho}\,d\theta.$$

If (5.41) holds, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} h(\theta)\,d\theta = \frac{b-a}{\pi}, \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\theta)\,e^{-i\theta}\,d\theta = \tfrac{1}{2}(a+b);$$

and so (B) implies (C).

None of the results (A), (B), (C) follow from (5.41) alone, as may be seen by means of the functions considered in Lemma 8. Let $\rho = 1$, and let $r_n = n\log n$ for $n \geqslant 2$. Further, suppose that all the zeros are real. Then we can make $S(r)$ oscillate between a and b, and it follows from the lemma that (5.41) is satisfied. But $\Sigma|\cos\theta_n|/r_n$ is divergent, $n(r) = O(r)$, and

* E. C. Titchmarsh, *Proc. London Math. Soc.* (2), 25 (1926), 283–302.

† M. L. Cartwright, *Proc. London Math. Soc.* (2), 33 (1931), 209–224, Theorem VI.

$c_1+\Sigma 1/z_n$ oscillates between a and b. Hence, *neither* (A), (B), *nor* (C) *follows from* (5.41). We can, however, obtain all three results by means of an additional hypothesis.

THEOREM X. *Suppose that $f(z)$ is of order 1 and mean type, and satisfies* (5.41). *Suppose also that*

$$\int_1^\infty \{\log|f(ir)|+\log|f(-ir)|\}^+ \frac{dr}{r^2}$$

is convergent. Then (A), (B), *and* (C) *hold.*

We apply Theorem III to $f(z)$ in the right- and left-half planes, and obtain (A) and (B) as in Theorem VII. As we have already remarked, (C) follows from (B), and we have the theorem complete.

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THE BOREL DERIVATES OF A FUNCTION

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Introduction.

1. It has been shown* that, if a continuous function $f(x)$ has a finite Borel derivative in a given set E of positive measure, then, almost everywhere in E, $f(x)$ has an approximate derivative equal to the Borel derivative. The converse of this is not true. Khintchine† has constructed a continuous function with a finite approximate derivative at points of a set of positive measure, which has an infinite one-sided Borel derivative at almost all points of the set.

It is the object of this note to generalize Khintchine's result by the consideration of upper and lower one-sided Borel derivates and to extend it to the case when $f(x)$ is not necessarily continuous, but is integrable in the special Denjoy sense. It will be shown that the existence of two finite right-hand Borel derivates in a measurable set E implies the existence almost everywhere in E of a Borel derivative, and of an approximate derivative equal to the Borel derivative.

Definition of the derivates.

2. We suppose throughout that $f(x)$ is finite, measurable, and integrable in the special Denjoy sense.

Since $1/(t-x)$, considered as a function of t, is of bounded variation in the interval $(x+\epsilon, x+h)$, whenever $0 < \epsilon < h$, the function

$$\{f(t)-f(x)\}/(t-x)$$

is integrable in the special Denjoy sense in this interval.

* Khintchine, **2**, 226–233 and 276–279.

† Khintchine, **2**, 233–238.

We define the upper right Borel derivate $BD^+f(x)$ and the lower right derivate $BD_+f(x)$ by the equations

$$BD^+f(x) = \overline{\lim_{h\to+0}}\ \overline{\lim_{\epsilon\to+0}}\ \frac{1}{h}\int_{x+\epsilon}^{x+h}\frac{f(t)-f(x)}{t-x}\,dt,$$

$$BD_+f(x) = \underline{\lim_{h\to+0}}\ \underline{\lim_{\epsilon\to+0}}\ \frac{1}{h}\int_{x+\epsilon}^{x+h}\frac{f(t)-f(x)}{t-x}\,dt.$$

There are corresponding definitions for the left derivates $BD^-f(x)$ and $BD_-f(x)$.

When the upper and lower right Borel derivates are equal, their common value is said to be the right-hand Borel derivative. When all four Borel derivates are equal, $f(x)$ has a Borel derivative $BDf(x)$.

Khintchine defined the asymptotic derivative of $f(x)$ to be the limit of $\{f(x+h)-f(x)\}/h$, as h tends to zero in such a way that $x+h$ belongs to a set E having unit density at x. This definition is equivalent to that of the approximate derivative given by Denjoy. The approximate λ-derivates and approximate derivates and derivatives have been clearly defined by Burkill and Haslam-Jones*.

Measurability.

3. LEMMA I. *If* $0<\epsilon<h$, *then* $\int_\epsilon^h \frac{f(x+t)}{t}\,dt$ *is a continuous function of* x.

Since $1/t$ decreases in the interval (ϵ, h), it follows from the second mean value theorem that there is a number T, dependent on ξ and x, such that $\epsilon \leqslant T \leqslant h$ and

$$\int_\epsilon^h \frac{f(x+\xi+t)}{t}\,dt - \int_\epsilon^h \frac{f(x+t)}{t}\,dt = \frac{1}{\epsilon}\int_\epsilon^T \{f(x+\xi+t)-f(x+t)\}\,dt$$

$$= \frac{1}{\epsilon}\int_{\epsilon+\xi}^{T+\xi} f(x+t)\,dt - \frac{1}{\epsilon}\int_\epsilon^T f(x+t)\,dt$$

$$= \frac{1}{\epsilon}\int_T^{T+\xi} f(x+t)\,dt - \frac{1}{\epsilon}\int_\epsilon^{\epsilon+\xi} f(x+t)\,dt.$$

Hence $$\lim_{\xi\to0}\int_\epsilon^h \frac{f(x+\xi+t)}{t}\,dt = \int_\epsilon^h \frac{f(x+t)}{t}\,dt,$$

so that $\int_\epsilon^h \frac{f(x+t)}{t}\,dt$ is a continuous function of x.

* Burkill and Haslam-Jones, 1, 347.

LEMMA II. *The functions*

$$\phi(x, h) = \overline{\lim_{\epsilon \to +0}} \int_{x+\epsilon}^{x+h} \frac{f(t)-f(x)}{t-x}\, dt,$$

$$\psi(x, h) = \underline{\lim_{\epsilon \to +0}} \int_{x+\epsilon}^{x+h} \frac{f(t)-f(x)}{t-x}\, dt$$

are, for $h > 0$, *measurable functions of* x *and continuous functions of* h.

Let $$F(x, \epsilon, h) = \int_{x+\epsilon}^{x+h} \frac{f(t)-f(x)}{t-x}\, dt.$$

Then $$\phi(x, h) = \overline{\lim_{\epsilon \to +0}}\, F(x, \epsilon, h),$$

$$\psi(x, h) = \underline{\lim_{\epsilon \to +0}}\, F(x, \epsilon, h).$$

It is clear that

$$F(x, \epsilon, h) = \int_{\epsilon}^{h} \frac{f(x+t)-f(x)}{t}\, dt$$

$$= \int_{\epsilon}^{h} \frac{f(x+t)}{t}\, dt - f(x) \int_{\epsilon}^{h} \frac{dt}{t}.$$

Hence, in view of Lemma I, $F(x, \epsilon, h)$ is the sum of a continuous function of x and a measurable function of x and is therefore measurable.

Since $F(x, \epsilon, h)$ is, for $\epsilon > 0$, a continuous function of ϵ, $\phi(x, h)$ and $\psi(x, h)$ are measurable functions of x.

If $0 < h$ and $0 < h+\eta$, then

$$\phi(x, h+\eta)-\phi(x, h) = \overline{\lim_{\epsilon \to +0}} \left\{ \int_{\epsilon}^{h} \frac{f(x+t)-f(x)}{t}\, dt + \int_{h}^{h+\eta} \frac{f(x+t)-f(x)}{t}\, dt \right\}$$

$$- \overline{\lim_{\epsilon \to +0}} \int_{\epsilon}^{h} \frac{f(x+t)-f(x)}{t}\, dt.$$

Since $\int_{h}^{h+\eta} \frac{f(x+t)-f(x)}{t}\, dt$ is independent of ϵ,

$$\phi(x, h+\eta)-\phi(x, h) = \int_{h}^{h+\eta} \frac{f(x+t)-f(x)}{t}\, dt. \tag{3.1}$$

In a similar way it can be shown that

$$\psi(x, h+\eta)-\psi(x, h) = \int_{h}^{h+\eta} \frac{f(x+t)-f(x)}{t}\, dt. \tag{3.2}$$

It follows at once from (3.1) and (3.2) that $\phi(x, h)$ and $\psi(x, h)$ are, for $h > 0$, continuous functions of h.

LEMMA III. *The Borel derivates are measurable functions of x.*

It is plain that

$$BD^+f(x) = \overline{\lim_{h\to+0}} \frac{\phi(x, h)}{h}, \tag{3.3}$$

$$BD_+f(x) = \underline{\lim_{h\to+0}} \frac{\psi(x, h)}{h}. \tag{3.4}$$

Since $\phi(x, h)$ and $\psi(x, h)$ are continuous functions of h for $h > 0$, so are $\phi(x, h)/h$ and $\psi(x, h)/h$. Hence $BD^+f(x)$ and $BD_+f(x)$ are measurable functions of x.

The following lemma is also required.

LEMMA IV. *If $BD^+f(x)$ and $BD_+f(x)$ are finite, then $\{f(t)-f(x)\}/(t-x)$, considered as a function of t, is integrable in the special Denjoy sense in the interval $(x, x+h)$, provided that $h > 0$, and*

$$BD^+f(x) = \overline{\lim_{h\to+0}} \frac{1}{h}\int_x^{x+h} \frac{f(t)-f(x)}{t-x}\,dt, \tag{3.5}$$

$$BD_+f(x) = \underline{\lim_{h\to+0}} \frac{1}{h}\int_x^{x+h} \frac{f(t)-f(x)}{t-x}\,dt. \tag{3.6}$$

If $BD^+f(x)$ and $BD_+f(x)$ are finite, it follows from (3.3) and (3.4) that

$$\overline{\lim_{h\to+0}}\, \phi(x, h) = 0,$$

$$\underline{\lim_{h\to+0}}\, \psi(x, h) = 0.$$

Since $\psi(x, h) \leqslant \phi(x, h)$, the functions $\phi(x, h)$ and $\psi(x, h)$ each tend to zero as $h \to +0$, and

$$\lim_{h\to+0} \{\phi(x, h)-\psi(x, h)\} = 0. \tag{3.7}$$

If $0 < h$ and $0 < h+\eta$, then

$$\phi(x, h+\eta)-\phi(x, h) = \psi(x, h+\eta)-\psi(x, h),$$

by (3.1) and (3.2), so that

$$\phi(x, h+\eta)-\psi(x, h+\eta) = \phi(x, h)-\psi(x, h).$$

The function $\phi(x, h)-\psi(x, h)$ is therefore, for $h > 0$, independent of h.

It follows from (3.7) that

$$\phi(x, h) = \psi(x, h),$$

whenever $h > 0$.

This means that $\{f(t)-f(x)\}/(t-x)$ is integrable in the interval $(x, x+h)$, provided that $h > 0$, and the right derivates are given by (3.5) and (3.6).

Existence of $ADf(x)$.

4. THEOREM I. *If $BD^+f(x)$ and $BD_+f(x)$ are finite at points of a measurable set E, then $f(x)$ has a finite approximate derivative almost everywhere in E.*

It has been proved* that the approximate derivates of a measurable function are measurable. Hence the sub-set H of E, in which $f(x)$ has no finite approximate derivative, is measurable.

We suppose that H is of measure greater than zero and obtain a contradiction.

By a result due to Burkill and Haslam-Jones†,

$$AD^+(f, x, \lambda) = +\infty, \tag{4.1}$$

whenever $0 < \lambda < 1$, at points x of a set H_1 of measure equal to that of H, where $AD^+(f, x, \lambda)$ denotes the upper right approximate λ-derivate of $f(x)$.

Since $BD^+f(x)$ and $BD_+f(x)$ are finite and measurable in H_1, we can fix M so that

$$-M \leqslant BD_+f(x) \leqslant BD^+f(x) \leqslant M \tag{4.2}$$

in a sub-set F of H_1 of positive measure.

By Lemma IV, $\{f(t)-f(x)\}/(t-x)$, considered as a function of t, is integrable in the special Denjoy sense in the interval $(x, x+h)$, provided that $h > 0$ and that x is a point of F. Let F_n be the set of points x of F for which

$$\left| \frac{1}{h} \int_x^{x+h} \frac{f(t)-f(x)}{t-x}\, dt \right| \leqslant 2M, \tag{4.3}$$

whenever $0 < h \leqslant 1/n$.

* Burkill and Haslam-Jones, 1, 347–349.

† Burkill and Haslam-Jones, 1, 349–351.

Since $\frac{1}{h}\int_x^{x+h}\frac{f(t)-f(x)}{t-x}\,dt$ is, for $h>0$, a continuous function of h, the set F_n is the same if only rational values of h are considered. By Lemma II $\frac{1}{h}\int_x^{x+h}\frac{f(t)-f(x)}{t-x}\,dt$ is a measurable function of x, provided that $h>0$. Hence F_n is the product of a denumerable number of measurable sets and is therefore measurable.

It follows from (4.2) that

$$F = F_1+F_2+F_3+\ldots+F_n+\ldots.$$

We can therefore find an integer ν such that the set F_ν is of measure greater than zero.

Let P be a perfect sub-set of F_ν such that every portion of P is of positive measure*.

Since $f(t)$ is integrable in the special Denjoy sense, there is a portion π of P, with complementary intervals (c_1, d_1), (c_2, d_2), ..., (c_n, d_n), ..., such that

$$\int_\pi |f(t)|\,dt < \infty \tag{4.4}$$

and

$$\sum_{n=1}^{\infty}\left\|\int_{\gamma_n}^{\delta_n} f(t)\,dt\right\| < \infty, \tag{4.5}$$

where $\left\|\int_{\gamma_n}^{\delta_n} f(t)\,dt\right\|$ denotes the upper bound of $\left|\int_{\gamma_n}^{\delta_n} f(t)\,dt\right|$ for

$$c_n \leqslant \gamma_n < \delta_n \leqslant d_n.$$

Let x_0 be a point of density of π. We shall show that

$$BD^+f(x_0) \geqslant 2M,$$

which contradicts (4.2).

Let δ be any positive number. In view of (4.1), we can find a positive number η such that

$$\eta < \min\left(\delta, \frac{1}{\nu}\right), \tag{4.6}$$

$$m\pi_\eta \geqslant \tfrac{3}{4}\eta, \tag{4.7}$$

and

$$mG > \tfrac{3}{4}\eta,$$

* For the existence of P, see G. C. Young, 3, 365–366.

where π_η denotes the portion of π contained in the interval $(x_0, x_0+\eta)$, and G is the set of points t of $(x_0, x_0+\eta)$ for which

$$\frac{f(t)-f(x_0)}{t-x_0} \geqslant 8M. \tag{4.8}$$

Let G_1 be a closed sub-set of G of measure greater than $\frac{3}{4}\eta$, and let

$$Q = \pi G_1.$$

Then Q is closed and, by (4.7),

$$mQ > \tfrac{1}{2}\eta. \tag{4.9}$$

Let a and b be the extreme points of Q, and let

$$(a_1, b_1),\ (a_2, b_2),\ \ldots,\ (a_n, b_n),\ \ldots$$

be the complementary intervals of Q. Since x_0 is not a point of G, a and x_0 do not coincide.

In view of (4.4) and (4.5), it can easily be shown that

$$\int_Q |f(t)|\,dt < \infty \tag{4.10}$$

and

$$\sum_{n=1}^{\infty} \left\| \int_{\alpha_n}^{\beta_n} f(t)\,dt \right\| < \infty, \tag{4.11}$$

where $\left\| \int_{\alpha_n}^{\beta_n} f(t)\,dt \right\|$ denotes the upper bound of $\left| \int_{\alpha_n}^{\beta_n} f(t)\,dt \right|$ for all intervals (α_n, β_n) contained in the closed interval (a_n, b_n).

By (4.10), since $1/(t-x_0)$ is bounded when t is a point of Q,

$$\int_Q \left| \frac{f(t)-f(x_0)}{t-x_0} \right| dt < \infty. \tag{4.12}$$

Suppose that $a_n \leqslant \alpha_n < \beta_n \leqslant b_n$. Then, since $1/(t-x_0)$ decreases in (α_n, β_n), it follows from the second mean value theorem that there is a point β_n' of (α_n, β_n) such that

$$\int_{\alpha_n}^{\beta_n} \frac{f(t)-f(x_0)}{t-x_0}\,dt = \frac{1}{\alpha_n - x_0} \int_{\alpha_n}^{\beta_n'} \{f(t)-f(x_0)\}\,dt.$$

Hence

$$\left|\int_{\alpha_n}^{\beta_n}\frac{f(t)-f(x_0)}{t-x_0}\,dt\right| \leqq \frac{1}{a-x_0}\left|\int_{\alpha_n}^{\beta_n'} f(t)\,dt\right| + \frac{\beta_n'-\alpha_n}{a-x_0}\,|f(x_0)|.$$

In view of (4.11) and $\sum_{n=1}^{\infty}(\beta_n'-\alpha_n)<\eta$,

$$\sum_{n=1}^{\infty}\left\|\int_{\alpha_n}^{\beta_n}\frac{f(t)-f(x_0)}{t-x_0}\,dt\right\| < \infty.$$

Hence

$$\int_a^{x_0+\eta}\frac{f(t)-f(x_0)}{t-x_0}\,dt$$

$$=\int_Q\frac{f(t)-f(x_0)}{t-x_0}\,dt+\sum_{n=1}^{\infty}\int_{a_n}^{b_n}\frac{f(t)-f(x_0)}{t-x_0}\,dt+\int_b^{x_0+\eta}\frac{f(t)-f(x_0)}{t-x_0}\,dt.$$

By Lemma IV, $\{f(t)-f(x_0)\}/(t-x_0)$ is integrable in (x_0, a), and therefore

$$\int_{x_0}^{x_0+\eta}\frac{f(t)-f(x_0)}{t-x_0}\,dt \tag{4.13}$$

$$=\int_Q\frac{f(t)-f(x_0)}{t-x_0}\,dt+\int_{x_0}^{a}\frac{f(t)-f(x_0)}{t-x_0}\,dt$$

$$+\sum_{n=1}^{\infty}\int_{a_n}^{b_n}\frac{f(t)-f(x_0)}{t-x_0}\,dt+\int_b^{x_0+\eta}\frac{f(t)-f(x_0)}{t-x_0}\,dt.$$

By (4.8), (4.9), and (4.12),

$$\int_Q\frac{f(t)-f(x_0)}{t-x_0}\,dt > 4M\eta. \tag{4.14}$$

By (4.3) and (4.6), since $a-x_0<\eta$ and x_0 is a point of F_ν,

$$\int_{x_0}^{a}\frac{f(t)-f(x_0)}{t-x_0}\,dt \geqq -2M(a-x_0). \tag{4.15}$$

It is clear that

$$\int_{a_n}^{b_n}\frac{f(t)-f(x_0)}{t-x_0}\,dt = \int_{a_n}^{b_n}\frac{f(t)-f(a_n)}{t-x_0}\,dt+\int_{a_n}^{b_n}\frac{f(a_n)-f(x_0)}{t-x_0}\,dt.$$

By (4.8),

$$f(a_n)-f(x_0)>0,$$

since a_n is a point of Q. Hence

$$\int_{a_n}^{b_n} \frac{f(t)-f(x_0)}{t-x_0}\,dt > \int_{a_n}^{b_n} \frac{f(t)-f(a_n)}{t-x_0}\,dt$$
$$= \int_{a_n}^{b_n} \frac{f(t)-f(a_n)}{t-a_n}\,\frac{t-a_n}{t-x_0}\,dt.$$

Since $(t-a_n)/(t-x_0)$ increases in (a_n, b_n) and $\{f(t)-f(a_n)\}/(t-a_n)$ is, by Lemma IV, integrable in (a_n, b_n), it follows from the second mean value theorem that there is a point ξ of (a_n, b_n) such that

$$\int_{a_n}^{b_n} \frac{f(t)-f(x_0)}{t-x_0}\,dt > \frac{b_n-a_n}{b_n-x_0}\int_{\xi}^{b_n} \frac{f(t)-f(a_n)}{t-a_n}\,dt$$
$$= \frac{b_n-a_n}{b_n-x_0}\left\{\int_{a_n}^{b_n} \frac{f(t)-f(a_n)}{t-a_n}\,dt - \int_{a_n}^{\xi} \frac{f(t)-f(a_n)}{t-a_n}\,dt\right\}$$
$$\geqslant \frac{b_n-a_n}{b_n-x_0}\{-2M(b_n-a_n)-2M(\xi-a_n)\},$$

by (4.3) and (4.6), since $b_n-a_n<\eta$ and a_n is a point of F_ν. Hence

$$\int_{a_n}^{b_n} \frac{f(t)-f(x_0)}{t-x_0}\,dt > -4M(b_n-a_n). \tag{4.16}$$

In the same way, since b is a point of Q and $b-x_0<\eta$, it can be shown that

$$\int_{b}^{x_0+\eta} \frac{f(t)-f(x_0)}{t-x_0}\,dt > -4M(x_0+\eta-b).$$

Together with (4.13), (4.14), (4.15), and (4.16), this yields

$$\int_{x_0}^{x_0+\eta} \frac{f(t)-f(x_0)}{t-x_0}\,dt > 4M\eta-4M\left\{(a-x_0)+\sum_{n=1}^{\infty}(b_n-a_n)+(x_0+\eta-b)\right\}$$
$$> 4M\eta-2M\eta,$$

by (4.9). Hence

$$\frac{1}{\eta}\int_{x_0}^{x_0+\eta} \frac{f(t)-f(x_0)}{t-x_0}\,dt > 2M.$$

Since $\eta<\delta$, where δ is arbitrary,

$$BD^+f(x_0) \geqslant 2M,$$

which contradicts (4.2).

The set H is therefore of measure zero and $f(x)$ is approximately differentiable almost everywhere in E.

Proof that $BD^-f(x)$ and $BD_-f(x)$ are finite almost everywhere in E.

5. THEOREM II. *If $BD^+f(x)$ and $BD_+f(x)$ are finite at points of a measurable set E, then $BD^-f(x)$ and $BD_-f(x)$ are finite almost everywhere in E.*

By Theorem I, $ADf(x)$ exists and is finite almost everywhere in E.

Let α be any positive number. Then, by a method similar to that used in Theorem I, we can find M, δ_1, and a sub-set F of E such that

$$mF > mE - \tfrac{1}{2}\alpha, \tag{5.1}$$

$$|ADf(x)| \leqslant M, \tag{5.2}$$

and

$$\left| \frac{1}{h} \int_x^{x+h} \frac{f(t)-f(x)}{t-x}\, dt \right| \leqslant 2M, \tag{5.3}$$

whenever $0 < h \leqslant \delta_1$ and x is a point of F.

Let F_1 be the set of points x of F such that

$$\frac{f(t)-f(x)}{t-x} < 2M \tag{5.4}$$

for all points t of $(x, x+h)$, with the possible exception of a set of measure less than $|h|/8$, whenever $0 < |h| \leqslant \delta_2$.

Let F_2 be the sub-set of F obtained by replacing (5.4) by

$$\frac{f(t)-f(x)}{t-x} > -2M. \tag{5.5}$$

It can be shown* that F_1 and F_2 are measurable.

In view of (5.2) we can choose δ_2 in such a way that

$$mF_1 > mF - \tfrac{1}{4}\alpha,$$

$$mF_2 > mF - \tfrac{1}{4}\alpha.$$

* See Burkill and Haslam-Jones, 1, 349, for the case when $0 < h \leqslant \delta_2$. The extension to $0 < |h| \leqslant \delta_2$ is easy.

Then, if G is the common part of F_1 and F_2,

$$mG > mF - \tfrac{1}{2}a$$

$$> mE - a,$$

by (5.1).

Let P be a perfect sub-set of G such that

$$mP > mE - a. \tag{5.6}$$

It follows from (5.4) and (5.5) that P consists of points x such that

$$-2M(t-x) < f(t) - f(x) < 2M(t-x)$$

for all points t of $(x, x+h)$, with the possible exception of a set of measure less than $|h|/4$, whenever $0 < |h| \leqslant \delta_2$.

Let x_1 and x_2 be any two points of P such that $0 < x_2 - x_1 \leqslant \delta_2$. Then we can find a point y of (x_1, x_2) such that

$$-2M(y-x_1) < f(y) - f(x_1) < 2M(y-x_1)$$

and

$$-2M(x_2-y) < f(x_2) - f(y) < 2M(x_2-y).$$

Hence

$$-2M(x_2-x_1) < f(x_2) - f(x_1) < 2M(x_2-x_1).$$

Thus

$$\left| \frac{f(x_2)-f(x_1)}{x_2-x_1} \right| < 2M, \tag{5.7}$$

whenever x_1 and x_2 belong to P and $0 < |x_2-x_1| \leqslant \delta_2$.

Let x_0 be a point of density of P. We can find a positive number $\delta \leqslant \min(\delta_1, \delta_2)$, such that $x_0 - \delta$ is a point of P and

$$mP_h > \tfrac{1}{2}|h|, \tag{5.8}$$

whenever $-\delta \leqslant h < 0$, where P_h denotes the common part of P and the closed interval (x_0, x_0+h).

Let h be such that $-\delta \leqslant h < 0$, and let (a_1, b_1), (a_2, b_2), ..., (a_n, b_n), ..., where $b_n < a_n$, be the complementary intervals of P_h. If x_0+h is not a point of P, suppose that it is contained in the complementary interval (a, b) of P, where $b < a$. Then, since $x_0 - \delta$ is a point of P, $x_0 - b \leqslant \delta$. If x_0+h is a point of P, we suppose that $a = b = x_0+h$.

By (5.7), since $|h| \leqslant \delta_2$,

$$\int_{P_h} \left| \frac{f(t)-f(x_0)}{t-x_0} \right| dt < 2M\,|h|. \tag{5.9}$$

Suppose that $b_n \leqslant \beta_n < a_n \leqslant a_n$. Then

$$(5.10) \qquad \int_{a_n}^{\beta_n} \frac{f(t)-f(x_0)}{t-x_0}\,dt = \int_{a_n}^{b_n} \frac{f(t)-f(x_0)}{t-x_0}\,dt - \int_{\beta_n}^{b_n} \frac{f(t)-f(x_0)}{t-x_0}\,dt.$$

Now

$$(5.11) \qquad \int_{a_n}^{b_n} \frac{f(t)-f(x_0)}{t-x_0}\,dt = \int_{a_n}^{b_n} \frac{f(t)-f(b_n)}{t-x_0}\,dt + \int_{a_n}^{b_n} \frac{f(b_n)-f(x_0)}{t-x_0}\,dt.$$

It is clear that

$$\int_{a_n}^{b_n} \frac{f(t)-f(b_n)}{t-x_0}\,dt = \int_{b_n}^{a_n} \frac{f(t)-f(b_n)}{t-b_n}\,\frac{t-b_n}{x_0-t}\,dt$$

$$= \frac{a_n-b_n}{x_0-a_n}\int_{\xi}^{a_n} \frac{f(t)-f(b_n)}{t-b_n}\,dt,$$

where $b_n \leqslant \xi \leqslant a_n$, by the second mean value theorem, since $(t-b_n)/(x_0-t)$ is positive and increases in (b_n, a_n) and $\{f(t)-f(b_n)\}/(t-b_n)$ is, by Lemma IV, integrable in (b_n, a_n). Hence

$$\left|\int_{a_n}^{b_n} \frac{f(t)-f(b_n)}{t-x_0}\,dt\right| = \frac{a_n-b_n}{x_0-a_n}\left|\int_{b_n}^{a_n} \frac{f(t)-f(b_n)}{t-b_n}\,dt - \int_{b_n}^{\xi} \frac{f(t)-f(b_n)}{t-b_n}\,dt\right|$$

$$\leqslant \frac{a_n-b_n}{x_0-a_n}\left\{\left|\int_{b_n}^{a_n} \frac{f(t)-f(b_n)}{t-b_n}\,dt\right| + \left|\int_{b_n}^{\xi} \frac{f(t)-f(b_n)}{t-b_n}\,dt\right|\right\}$$

$$\leqslant \frac{a_n-b_n}{x_0-a_n}\{2M(a_n-b_n)+2M(\xi-b_n)\},$$

by (5.3). In view of (5.8), $(a_n-b_n)/(x_0-a_n) < 1$, so that

$$(5.12) \qquad \left|\int_{a_n}^{b_n} \frac{f(t)-f(b_n)}{t-x_0}\,dt\right| < 4M(a_n-b_n).$$

Also

$$\left|\int_{a_n}^{b_n} \frac{f(b_n)-f(x_0)}{t-x_0}\,dt\right| \leqslant \left|\frac{f(b_n)-f(x_0)}{a_n-x_0}\right|(a_n-b_n)$$

$$= \left|\frac{f(b_n)-f(x_0)}{b_n-x_0}\right|\frac{x_0-b_n}{x_0-a_n}\,(a_n-b_n)$$

$$< 2M\,\frac{x_0-b_n}{x_0-a_n}\,(a_n-b_n),$$

by (5.7), since b_n and x_0 belong to P and $x_0-b_n \leqslant \delta_2$. It follows from

(5.8) that $(x_0-b_n)/(x_0-a_n)<2$. Hence

$$\left|\int_{a_n}^{b_n}\frac{f(b_n)-f(x_0)}{t-x_0}\,dt\right|<4M(a_n-b_n).$$

Combined with (5.11) and (5.12), this yields

$$\left|\int_{a_n}^{b_n}\frac{f(t)-f(x_0)}{t-x_0}\,dt\right|<8M(a_n-b_n).$$

In the same way,

$$\left|\int_{\beta_n}^{b_n}\frac{f(t)-f(x_0)}{t-x_0}\,dt\right|<8M(a_n-b_n).$$

Hence, in view of (5.10),

$$\left|\int_{a_n}^{\beta_n}\frac{f(t)-f(x_0)}{t-x_0}\,dt\right|<16M(a_n-b_n). \tag{5.13}$$

Since $b\leqslant x_0+h\leqslant a$ and $x_0-b\leqslant\delta$, it can be proved in exactly the same way that

$$\left|\int_{a}^{x_0+h}\frac{f(t)-f(x_0)}{t-x_0}\,dt\right|\leqslant 16M(a-b). \tag{5.14}$$

It follows from (5.13) that

$$\sum_{n=1}^{\infty}\left|\int_{a_n}^{\beta_n}\frac{f(t)-f(x_0)}{t-x_0}\,dt\right|<\infty.$$

Together with (5.9) this shows that $\{f(t)-f(x_0)\}/(t-x_0)$ is integrable in (x_0, x_0+h) and

$$\int_{x_0}^{x_0+h}\frac{f(t)-f(x_0)}{t-x_0}\,dt$$

$$=\int_{P_h}\frac{f(t)-f(x_0)}{t-x_0}\,dt+\sum_{n=1}^{\infty}\int_{a_n}^{b_n}\frac{f(t)-f(x_0)}{t-x_0}\,dt+\int_{a}^{x_0+h}\frac{f(t)-f(x_0)}{t-x_0}\,dt.$$

Hence

$$\left|\int_{x_0}^{x_0+h}\frac{f(t)-f(x_0)}{t-x_0}\,dt\right|$$

$$\leqslant\left|\int_{P_h}\frac{f(t)-f(x_0)}{t-x_0}\,dt\right|+\sum_{n=1}^{\infty}\left|\int_{a_n}^{b_n}\frac{f(t)-f(x_0)}{t-x_0}\,dt\right|+\left|\int_{a}^{x_0+h}\frac{f(t)-f(x_0)}{t-x_0}\,dt\right|$$

$$<2M|h|+16M\sum_{n=1}^{\infty}(a_n-b_n)+16M(a-b),$$

by (5.9), (5.13), and (5.14).

Since $x_0-b<\delta$, it follows from (5.8) that

$$\sum_{n=1}^{\infty}(a_n-b_n)+(a-b)<x_0-a\leqslant|h|,$$

so that
$$\left|\int_{x_0}^{x_0+h}\frac{f(t)-f(x_0)}{t-x_0}\,dt\right|<18M|h|.$$

Thus
$$\left|\frac{1}{h}\int_{x_0}^{x_0+h}\frac{f(t)-f(x_0)}{t-x_0}\,dt\right|<18M,$$

provided that $-\delta\leqslant h<0$. Hence

$$-18M\leqslant BD_-f(x_0)\leqslant BD^-f(x_0)\leqslant 18M,$$

whenever x_0 is a point of density of P.

The left derivates $BD_-f(x)$ and $BD^-f(x)$ are, therefore, finite almost everywhere in P, and hence, by (5.6), since a is arbitrary, almost everywhere in E.

Existence of the Borel derivative.

6. It can now be proved that the Borel derivates are equal to the approximate derivative almost everywhere in E. The proof is similar to that given by Khintchine* for the finite Borel derivative $BDf(x)$ of a continuous function $f(x)$.

We first consider the case when the approximate derivative is zero.

THEOREM III. *If $ADf(x)$ is zero at points of a measurable set E in which $BD^+f(x)$ and $BD_+f(x)$ are finite, then $f(x)$ has a Borel derivative equal to zero almost everywhere in E.*

By Theorem II, $BD^-f(x)$ and $BD_-f(x)$ are finite in a sub-set E_1 of E of measure equal to that of E.

Let a be any positive number. We can find M, δ_1, and a sub-set F of E_1 such that

$$mF>mE-\tfrac{1}{2}a,$$

and, if x is a point of F,

$$\left|\frac{1}{h}\int_{x}^{x+h}\frac{f(t)-f(x)}{t-x}\,dt\right|\leqslant 2M, \tag{6.1}$$

whenever $0<|h|\leqslant\delta_1$.

* Khintchine, **2**, 229–233.

Since $ADf(x)$ is zero in F, it follows from a lemma due to Khintchine* that there is a perfect sub-set P of F such that

$$mP > mE - a, \tag{6.2}$$

and, whenever x is a point of P,

$$\frac{f(t)-f(x)}{t-x} \to 0 \tag{6.3}$$

as $t \to x$, t being a point of P.

Let x_0 be a point of density of P.

In view of (6.3), we can find a positive number δ_2 such that

$$\left|\frac{f(t)-f(x_0)}{t-x_0}\right| < 2M, \tag{6.4}$$

whenever t is a point of P and $0 < |t-x_0| \leqslant \delta_2$.

Let δ, P_h, and the complementary intervals

$$(a, b), \quad (a_1, b_1), \quad (a_2, b_2), \quad \ldots, \quad (a_n, b_n), \quad \ldots,$$

be defined exactly as in Theorem II.

It follows from (6.1) and (6.4) that, provided that $-\delta \leqslant h < 0$,

$$\int_{x_0}^{x_0+h} \frac{f(t)-f(x_0)}{t-x_0}\,dt = \int_{P_h} \frac{f(t)-f(x_0)}{t-x_0}\,dt + \sum_{n=1}^{\infty} \int_{a_n}^{b_n} \frac{f(t)-f(x_0)}{t-x_0}\,dt + \int_{a}^{x_0+h} \frac{f(t)-f(x_0)}{t-x_0}\,dt \tag{6.5}$$

and

$$\left| \sum_{n=1}^{\infty} \int_{a_n}^{b_n} \frac{f(t)-f(x_0)}{t-x_0}\,dt + \int_{a}^{x_0+h} \frac{f(t)-f(x_0)}{t-x_0}\,dt \right| < 16M \left\{ \sum_{n=1}^{\infty} (a_n - b_n) + (a-b) \right\}. \tag{6.6}$$

The proof is the same as in Theorem II.

Since x_0 is a point of density of P, it follows from (6.6) that

$$\lim_{h \to -0} \left\{ \frac{1}{h} \sum_{n=1}^{\infty} \int_{a_n}^{b_n} \frac{f(t)-f(x_0)}{t-x_0}\,dt + \frac{1}{h} \int_{a}^{x_0+h} \frac{f(t)-f(x_0)}{t-x_0}\,dt \right\} = 0. \tag{6.7}$$

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* Khintchine, 2, 228.

By (6.3),

$$\lim_{h\to -0}\frac{1}{h}\int_{P_h}\frac{f(t)-f(x_0)}{t-x_0}\,dt=0. \tag{6.8}$$

In view of (6.5), (6.7), and (6.8),

$$\lim_{h\to -0}\frac{1}{h}\int_{x_0}^{x_0+h}\frac{f(t)-f(x_0)}{t-x_0}\,dt=0$$

Similarly it can be proved that

$$\lim_{h\to +0}\frac{1}{h}\int_{x_0}^{x_0+h}\frac{f(t)-f(x_0)}{t-x_0}\,dt=0.$$

Thus

$$BDf(x_0)=0,$$

whenever x_0 is a point of density of P. In view of (6.2), since a is arbitrary, $f(x)$ has a Borel derivative equal to zero almost everywhere in E.

We are now able to prove the main theorem.

THEOREM. *If $f(x)$ is integrable in the special Denjoy sense and $BD^+f(x)$ and $BD_+f(x)$ are finite at points of a measurable set E, then, almost everywhere in E, $f(x)$ has a Borel derivative, and an approximate derivative equal in value to the Borel derivative.*

By Theorem I, $ADf(x)$ exists and is finite in a sub-set E_1 of E of measure equal to that of E. The function $f_1(x)$, which is equal to $ADf(x)$ in E_1 and is zero outside E_1, is measurable and finite.

By a theorem due to Lusin, there is a continuous function $F(x)$ such that

$$F'(x)=f_1(x)$$

almost everywhere. Let E_2 be the set of points of E_1 at which this equation is satisfied. Then

$$mE_2=mE,$$

and

$$F'(x)=ADf(x), \tag{6.9}$$

if x is a point of E_2.

Let

$$g(x)=F(x)-f(x).$$

It is easily seen that

$$BD^+g(x)=F'(x)-BD_+f(x)$$

and

$$BD_+g(x)=F'(x)-BD^+f(x),$$

so that $BD^+g(x)$ and $BD_+g(x)$ are finite in E_2.

Also, by (6.9), $$ADg(x) = 0,$$

whenever x is a point of E_2.

Hence, by Theorem III, $g(x)$ has a Borel derivative equal to zero almost everywhere in E_2. It follows from (6.9) that $f(x)$ has a Borel derivative equal to $ADf(x)$ almost everywhere in E_2, and hence almost everywhere in E.

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ON FOURIER TRANSFORMS

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1. *Introduction.*

It has been shown by Fekete† that a necessary and sufficient condition for a given trigonometrical series

$$\sum_1^\infty (a_n \cos n\theta + b_n \sin n\theta) \tag{1.1}$$

to be the Fourier series of a bounded function is that

$$\left| \sum_{n<\omega} \left(1-\frac{n}{\omega}\right)(a_n \cos n\theta + b_n \sin n\theta) \right| \leqslant M, \tag{1.2}$$

for all $\omega \geqslant 0$ and $0 \leqslant \theta \leqslant 2\pi$, M being an absolute constant. The corresponding result for power series had been previously published by Landau‡.

The object of this paper is to investigate the corresponding problem in the theory of Fourier transforms. A function $F(x)$ is said to be the Fourier cosine transform of another function $f(x)$ if

$$F(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \cos xu f(u)\, du, \tag{1.31}$$

where the integral is interpreted in a generalized sense§. The theory

† Fekete, **3**.

‡ Landau, **9**, 22.

§ See, *e.g.*, Hobson, **8**; Bochner, **3**, 169–178; Titchmarsh, **13**; Plancherel, **10**; Wiener, **14**, 46–71. The theory has recently been extended by Berry, **1**. However, the reader need have only a slight acquaintance with the ordinary theory.

usually requires the existence of the inverse relation

$$f(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \cos xu\, F(u)\, du \tag{1.32}$$

in some generalized sense.

Fekete's condition (1.1) suggests the following definition.

DEFINITION 1. *A real even function $f(x)$ will be said to belong to the class H when it is integrable L in every finite range and such that*

$$\left| \int_0^\omega \left(1 - \frac{u}{\omega}\right) \cos xu f(u)\, du \right| \leqslant M, \tag{H}$$

for all $\omega \geqslant 0$ and $x \geqslant 0$.

We show, in §2, Theorem 1, that, when $f(x)$ belongs to H, the integral (1.31) is summable $(C, 1)$ almost everywhere to a bounded function $F(x)$, and, further, the integral (1.32) is summable $(C, 2)$ almost everywhere to $f(x)$. It is consistent with the ordinary theory† to say that $F(x)$ is the cosine transform of the function $f(x)$ of the class H. This corresponds, in Fekete's problem, to the result that (1.2) is sufficient for (1.1) to be the Fourier series of a bounded function. The proof of our Theorem 1 has, however, little in common with Fekete's method. But it is possible to construct a proof of Fekete's theorem on the lines of Landau's proof of the corresponding theorem for power series. The proof of Theorem 1 is analogous to this.

A difficulty arises when we consider the converse problem. A bounded integrable function has a Fourier series, but it need not have a Fourier transform. However, if $F(x)$ is bounded, then $F(x)\, x$ has its square integrable in (δ, ∞), $\delta > 0$. Hence, by the theory of Fourier transforms, the integral‡

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \frac{\sin xu}{u} F(u)\, du$$

is summable $(C, 1)$ for almost all x to a function which we denote by $\mathfrak{F}_1(x)$. The summability will be uniform in any interval in which $\mathfrak{F}_1(x)$ is continuous. This suggests the following definition.

DEFINITION 2. *A bounded function $F(x)$ will be said to belong to B^* if it is such that the function $\mathfrak{F}_1(x)$ defined by*

$$\mathfrak{F}_1(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \frac{\sin xu}{u} F(u)\, du \qquad (C, 1) \tag{1.41}$$

† This definition is, in fact, a slight extension of the case $p = \infty$ of Berry's definition, **1**, 230.

‡ Hobson, **8**, 748. Plancherel, **11**, 321.

is an indefinite integral, i.e. if there exists $\mathfrak{F}(x)$ *such that*

$$\mathfrak{F}_1(x) = \int_0^x \mathfrak{F}(u)\,du. \tag{1.42}$$

We show, in § 3, Theorem 5, that, when $F(x)$ belongs to B^*, the function $\mathfrak{F}(x)$, defined by (1.4), belongs to H and has $F(x)$ for its cosine transform. Thus we show that a function $f(x)$ of H has a cosine transform which belongs to B^* (Theorem 2), and a function $F(x)$ of B^* is the cosine transform of a function which belongs to H.

It is interesting to consider the following problem in the light of Theorems 1 and 5.

What are necessary and sufficient conditions for the Fourier transform of a given function to be bounded?

Condition H is shown to be sufficient for the cosine transform of the given function to exist and to be bounded. We cannot, however, state that the condition is necessary until we have settled the sense in which the transform of the given function exists. But it will be seen from the results of § 3 that no theory of Fourier transforms has been developed in which condition H fails to be a necessary condition for boundedness†.

The relation between the functions of H and their cosine transforms of B^* is not symmetrical. This leads us to the following definition.

DEFINITION 3. *If a function* $f(x)$ *is bounded and belongs to* H, *it will be said to belong to* HB.

In § 4, Theorem 8, we show that the cosine transform of a function of HB also belongs to HB. Thus there is complete symmetry in the class HB.

Finally, in § 5, we determine the class of all the self-reciprocal functions of HB, *i.e.* the class of all the functions of HB which satisfy

$$f(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \cos xu f(u)\,du, \tag{C, 1}$$

or, what is the same thing, the class of all the bounded functions which satisfy

$$f(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{d}{dx} \int_0^\infty \frac{\sin xu}{u} f(u)\,du.$$

† Necessary and sufficient conditions for the Fourier transform of a given function to belong to certain classes have also been considered by Berry, 2.

A theory of self-reciprocal functions has been developed by Hardy and Titchmarsh†, and our results depend on theirs.

There are corresponding results for the sine transforms, but, since they are so completely analogous to their cosine analogues, we shall not consider them. I intend to devote a second paper‡ to the extension of these results to Hankel transforms. These will, of course, include the theorems for the sine transforms.

My warmest thanks are due to Prof. G. H. Hardy for his interest in the problem. Prof. Hardy kindly emphasized the connection between the problem for integrals and the corresponding one for Fourier series and also called my attention to Pollard's paper.

2. *The class H.*

In this section we establish the existence of the cosine transforms of functions of the class H. The results may be expressed in the following theorems.

THEOREM 1. *If $f(x)$ belongs to H, then the integral*

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \cos xu f(u)\, du$$

is summable $(C, 1)$, almost everywhere, to a bounded function which we denote by $F(x)$. The function $F(x)$ is the cosine transform of $f(x)$ and

$$f(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \cos xu\, F(u)\, du, \qquad (C, 2) \tag{2.1}$$

almost everywhere.

This is a consequence of Lemmas 3 and 4 below. An alternative form of Theorem 1 is

THEOREM 2. *If $f(x)$ belongs to H, then*

$$F(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{d}{dx} \int_0^\infty \frac{\sin xu}{u} f(u)\, du \qquad (C, 1)$$

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† Hardy and Titchmarsh, 6.

‡ "On Hankel transforms", to appear in these *Proceedings*.

almost everywhere, $F(x)$ belongs to B^*, and

$$f(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{d}{dx} \int_0^\infty \frac{\sin xu}{u} F(u)\, du, \qquad (C, 1)$$

almost everywhere.

The next result is a form of Parseval's theorem for functions of the class H.

THEOREM 3. *If $f(x)$ belongs to H, $G(x)$ belongs to $L(0, \infty)$, and if $g(x)$ is the cosine transform of $G(x)$, then*

$$\int_0^\infty F(x)\, G(x)\, dx = \int_0^\infty f(x)\, g(x)\, dx. \qquad (C, 1)$$

In certain circumstances the integral (2.1) will be summable $(C, 1)$ almost everywhere.

THEOREM 4. *If $f(x)$ belongs to H, and if $\int_1^\infty \frac{|f(u)|}{u^2}\, du$ is convergent, then the integral (2.1) is summable $(C, 1)$, almost everywhere.*

We require the following lemmas.

LEMMA 1. *If $f(x)$ belongs to H, then the integral*

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty e^{-\eta u} \cos xu f(u)\, du \qquad (2.2)$$

is summable $(C, 1)$, for all $\eta > 0$, to a function $F(\eta, x)$. Further, $F(\eta, x)$ is bounded, for all $\eta > 0$, and as $\eta \to 0$ $F(\eta, x)$ tends boundedly to a limit $F(x)$, for almost all x.

Proof. Write

$$f_1(\omega) = \int_0^\omega f(u)\, du, \qquad f_2(\omega) = \int_0^\omega f_1(u)\, du,$$

$$f_1(\omega, x) = \int_0^\omega \cos xu f(u)\, du, \qquad f_2(\omega, x) = \int_0^\omega f_1(u, x)\, du.$$

Then

$$\int_0^\omega (\omega - u) \cos xu f(u)\, du = \int_0^\omega f_1(u, x)\, du = f_2(\omega, x).$$

Hence, by (H),

$$|f_2(\omega, x)| \leqslant M\omega, \qquad (2.4)$$

for all x. Consider

$$\int_0^\omega (\omega-u)e^{-\eta u}\cos xu f(u)\,du$$

$$= -\int_0^\omega \frac{d}{du}\{(\omega-u)e^{-\eta u}\}f_1(u,x)\,du$$

$$= e^{-\eta\omega}f_2(\omega,x)+2\eta\int_0^\omega e^{-\eta u}f_2(u,x)\,du+\eta^2\int_0^\omega(\omega-u)e^{-\eta u}f_2(u,x)\,du$$

$$= I_1+I_2+I_3.$$

By (2.4), η being positive, $I_1 = o(1)$ and $I_2 = O(1)$ as $\omega\to\infty$. Hence

$$\lim_{\omega\to\infty}\int_0^\omega\left(1-\frac{u}{\omega}\right)e^{-\eta u}\cos xu\, f(u)\,du = \eta^2\lim_{\omega\to\infty}\int_0^\omega\left(1-\frac{u}{\omega}\right)e^{-\eta u}f_2(u,x)\,du$$

$$= \eta^2\int_0^\infty e^{-\eta u}f_2(u,x)\,du,$$

since this integral converges, by (2.4). This proves that the integral (2.2) is summable $(C,1)$, for all $\eta>0$, to the function

$$F(\eta,x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\eta^2\int_0^\infty e^{-\eta u}f_2(u,x)\,du.$$

Further, by (2.4),

$$|F(\eta,x)| \leqslant \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\eta^2 M\int_0^\infty ue^{-\eta u}\,du = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}M,$$

i.e. $F(\eta,x)$ is bounded in the half plane $\eta>0$. Now, by definition, $F(\eta,x)$ is the real part of the function

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty e^{-su}f(u)\,du,$$

where $s=\eta+ix$. Hence $F(\eta,x)$ is a bounded harmonic function in the half plane $\eta>0$. It follows by Fatou's theorem that $F(\eta,x)$ tends boundedly to a limit $F(x)$, as η tends to zero, for almost all x. This completes the proof of the lemma.

LEMMA 2. *Let $f(x)$ belong to H, and let $F(x)$ be defined as in Lemma 1. Further, let $G(x)$ belong to $L(0,\infty)$, and let its cosine transform be $g(x)$. Then*

$$\int_0^\infty F(x)\,G(x)\,dx = \lim_{\eta\to0}\lim_{\omega\to\infty}\int_0^\omega\left(1-\frac{u}{\omega}\right)e^{-\eta u}f(u)\,g(u)\,du.$$

Proof. Since $G(x)$ belongs to $L(0, \infty)$, we have, by Lemma 1,

$$\int_0^\infty F(x)\,G(x)\,dx = \lim_{\eta\to 0}\int_0^\infty F(\eta, x)\,G(x)\,dx$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \lim_{\eta\to 0}\lim_{\omega\to\infty}\int_0^\infty G(x)\,dx\int_0^\omega \left(1-\frac{u}{\omega}\right)e^{-\eta u}\cos xu\,f(u)\,du,$$

as, for fixed η, the integral with respect to u tends boundedly to a limit as ω tends to infinity. Hence

$$\int_0^\infty F(x)\,G(x)\,dx = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \lim_{\eta\to 0}\lim_{\omega\to\infty}\int_0^\omega \left(1-\frac{u}{\omega}\right)e^{-\eta u} f(u)\,du\int_0^\infty \cos ux\,G(x)\,dx$$

$$= \lim_{\eta\to 0}\lim_{\omega\to\infty}\int_0^\omega \left(1-\frac{u}{\omega}\right)e^{-\eta u} f(u)\,g(u)\,du,$$

the required result.

LEMMA 3. *If $f(x)$ belongs to H, and if $F(x)$ is defined as in Lemma 1, then*

$$F(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty \cos xu\,f(u)\,du, \qquad (C, 1), \tag{2.5}$$

almost everywhere.

Proof. In Lemma 2 take

$$g(u) = \begin{cases} \left(\dfrac{2}{\pi}\right)^{\frac{1}{2}}\left(1-\dfrac{u}{\omega}\right)\cos xu & (0\leqslant u\leqslant\omega), \\ 0 & (\omega\leqslant u). \end{cases} \tag{2.61}$$

Then

$$G(u) = \frac{2}{\pi\omega}\left[\frac{\sin^2\frac{1}{2}\omega(x+u)}{(x+u)^2} + \frac{\sin^2\frac{1}{2}\omega(x-u)}{(x-u)^2}\right], \tag{2.62}$$

and the hypotheses of Lemma 2 are obviously satisfied. Hence

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\omega\left(1-\frac{u}{\omega}\right)\cos xu\,f(u)\,du = \frac{2}{\pi\omega}\int_{-\infty}^\infty F(x+\xi)\,\frac{\sin^2\frac{1}{2}\omega\xi}{\xi^2}\,d\xi,$$

since in this case the integral on the right is obviously convergent. We divide this integral into three parts, viz.

$$\frac{2}{\pi\omega}\left\{\int_{-\infty}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\infty}\right\} = I_1 + I_2 + I_3.$$

Now $F(x)$ is bounded, so, if $\delta > 0$, we get

$$|I_3| \leqslant \frac{M}{\omega} \int_\delta^\infty \frac{d\xi}{\xi^2} \to 0$$

as $\omega \to \infty$. Similarly, $I_1 \to 0$ as $\omega \to \infty$. Again,

$$I_2 = \frac{2}{\pi\omega} \int_0^\delta \{F(x+\xi) + F(x-\xi) - 2F(x)\} \frac{\sin^2 \frac{1}{2}\omega\xi}{\xi^2} d\xi + \frac{2}{\pi} F(x) \int_0^{\omega\delta} \left(\frac{\sin\theta}{\theta}\right)^2 d\theta,$$

and it follows by a classical argument† that the first integral tends to zero at every point at which

$$\frac{1}{\xi} \int_0^\xi |F(x+u) + F(x-u) - 2F(x)|\, du \to 0$$

as ξ tends to zero, and so almost everywhere. This gives (2.5).

Proof of Theorem 3. By Lemma 3 and (H), and using the fact that $G(x)$ belongs to $L(0, \infty)$, we have

$$\begin{aligned}
\int_0^\infty G(x)\, F(x)\, dx &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \lim_{\omega\to\infty} \int_0^\infty G(x)\, dx \int_0^\omega \left(1 - \frac{u}{\omega}\right) \cos xu\, f(u)\, du \\
&= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \lim_{\omega\to\infty} \int_0^\omega \left(1 - \frac{u}{\omega}\right) f(u)\, du \int_0^\infty \cos ux\, G(x)\, dx \\
&= \lim_{\omega\to\infty} \int_0^\omega \left(1 - \frac{u}{\omega}\right) f(u)\, g(u)\, du,
\end{aligned}$$

the required result.

LEMMA 4. *If $f(x)$ belongs to H, and if $F(x)$ is defined as in Lemma* 1, *then*

$$f(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \cos xu\, F(u)\, du, \qquad (C, 2)$$

almost everywhere.

Proof. In Theorem 3 take

$$G(u) = \begin{cases} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(1 - \frac{u}{z}\right)^2 \cos xu & (0 \leqslant u \leqslant z), \\ 0 & (z \leqslant u). \end{cases}$$

† Hobson, **8**, 561.

Then

$$g(u) = \frac{2}{\pi}\int_0^z \left(1-\frac{\xi}{z}\right)^2 \cos x\xi \cos u\xi \, d\xi$$

$$= \frac{2}{\pi z^2}\left[\frac{z}{(x+u)^2}+\frac{z}{(x-u)^2}-\left\{\frac{\sin z(x+u)}{(x+u)^3}+\frac{\sin z(x-u)}{(x-u)^3}\right\}\right].$$

Now, by Theorem 3, we easily see that

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^z \left(1-\frac{u}{z}\right)^2 \cos xu\, F(u)\, du$$

$$= \int_0^\infty g(u) f(u)\, du \qquad (C, 1)$$

$$= \frac{2}{\pi z^2}\lim_{\omega\to\infty}\int_{-\omega}^{\omega}\left(1-\frac{|u|}{\omega}\right)\left\{\frac{z}{u^2}-\frac{\sin zu}{u^3}\right\} f(x+u)\, du,$$

provided that this limit exists. We divide this integral into five parts, viz.:

$$\frac{2}{\pi z^2}\int_{-\delta}^{\delta}\left(1-\frac{|u|}{\omega}\right)\left\{\frac{z}{u^2}-\frac{\sin zu}{u^3}\right\} f(x+u)\, du+\frac{2}{\pi z}\int_\delta^\omega\left(1-\frac{u}{\omega}\right)\frac{1}{u^2} f(x+u)\, du$$

$$-\frac{2}{\pi z^2}\int_\delta^\omega\left(1-\frac{u}{\omega}\right)\frac{\sin zu}{u^3} f(x+u)\, du+\ldots$$

$$= \frac{2}{\pi z}\left\{\frac{1}{z} I_1(\omega)+I_2(\omega)+\frac{1}{z} I_3(\omega)+I_4(\omega)+\frac{1}{z} I_5(\omega)\right\}.$$

Now, by two integrations by parts,

$$I_2(\omega) = -\left(1-\frac{\delta}{\omega}\right)\frac{1}{\delta^2} f_1(x+\delta)+\frac{1}{\omega^3} f_2(x+\omega)-\frac{1}{\omega} f_2(x+\delta)\left\{\frac{1}{\delta^2}+2\,\frac{\omega-\delta}{\delta^3}\right\}$$

$$+\frac{1}{\omega}\int_\delta^\omega f_2(x+u)\,\frac{d^2}{du^2}\left\{(\omega-u)\,\frac{1}{u^2}\right\} du.$$

But, by (2.4), $|f_2(x+u)| \leqslant M(x+u)$, and hence

$$\lim_{\omega\to\infty} I_2(\omega) = -\frac{1}{\delta^2} f_1(x+\delta)-\frac{2}{\delta^3} f_2(x+\delta)+6\int_\delta^\infty \frac{f_2(x+u)}{u^4}\, du.$$

Therefore

$$\frac{2}{\pi z}\lim_{\omega\to\infty} I_2(\omega)\to 0,$$

as $z\to\infty$, uniformly for $0 \leqslant x \leqslant X$ and every finite X.

Similarly we have, using (2.4),

$$\frac{2}{\pi z^2}\lim_{\omega\to\infty} I_3(\omega) = -\frac{2}{\pi z^2}\lim_{\omega\to\infty}\frac{1}{\omega}\int_\delta^\omega f_2(x+u)\frac{d^2}{du^2}\left\{(\omega-u)\frac{\sin zu}{u^3}\right\}du+o(1)$$

$$= -\frac{2}{\pi z^2}\int_\delta^\infty f_2(x+u)\frac{d^2}{du^2}\left(\frac{\sin zu}{u^3}\right)du+o(1)$$

$$= \frac{2}{\pi}\int_\delta^\infty f_2(x+u)\frac{\sin zu}{u^3}\,du+o(1),$$

and this term tends to zero as z tends to infinity by the Riemann-Lebesgue theorem†. Similarly we can deal with I_4 and I_5. Hence

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^z\left(1-\frac{u}{z}\right)^2\cos xu\,F(u)\,du$$

$$= \frac{2}{\pi z^2}\int_{-\delta}^{\delta}\left\{\frac{z}{u^2}-\frac{\sin zu}{u^3}\right\}f(x+u)\,du+o(1)$$

$$= \frac{2}{\pi z^2}\int_0^\delta\{f(x+u)+f(x-u)-2f(x)\}\left\{\frac{z}{u^2}-\frac{\sin zu}{u^3}\right\}du$$

$$+\frac{4}{\pi z^2}f(x)\int_0^{z\delta}\left\{\frac{1}{\theta^2}-\frac{\sin\theta}{\theta^3}\right\}d\theta+o(1).$$

Of the two terms in the second member the second tends to $f(x)$ as z tends to infinity, while it follows, by a classical argument, that the first tends to zero at every point at which

$$\frac{1}{\xi}\int_0^\xi\{f(x+u)+f(x-u)-2f(x)\}\,du\to 0$$

as ξ tends to zero, and so almost everywhere. Hence we get

$$\lim_{z\to\infty}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^z\left(1-\frac{u}{z}\right)^2\cos xu\,F(u)\,du = f(x),$$

almost everywhere, and this completes the proof of the lemma.

Proof of Theorem 2. It follows at once from (H) and (2.5) that

$$\int_0^x F(u)\,du = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\lim_{\omega\to\infty}\int_0^\omega\left(1-\frac{u}{\omega}\right)\frac{\sin xu}{u}f(u)\,du,$$

i.e.

$$F(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{d}{dx}\int_0^\infty\frac{\sin xu}{u}f(u)\,du, \qquad (C,\,1)$$

† Hobson, **8**, 515.

as required. We can prove the remaining part either along the lines of Lemma 4 or we can deduce it from the Plancherel-Titchmarsh theory as follows. Since $F(x)$ is bounded, $F(x)/x$ belongs to $L^2(1, \infty)$, and therefore the integral

$$\int_1^\infty \frac{\sin xu}{u} F(u)\,du$$

converges in mean square†. Hence, if

$$h(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \frac{\sin xu}{u} F(u)\,du,$$

then

$$\int_0^x h(u)\,du = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \frac{1-\cos xu}{u^2} F(u)\,du$$

$$= \frac{2}{\pi} \lim_{\omega\to\infty} \int_0^\omega \left(1-\frac{u}{\omega}\right) f(u)\,du \int_0^\infty \frac{1-\cos xt}{t^2} \cos ut\,dt.$$

But

$$\int_0^\infty \frac{1-\cos xt}{t^2} \cos ut\,dt = \begin{cases} \frac{1}{2}\pi(x-u) & (0 \leqslant u \leqslant x), \\ 0 & (x \leqslant u). \end{cases}$$

Therefore

$$\int_0^x h(u)\,du = \lim_{\omega\to\infty} \int_0^x \left(1-\frac{u}{\omega}\right)(x-u) f(u)\,du$$

$$= \int_0^x (x-u) f(u)\,du = f_2(x).$$

Hence, if we take $h(x)$ to be the derivative of its integral, we get $h(x) = f_1(x)$, *i.e.*

$$f_1(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \frac{\sin xu}{u} F(u)\,du, \qquad (C, 1)$$

and this gives the required result.

Proof of Theorem 4. In Theorem 3 take

$$G(u) = \begin{cases} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(1-\frac{u}{\omega}\right) \cos xu & (0 \leqslant u \leqslant \omega), \\ 0 & (\omega \leqslant u). \end{cases}$$

† Hobson, **8**, 745. Titchmarsh, **13**. Wiener, **14**, 69.

Then, as before,

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^{\omega}\left(1-\frac{u}{\omega}\right)\cos xu\,F(u)\,du=\frac{2}{\pi\omega}\int_{-\infty}^{\infty}f(x+u)\,\frac{\sin^2\frac{1}{2}\omega u}{u^2}\,du$$

$$=\frac{2}{\pi\omega}\left\{\int_{-\infty}^{-\delta}+\int_{-\delta}^{\delta}+\int_{\delta}^{\infty}\right\}$$

$$=I_1+I_2+I_3.$$

But $$|I_3|\leqslant\frac{2}{\pi\omega}\int_{\delta}^{\infty}\frac{|f(x+u)|}{u^2}\,du\to 0$$

as ω tends to infinity by hypothesis. Similarly for I_1, while I_2 is Fejér's integral and tends to $f(x)$ as ω tends to infinity, almost everywhere, as in the proof of Lemma 3. Hence

$$f(x)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^{\infty}\cos xu\,F(u)\,du,\qquad (C,\,1),$$

almost everywhere. This is the required result.

3. *The class B^*.*

It has already been remarked that, if $F(x)$ belongs to B^*, then the integral

$$\int_0^{\infty}\frac{\sin xu}{u}\,F(u)\,du$$

is uniformly summable $(C,\,1)$, for $0\leqslant x\leqslant X$ and every finite X. The principal result of this section is contained in the following theorem. It is a converse of Theorem 2.

THEOREM 5. *If $F(x)$ belongs to B^*, and if*

$$f(x)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{d}{dx}\int_0^{\infty}\frac{\sin xu}{u}\,F(u)\,du,\qquad (3.1)$$

then $f(x)$ belongs to H and has $F(x)$ for its cosine transform, i.e. we have

$$F(x)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^{\infty}\cos xu\,f(u)\,du,\qquad (C,\,1)$$

almost everywhere.

Proof. In virtue of Theorems 1 and 2, it will be sufficient to show that, when (3.1) is satisfied, $f(x)$ belongs to H. We have

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\omega \left(1-\frac{u}{\omega}\right) \cos xu f(u)\,du$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{1}{\omega} \int_0^\omega f_1(u)\{\cos xu + (\omega-u)\,x \sin xu\}\,du$$

$$= \frac{2}{\pi\omega} \int_0^\omega \{\cos xu + (\omega-u)\,x \sin xu\}\,du \lim_{T\to\infty} \int_0^T \left(1-\frac{t}{T}\right) \frac{\sin ut}{t} F(t)\,dt$$

$$= \frac{2}{\pi\omega} \lim_{T\to\infty} \int_0^T \left(1-\frac{t}{T}\right) \frac{F(t)}{t}\,dt \int_0^\omega \{\cos xu + (\omega-u)\,x \sin xu\} \sin tu\,du$$

$$= \frac{1}{\pi\omega} \lim_{T\to\infty} \int_{-T}^T \left(1-\frac{|t|}{T}\right) F(t)\, \frac{1-\cos\omega(t-x)}{(t-x)^2}\,dt$$

$$= \frac{2}{\pi\omega} \int_{-\infty}^\infty \frac{\sin^2 \frac{1}{2}\omega t}{t^2} F(x+t)\,dt.$$

Since $F(x)$ is bounded, this integral converges. Hence

$$\left|\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\omega \left(1-\frac{u}{\omega}\right) \cos xu f(u)\,du\right| \leqslant \frac{2}{\pi} M \int_0^\infty \left(\frac{\sin\theta}{\theta}\right)^2 d\theta = M.$$

Therefore $f(x)$ belongs to H and this completes the proof.

The following is an immediate corollary of Theorems 1 and 5.

COROLLARY. *If $f(x)$ belongs to either H or B^*, then*

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos xu\,du \int_0^\infty \cos uv f(v)\,dv,$$

both integrals being summable $(C, 2)$.

We shall now consider certain cases in which a bounded function $F(x)$ will belong to B^*. Notice that, if $F(x)$ is bounded and such that

$$\int_0^\infty f(x)\,g(x)\,dx = \int_0^\infty F(x)\,G(x)\,dx, \tag{3.2}$$

where $g(x)$ and $G(x)$ are defined by (2.61) and (2.62), so that both integrals converge, then it follows at once that $f(x)$ belongs to H and therefore $F(x)$ belongs to B^*. Now, with these values of $g(x)$ and $G(x)$ the relation (3.2) is true† for all classes of functions for which a theory of Fourier transforms has been developed‡.

† Hardy and Titchmarsh, 7.

‡ See, *e.g.*, Hobson, 8, and Berry, 1.

The next two theorems give sufficient conditions for a given bounded function $F(x)$ to be of the class B^*.

THEOREM 6. *If $F(x)$ is bounded, and if*

$$\left|\int_0^\omega \left(1-\frac{u}{\omega}\right)\cos xu\, F(u)\,du\right| \leqslant \chi(x) \tag{3.3}$$

for all $\omega \geqslant 0$, $x \geqslant 0$, where $\chi(x)$ is integrable L in every finite range, then $F(x)$ belongs to B^*.

Proof. Let

$$h(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \frac{\sin xu}{u}\, F(u)\,du. \qquad (C, 1)$$

Then

$$\begin{aligned} |h(x_2)-h(x_1)| &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \lim_{\omega\to\infty} \left|\int_0^\omega \left(1-\frac{u}{\omega}\right) F(u)\,du \int_{x_1}^{x_2} \cos ut\,dt\right| \\ &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \lim_{\omega\to\infty} \left|\int_{x_1}^{x_2} dt \int_0^\omega \left(1-\frac{u}{\omega}\right)\cos tu\, F(u)\,du\right| \\ &\leqslant \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_{x_1}^{x_2} \chi(t)\,dt, \end{aligned}$$

and it follows that $h(x)$ is absolutely continuous in every finite range. This is the required result.

We deduce as a corollary that, when (3.3) is satisfied, the integral

$$\int_0^\infty \cos xu\, F(u)\,du$$

is summable $(C, 2)$ almost everywhere.

THEOREM 7. *If $F(x)$ is bounded, and if*

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \cos xu\, F(u)\,du$$

converges everywhere to a function which we denote by $f(x)$, and if $f(x)$ is everywhere finite, then $F(x)$ belongs to B^*.

Proof. Pollard† has shown that under these hypotheses

$$f_2(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \frac{1-\cos xu}{u^2}\, F(u)\,du.$$

† Pollard, **12**, 456–8.

From this it follows, as in Theorem 2, that

$$f(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{d}{dx} \int_0^\infty \frac{\sin xu}{u} F(u)\, du,$$

which is the required result.

4. *The class HB.*

Consider the class HB, *i.e.* the class of all the bounded functions which belong to H.

THEOREM 8. *If $f(x)$ belongs to HB, then its cosine transform also belongs to HB.*

Proof. By Theorem 1, if $f(x)$ belongs to H, its cosine transform $F(x)$ exists and is bounded. It remains only to show that, if $f(x)$ is bounded, then $F(x)$ belongs to H. This follows from Theorem 3. As in the proof of Theorem 4, we have

$$\left|\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\omega \left(1 - \frac{u}{\omega}\right) \cos xu\, F(u)\, du\right| = \left|\frac{2}{\pi\omega} \int_{-\infty}^{\infty} \frac{\sin^2 \frac{1}{2}\omega u}{u^2} f(x+u)\, du\right|$$

$$\leqslant \frac{2M}{\pi} \int_0^\infty \left(\frac{\sin\theta}{\theta}\right)^2 d\theta = M,$$

which completes the proof.

For functions of HB, Theorem 4 applies. Also, since the cosine transform of a function of H belongs to B^*, it follows that the class HB is identical with the class HB^*, *i.e.* with the class of all the functions which belong simultaneously to H and B^*.

5. *Self-reciprocal functions.*

It was shown in the last section that the cosine transform of a function of the class HB also belongs to HB. In this section we shall find necessary and sufficient conditions for a function of HB to be its own cosine transform. The completeness of the solution is possible only in virtue of the symmetry in the class HB just described. It has already been remarked that the problem of self-reciprocal functions has been considered by Hardy and Titchmarsh†. We shall make an application of their first

† Hardy and Titchmarsh, 6.

solution† and obtain the result corresponding to their Theorem 1. The method employed is analogous to theirs.

Let $\chi(t)$ be a function (not necessarily real) of the real variable t, defined for all positive and negative values of t and let it be integrable L in every finite range. Further, suppose that $\chi(t)$ satisfies

$$\left|\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)\chi(t)\,x^{-\frac{1}{2}-it}\,dt\right| \leqslant M \tag{5.11}$$

for all $T \geqslant 0$ and $x \geqslant 0$, and that

$$\frac{\chi(t)}{2^{\frac{1}{4}+\frac{1}{2}it}\,\Gamma(\frac{1}{4}+\frac{1}{2}it)} = \textit{even function of } t. \tag{5.12}$$

THEOREM 9. *A necessary and sufficient condition that a function $f(x)$ of HB should be its own cosine transform is that it should be of the form*

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\chi(t)\,x^{-\frac{1}{2}-it}\,dt, \tag{5.2}$$

where the integral is summable $(C, 1)$ almost everywhere and $\chi(t)$ satisfies (5.11) *and* (5.12).

Proof that the conditions are sufficient. We see at once from (5.11) that $f(x)$ is bounded. Hence $f(x)/x$ belongs to $L^2(1, \infty)$. Therefore, by the theory of transforms, we can write

$$g_1(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^1 \frac{\sin xu}{u}\,f(u)\,du + \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_1^{\infty}\frac{\sin xu}{u}\,f(u)\,du, \tag{5.21}$$

where the second integral converges in mean square. Hence

$$\begin{aligned} g_2(x) &= \int_0^x g_1(y)\,dy = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^{\infty}\frac{1-\cos xu}{u^2}\,f(u)\,du \\ &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{1}{2\pi}\lim_{T\to\infty}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)\chi(t)\,dt\int_0^{\infty}\frac{1-\cos xu}{u^2}\,u^{-(\frac{1}{2}+it)}\,du, \end{aligned}$$

by (5.11) and (5.2). But

$$\frac{2}{\pi}\int_0^{\infty}\frac{1-\cos xu}{u^2}\,u^{-(\frac{1}{2}+it)}\,du = \frac{x^{\frac{3}{2}+it}}{(\frac{1}{2}+it)(\frac{3}{2}+it)\cos(\frac{1}{4}\pi+\frac{1}{2}\pi it)\,\Gamma(\frac{1}{2}+it)}.$$

† Hardy and Titchmarsh, 6, 198.

Therefore

$$g_2(x) = \tfrac{1}{4}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \frac{\chi(t)}{\cos(\frac{1}{4}\pi + \frac{1}{2}\pi i t)\,\Gamma(\frac{1}{2}+it)} \frac{x^{\frac{3}{2}+it}}{(\frac{1}{2}+it)(\frac{3}{2}+it)}\,dt \quad (C,1)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \chi(t)\,\frac{x^{\frac{3}{2}-it}}{(\frac{1}{2}-it)(\frac{3}{2}-it)}\,dt, \quad (C,1)$$

by (5.12). On the other hand, from (5.11) and (5.2),

$$f_1(x) = \int_0^x f(u)\,du = \frac{1}{2\pi}\lim_{T\to\infty}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)\chi(t)\,\frac{x^{\frac{1}{2}-it}}{\frac{1}{2}-it}\,dt,$$

and, since this integral is boundedly summable for $0 \leqslant x \leqslant \chi$ and every finite X, we get

$$f_2(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\chi(t)\,\frac{x^{\frac{3}{2}-it}}{(\frac{1}{2}-it)(\frac{3}{2}-it)}\,dt. \quad (C,1)$$

Therefore $$f_2(x) = g_2(x),$$

and so $$f_1(x) = g_1(x).$$

It follows, therefore, by (5.21), that

$$f(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{d}{dx}\int_0^{\infty}\frac{\sin xu}{u}\,f(u)\,du.$$

Hence, by Theorem 2, $f(x)$ is its own cosine transform and we have

$$f(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^{\infty}\cos xu\,f(u)\,du, \quad (C,1)$$

almost everywhere.

Proof that the conditions are necessary. Since

$$\int_0^{\infty} f(u)\,du$$

is bounded $(C,1)$, it follows that

$$\int_0^{\infty} x^{s-1}f(x)\,dx,$$

where $s = \sigma + it$ is uniformly summable $(C,1)$ throughout any region $0 < \alpha \leqslant \sigma \leqslant \beta < 1$, $-T \leqslant t \leqslant T$, and represents an analytic function $\phi(s)$ in the strip $0 < \sigma < 1$. Write

$$\chi(t) = \phi(\tfrac{1}{2}+it) = \int_0^{\infty} x^{-\frac{1}{2}+it}f(x)\,dx. \quad (C,1)$$

We can invert this by Mellin's formula and obtain

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x^{-\frac{1}{2}-it}\chi(t)\,dt. \qquad (C, 1)$$

This can be proved by a slight modification of an argument used by Hardy†. It also follows from Hardy's argument that

$$\left|\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)\chi(t)\,x^{-\frac{1}{2}-it}\,dt\right| \leqslant M,$$

for all $T \geqslant 0$ and all x. We have thus shown that, when $f(x)$ belongs to HB, conditions (5.11) and (5.2) are satisfied. We shall now make use of the self-reciprocal property of $f(x)$ and show that (5.12) is satisfied by justifying the formal analysis given by Hardy and Titchmarsh‡. We have

$$\begin{aligned}\int_0^{\omega}(\omega-x)x^{s-1}f(x)\,dx &= \int_0^{\delta}(\omega-u)\,x^{s-1}f(x)\,dx-(\omega-\delta)\,\delta^{s-1}f_1(\delta)+\omega^{s-1}f_2(\omega)\\ &\quad -\{\delta^{s-1}+(1-s)\,(\omega-\delta)\,\delta^{s-2}\}f_2(\delta)\\ &\quad +\int_{\delta}^{\omega}\{2(1-s)\,x^{s-2}+(1-s)\,(2-s)\,(\omega-x)\,x^{s-3}\}f_2(x)\,dx.\end{aligned}$$

Hence, by (2.4),

$$\begin{aligned}\phi(s) &= \lim_{\omega\to\infty}\int_0^{\omega}\left(1-\frac{x}{\omega}\right)x^{s-1}f(x)\,dx\\ &= \int_0^{\delta}x^{s-1}f(x)\,dx-\delta^{s-1}f_1(\delta)-(1-s)\,\delta^{s-2}f_2(\delta)\\ &\quad +(1-s)\,(2-s)\int_{\delta}^{\infty}x^{s-3}f_2(x)\,dx.\end{aligned}$$

But

$$I = \int_{\delta}^{\infty}x^{s-3}f_2(x)\,dx = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_{\delta}^{\infty}x^{s-3}\,dx\int_0^{\infty}\frac{1-\cos xu}{u^2}f(u)\,du$$

$$= \int_{\delta}^{\infty}\int_0^{\delta}+\int_{\delta}^{\infty}\int_{\delta}^{\infty}.$$

The second of these two integrals is absolutely convergent, while the integral obtained on inverting the first is uniformly convergent, and so in each case

† Hardy, **5**, 179.

‡ Hardy and Titchmarsh, **6**, 198.

inversion is permissible. Hence

$$(1-s)(2-s)I = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}(1-s)(2-s)\int_0^\infty \frac{f(u)}{u}\,du\int_\delta^\infty (1-\cos ux)\,x^{s-3}\,dx$$

$$= (1-s)\delta^{s-2}f_2(\delta)+\delta^{s-1}f_1(\delta)+\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty f(u)\,du\int_\delta^\infty x^{s-1}\cos ux\,dx.$$

Again,

$$\int_0^\delta x^{s-1}f(x)\,dx = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\lim_{\omega\to\infty}\int_0^\omega\left(1-\frac{u}{\omega}\right)f(u)\,du\int_0^\delta x^{s-1}\cos ux\,dx.$$

Now
$$\int_0^\infty x^{s-1}\cos ux\,dx = u^{-s}\Gamma(s)\cos\tfrac{1}{2}s\pi \quad (0<\sigma<1).$$

Hence
$$\phi(s) = \Gamma(s)\cos\tfrac{1}{2}s\pi\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty x^{-s}f(x)\,dx \qquad (C,\,1)$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\Gamma(s)\cos\tfrac{1}{2}s\pi\,\phi(1-s).$$

So, taking $s=\frac{1}{2}+it$, we get

$$\chi(t) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\Gamma(\tfrac{1}{2}+it)\cos(\tfrac{1}{4}+\tfrac{1}{2}it)\,\pi\chi(-t),$$

which proves (5.12).

Combining Theorems 5 and 9, we deduce

THEOREM 10. *A necessary and sufficient condition for a bounded function $f(x)$ to be a solution of the equation*

$$f(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{d}{dx}\int_0^\infty \frac{\sin xu}{u}\,f(u)\,du$$

is that it should be of the form (5.2), *where $\chi(t)$ satisfies* (5.11) *and* (5.12).

Another set of necessary and sufficient conditions can be obtained from Hardy and Titchmarsh's second solution, but they are very complicated and we do not give them. It might be worth mentioning, however, that for functions of HB the formal analysis used to obtain the second solution is valid practically as it stands.

[*Added in proof*, 21 *September*, 1934. I have now succeeded in proving that the integral (2.1) of Theorem 1 is summable $(C,\,1)$, almost everywhere, to $f(x)$. The additional hypothesis in Theorem 4 is therefore not needed. I must defer the proof for another occasion. Since

the communication of this paper, a result very similar to Theorem 1 has been published by S. Verblunsky, *Proc. London Math. Soc.* (2), 38 (1934), 1–48, 19, Theorem VII. He supposes that

$$\int_0^\infty e^{-\eta t}\{A(t)\cos xt+B(t)\sin xt\}\,dt$$

converges absolutely to a bounded function for $\eta>0$ and all x. It is easy to show that this implies

$$\left|\int_0^T\left(1-\frac{t}{T}\right)\{A(t)\cos xt+B(t)\sin xt\}\,dt\right|\leqslant M$$

for $T\geqslant 0$ and all x.]

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THE CLASSIFICATION OF IDEAL NUMBERS IN A CUBIC FIELD

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1. *Introductory.*

It was recognized early in the nineteenth century that the unique factorization into primes of the ordinary (rational) integers generally breaks down when the domain of the integers is extended to incorporate algebraic numbers. Various systems of *ideal* numbers were devised with the object of restoring to a field of algebraic numbers the quality of unique factorization into primes, but these at first were found to be wanting in some particular. Thus Zolotareff's theory* fails to account satisfactorily for the ideal factors of a divisor of the discriminant of a member of the field. Dedekind† in 1875 (five years before Zolotareff's paper just cited) first presented a theory of ideal numbers which overcame the defects of earlier attempts. To Dedekind is also due the result that the ideals can be placed in a finite number of classes of equivalent ideals. The existence of units in a field of algebraic numbers had been known since the time of Dirichlet‡, who showed that, in a field which is one of a conjugate set of r real and $2s$ complex ones, every unit is a product of powers of $r+s-1$ independent units. Dirichlet did not indicate a method of finding the units in an assigned field, and since the integers known to him were not the same as those afterwards defined by Dedekind, Dirichlet's set of $r+s-1$ independent units was not generally the same as that given by Dedekind's later theory.

When all the units of the field, apart from combinations with cyclotomic ones, are powers of a single unit ϵ, each class contains a finite number

* G. Zolotareff, *Journal de Math.* (3), 6 (1880), 51–84, 129–166.

† R. Dedekind, *Dirichlet's Zahlentheorie*, 3rd ed. (1879), Supp. XI.

‡ P. G. Lejeune-Dirichlet, *Bericht der K. Preuss. Akad. der Wiss. zu Berlin*, 1846, 103–107. [*Werke*, 1 (1889), 641–644.]

of "reduced" ideals, which can be linked in a closed cycle, and each cycle generates the primitive unit ϵ. The possible cases are

$$r = 2, \quad s = 0; \qquad r = 1, \quad s = 1; \qquad r = 0, \quad s = 2.$$

For a real quadratic field, when $r = 2$, $s = 0$, there is a certain analogy between the cycles of reduced ideals and the cycles of "reduced" quadratic forms, though the latter, which long preceded Dedekind's theory of ideals, do not always generate the primitive unit. Thus the Gaussian cycle of reduced quadratic forms $ax^2+2bxy+cy^2$ for the determinant $D = 5$ is

$$(1, 2, -1) \sim (-1, 2, 1),$$

which generates the unit $9+4\sqrt{5} = \epsilon^6$, whereas the half-cycle gives $2+\sqrt{5} = \epsilon^3$. On the other hand, taking Kronecker's forms

$$ax^2+bxy+cy^2,$$

the cycle of reduced forms

$$(1, 1, -1) \sim (-1, 1, 1)$$

leads to the unit $\frac{1}{2}(3+\sqrt{5}) = \epsilon^2$, while the half of it generates $\frac{1}{2}(1+\sqrt{5}) = \epsilon$. The continued fraction expansion for $\sqrt{5}$ given by Degen in the *Canon Pellianus* gives the unit $2+\sqrt{5} = \epsilon^3$. Finally, the cycle of reduced ideals in $(\sqrt{5})$ only contains one member $[1, \frac{1}{2}(1+\sqrt{5})]$ which immediately defines the primitive unit $\frac{1}{2}(1+\sqrt{5}) = \epsilon$.

In cubic fields of negative discriminant ($r = 1$, $s = 1$) it has been shown by the present writer* that reduced ideals can be so defined as to link themselves in cycles, that each class contains one cycle, and that each cycle generates the primitive unit. Several writers† have given tables of units but these are not all fundamental, partly due to the fact that the integral basis is not always correctly chosen.

There is a similar arrangement of reduced ideals in a quartic field, when all four conjugates are complex ($r = 0$, $s = 2$). A few units, not all of which are primitive, in such fields have been calculated by Pocklington†.

Passing to cubic fields of positive discriminant, with which this paper is primarily concerned, there are two independent units, and a primitive

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* W. E. H. Berwick, *Proc. London Math. Soc.* (2), 12 (1913), 393–429.

† A. Markoff, *Mem. Acad. Imp. Sci. St. Petersbourg* (7), 38 (1892), No. 9, (37 pp.); L. Kollros, *Diss. Zürich* (Genève, Soullier, 1905), 1–47; L. W. Reid, *American J. of Math.*, 23 (1901), 68–84; H. C. Pocklington, *Proc. Camb. Phil. Soc.*, 24 (1928), 471–476; P. H. Daus, *American. J. of Math.*, 51 (1929), 67–98.

pair can be chosen in an unlimited number of ways. I have recently shown* how to define a particular set of three units ϵ_1, ϵ_2, ϵ_3 such that $\epsilon_1\epsilon_2\epsilon_3 = \pm 1$, and that every unit of the field is a product of positive integral powers of two of them. It now seems unlikely that reduced ideals can be so defined as to arrange themselves in cycles to generate a primitive pair of units in the same way as in the one-unit fields. The present purpose is to show that reduced ideals can be defined to arrange themselves in a wider configuration, which will be called a *cluster*, so that:

(i) the number of reduced ideals in the field is finite (§§ 2, 5),

(ii) each class of ideals contains one and only one cluster (§§ 3, 4),

(iii) the configuration of reduced ideals in every cluster determines the fundamental set of units ϵ_1, ϵ_2, ϵ_3 (§ 6).

Numerical examples are given (§ 7), showing the clusters of reduced ideals in selected fields, and how they lead to the fundamental set of units defined in my former paper.

Mathematical journals of the last twenty-five years contain several papers on three-dimensional continued fractions, but, since none of the authors appears to attempt an adequate solution of the problem here discussed, it seems unnecessary to give a detailed list of references.

2. *The semi-reduced ideals.*

An ideal $\mathfrak{j}$ in a cubic field (θ) possesses two rational invariants, its norm $n(\mathfrak{j})$, and $l(\mathfrak{j})$, which is the least positive rational integer contained in it. We assume that $\mathfrak{j}$ is not divisible by any principal ideal (m) where $m > 1$. Each prime ideal factor $\mathfrak{p}$ of $\mathfrak{j}$ is such that

$$l(\mathfrak{p}) = p, \quad n(\mathfrak{p}) = p \quad \text{or} \quad p^2,$$

$$l(\mathfrak{p}) \leqslant n(\mathfrak{p}) \leqslant l(\mathfrak{p})^2,$$

whence, by multiplication, in all the relevant cases,

$$l(\mathfrak{j}) \leqslant n(\mathfrak{j}) \leqslant l(\mathfrak{j})^2.$$

(i) It will first be proved that there is only a finite number of ideals $\mathfrak{j}$ in the field containing no integer γ such that

$$|\gamma| < l(\mathfrak{j}), \quad |\gamma'| < l(\mathfrak{j}), \quad |\gamma''| < l(\mathfrak{j}). \tag{1}$$

* W. E. H. Berwick, *Proc. London Math. Soc.* (2), 34 (1932), 360–378.

Any ideal $\mathfrak{j}$ in the field has a basis $\{l(\mathfrak{j}), \alpha, \beta\}$, and the volume of the parallelepiped bounded by the six planes

$$xl(\mathfrak{j})+y\alpha+z\beta = \pm l(\mathfrak{j}), \quad xl(\mathfrak{j})+y\alpha'+z\beta' = \pm l(\mathfrak{j}),$$
$$xl(\mathfrak{j})+y\alpha''+z\beta'' = \pm l(\mathfrak{j})$$

is $8l(\mathfrak{j})^3/n(\mathfrak{j})\sqrt{D}$. If

$$l(\mathfrak{j}).l(\mathfrak{j})^2/n(\mathfrak{j}) > \sqrt{D},$$

which is certainly satisfied when $l(\mathfrak{j}) > \sqrt{D}$, it follows, from Minkowski's lemma on linear forms, that there is a member γ of the ideal satisfying (1). An ideal which contains no member γ satisfying the same inequalities will be called semi-reduced, and since for every such ideal $l(\mathfrak{j}) < \sqrt{D}$, the number of semi-reduced ideals is finite.

The reduced ideals are some of the semi-reduced ones as defined in § 5.

(ii) Next it will be shown that every ideal in the field is equivalent to a semi-reduced one. Among the various members of $\mathfrak{j} = \{l(\mathfrak{j}), \alpha, \beta\}$, an ideal which is not semi-reduced, there are a finite number such that

$$|\gamma| < l(\mathfrak{j}), \quad |\gamma'| < l(\mathfrak{j}), \quad |\gamma''| < l(\mathfrak{j}),$$

and of these there is *at least* one, $\lambda = pl(\mathfrak{j})+q\alpha+r\beta$, such that there is no other member of $\mathfrak{j}$ satisfying simultaneously

$$|\kappa| < |\lambda|, \quad |\kappa'| < |\lambda'|, \quad |\kappa''| < |\lambda''|.$$

Since, necessarily, $dv(p, q, r) = 1$, there exist rational integers p', etc., such that $|pq'r''| = \pm 1$, and

$$\{pl(\mathfrak{j})+q\alpha+r\beta,\ p'l(\mathfrak{j})+q'\alpha+r'\beta,\ p''l(\mathfrak{j})+q''\alpha+r''\beta\} = (\lambda, \mu, \nu)$$

is a basis of $\mathfrak{j}$. If

$$\rho = \pm\{pl(\mathfrak{j})+q\alpha'+r\beta'\}\{pl(\mathfrak{j})+q\alpha''+r\beta''\},$$

the sign being so chosen that $\lambda\rho = M$, is positive,

$$\rho\mathfrak{j} = (\rho\lambda, \rho\mu, \rho\nu) = (M, \delta, \phi).$$

Taking G to be the greatest rational integer which divides M, δ, ϕ, and

$$\rho\mathfrak{j} = G(l_0, \alpha_0, \beta_0),$$

the ideal (l_0, α_0, β_0) is semi-reduced. For, if this ideal contains an integer μ such that

$$|\mu| < l_0, \quad |\mu'| < l_0, \quad |\mu''| < l_0,$$

then must $$|G\mu| < M, \quad |G\mu'| < M, \quad |G\mu''| < M,$$

and $$|G\mu/\rho| < |\lambda|, \quad |G\mu'/\rho'| < |\lambda'|, \quad |G\mu''/\rho''| < |\lambda''|,$$

contrary to supposition, since $G\mu/\rho$ is a member of $\mathfrak{j}$. It follows that (l_0, α_0, β_0) is semi-reduced as required.

It will be noticed that λ can generally be chosen in more ways than one (*cf.* 4 below), and hence that (l_0, α_0, β_0) is not unique.

(iii) A member λ of an ideal $\mathfrak{j}$, such that there is no other integer μ contained in it for which

$$|\mu| < |\lambda|, \quad |\mu'| < |\lambda'|, \quad |\mu''| < |\lambda''|,$$

will be called a convergent to $\mathfrak{j}$. In the geometrical representation the point $(\lambda, \lambda', \lambda'')$ will be called a convergent point for the ideal. From the argument given in (ii) it follows that every ideal has convergents. If λ is a convergent and ϵ any unit, $\lambda\epsilon$ is also a convergent, and, since

$$n(\lambda) < n(\mathfrak{j})\sqrt{D},$$

the number of convergents, apart from combinations with units, is finite. Further, if λ is a convergent to $\mathfrak{j}$, $\rho\lambda$ is a convergent to the ideal $\rho\mathfrak{j}$.

When an ideal is semi-reduced it has a rational convergent $l(\mathfrak{j})$. In fact, the condition for a semi-reduced ideal can be otherwise stated to be that it is one which has a rational convergent.

3. *Arrangement of semi-reduced ideals in a cluster.*

Taking $\mathfrak{j} = \{l(\mathfrak{j}), \alpha, \beta\}$ to be any semi-reduced ideal, of norm $n(\mathfrak{j})$, there is, by Minkowski's lemma on linear forms, at least one integer point inside the region

$$0 < l(\mathfrak{j})x + \alpha y + \beta y = \phi < n(\mathfrak{j})\sqrt{D}/l(\mathfrak{j})^2,$$

$$-l(\mathfrak{j}) < l(\mathfrak{j})x + \alpha' y + \beta' z < l(\mathfrak{j}), \quad -l(\mathfrak{j}) < l(\mathfrak{j})x + \alpha'' y + \beta'' z < l(\mathfrak{j}).$$

Of the several sets of values of (x, y, z) satisfying these inequalities, take (x_1, y_1, z_1) to be that for which ϕ is least. It will be proved that (y_1, z_1) are co-prime. The region bounded by the six planes

$$l(\mathfrak{j})x + \alpha y + \beta z = 0, \quad n(\mathfrak{j})\sqrt{D}/l(\mathfrak{j})^2,$$

$$l(\mathfrak{j})x + \alpha' y + \beta' z = \pm l(\mathfrak{j}), \quad l(\mathfrak{j})x + \alpha'' y + \beta'' z = \pm l(\mathfrak{j})$$

is convex in Minkowski's sense, and so is the section of it by the plane II through the three *nodes* (points with integral coordinates)

$$O \equiv (0, 0, 0), \quad A \equiv (1, 0, 0), \quad P \equiv (x_1, y_1, z_1).$$

The fourth vertex Q of the parallelogram $OAQP$ has the coordinates $(1+x_1, y_1, z_1)$, and the whole plane II can be cut up into parallelograms contiguous with each other and congruent with $OAQP$. Every node in II is either a common vertex of four of these parallelograms, or lies on a side common to two or else is inside one. If there is a node M inside or on a side of any parallelogram, there will be a homothetic node M_0 inside or on a side of $OAPQ$, and hence a node (x', y', z') inside or on a side of OAP. Since the triangular area OAP is convex,

$$0 < l(\mathfrak{j})\,x' + \alpha y' + \beta z' < l(\mathfrak{j})\,x_1 + \alpha y_1 + \beta z_1,$$

$$-l(\mathfrak{j}) < l(\mathfrak{j})\,x' + \alpha' y' + \beta' z' < l(\mathfrak{j}), \quad -l(\mathfrak{j}) < l(\mathfrak{j})\,x' + \alpha'' y' + \beta'' z' < l(\mathfrak{j}),$$

contrary to the assumption that (x_1, y_1, z_1) comes first in the approximation. It follows that all the nodes in II belong to the lattice based on OA, OP. Hence, if $C(x_0, y_0, z_0)$ is on one of the two node-planes parallel and nearest to II, all the nodes of the three-dimensional lattice belong to the system OA, OP, OC, *i.e.* every member of the ideal $\mathfrak{j}$ is

$$l(\mathfrak{j})\,X + \{l(\mathfrak{j})\,x_1 + \alpha y_1 + \beta z_1\}\,Y + \{l(\mathfrak{j})\,x_0 + \alpha y_0 + \beta z_0\}\,Z.$$

Hence $$\mathfrak{j} = \{l(\mathfrak{j}),\ l(\mathfrak{j})\,x_1 + \alpha y_1 + \beta z_1,\ l(\mathfrak{j})\,x_0 + \alpha y_0 + \beta z_0\},$$

where $$\begin{vmatrix} 1 & . & . \\ x_1 & y_1 & z_1 \\ x_0 & y_0 & z_0 \end{vmatrix} = \pm 1,$$

and this is only possible when y_1, z_1 are co-prime as stated.

If we put

$$\gamma = l(\mathfrak{j})\,x_1 + \alpha y_1 + \beta z_1, \quad \delta = l(\mathfrak{j})\,x_0 + \alpha y_0 + \beta z_0,$$

a basis of the ideal is

$$\mathfrak{j} = \{l(\mathfrak{j}),\ \gamma,\ \delta\},$$

in which representation the first and second members, being consecutive convergents to the ideal, are unique, and the last is so chosen that the three are a basis of $\mathfrak{j}$.

[From the fact that OAP is a unit triangle in the above, it is easy to conjecture that the numbers $l(\mathfrak{j})$, γ have no common ideal factor apart from $\mathfrak{j}$; this inference is incorrect, since, in the example III below,

$$4,\ -20 + 4\eta + 2\lambda$$

are consecutive convergents to the ideal $(4,\ \eta,\ 2\lambda)$.]

Further, putting

$$\rho_1 = \gamma'\gamma'', \quad \rho_1 \mathfrak{j} = \{\rho_1\gamma, \rho_1 l(\mathfrak{j}), \rho_1\delta\},$$

and taking k_1 to be the greatest rational integer dividing the last ideal, we have

$$k_1 l_1 = \gamma\rho_1, \quad k_1\alpha_1 = \rho_1 l(\mathfrak{j}), \quad k_1\beta_1 = \rho_1\delta,$$

$$\rho_1\mathfrak{j} = k_1\mathfrak{j}_1 = k_1(l_1, \alpha_1, \beta_1).$$

Reciprocally, $\gamma\mathfrak{j}_1 = g_1\mathfrak{j}$, where $g_1 k_1 = \rho_1\gamma = \pm n(\gamma)$.

It is easily seen that the ideal $\mathfrak{j}_1$ belongs to the semi-reduced system, for, since

$$\frac{l}{\alpha_1} = \frac{\gamma}{l_1} = \frac{\delta}{\beta_1},$$

if there are rational integers u, v, w such that

$$0 < l_1 u + \alpha_1 v + \beta_1 w < l_1, \quad -l_1 < l_1 u + \alpha_1' v + \beta_1' w < l_1,$$

$$-l_1 < l_1 u + \alpha_1'' v + \beta_1'' w < l_1,$$

we must also have $\quad 0 < u + l(\mathfrak{j}) v + \delta w < l(\mathfrak{j}),$

$$|\gamma' u + l(\mathfrak{j}) v + \delta' w| < \gamma' < l(\mathfrak{j}), \quad |\gamma'' u + l(\mathfrak{j}) v + \delta'' w| < \gamma'' < l(\mathfrak{j}),$$

contrary to definition of

$$l(\mathfrak{j}) x_1 + \alpha y_1 + \beta z_1.$$

Two other ideals adjacent to $\mathfrak{j}$ are obtained as follows. For $\mathfrak{j}_2$ the inequalities are taken to be

$$-l(\mathfrak{j}) < l(\mathfrak{j}) x_2 + \alpha y_2 + \beta z_2 < l(\mathfrak{j}), \quad -l(\mathfrak{j}) < l(\mathfrak{j}) x_2 + \alpha'' y_2 + \beta'' z_2 < l(\mathfrak{j}),$$

$$0 < l(\mathfrak{j}) x_2 + \alpha' y_2 + \beta' z_2 = \phi_2' < n(\mathfrak{j})\sqrt{D}/l(\mathfrak{j})^2, \quad \text{and} \quad \phi_2' \text{ least.}$$

Similarly, $\mathfrak{j}_3$ is defined by the conditions

$$-l(\mathfrak{j}) < l(\mathfrak{j}) x_3 + \alpha y_3 + \beta z_3 < l(\mathfrak{j}), \quad -l(\mathfrak{j}) < l(\mathfrak{j}) x_3 + \alpha' y_3 + \beta' z_3 < l(\mathfrak{j}),$$

$$0 < l(\mathfrak{j}) x_3 + \alpha'' y_3 + \beta'' z_3 = \phi_3'' < n(\mathfrak{j})\sqrt{D}/l(\mathfrak{j})^2 \quad \text{and} \quad \phi_3'' \text{ least.}$$

As before, y_2, z_2 are co-prime, also y_3, z_3.

It is evident that $\mathfrak{j}$ defines the three adjacent ideals $\mathfrak{j}_1, \mathfrak{j}_2, \mathfrak{j}_3$ uniquely, but $\mathfrak{j}$ is not uniquely determined by $\mathfrak{j}_1$. For, given $\mathfrak{j}_1 = (l_1, \alpha_1, \beta_1)$, the ideal $\mathfrak{j} = (l, \alpha, \beta)$ is fixed by

$$\frac{l}{\alpha_1} = \frac{\gamma}{l_1} = \frac{\delta}{\beta_1} \quad \text{and} \quad l \equiv 0 \quad [\text{mod } (l, \gamma, \delta)].$$

Thus l has a factor in common with $n(\gamma)$, and, since $l < \sqrt{D}$, there is only a limited choice for $\mathfrak{j}$. A knowledge of $\mathfrak{j}_2$ or $\mathfrak{j}_3$ also fixes $\mathfrak{j}$ within a limited range, and the choice becomes narrower when two or all of the ideals $\mathfrak{j}_1$, $\mathfrak{j}_2$, $\mathfrak{j}_3$ are known.

It is convenient here to give a representation of any semi-reduced ideal which is of use in the sequel, viz.:

$$\mathfrak{j} = \begin{pmatrix} l(\mathfrak{j}), & \alpha, & \beta \\ \phi_1, & \phi_2, & \phi_3 \end{pmatrix},$$

where the terms in the first line form an ordinary Dedekind basis, and those in the second line are the three convergents immediately succeeding $l(\mathfrak{j})$.

Adjacent to $\mathfrak{j}_r$ are three ideals $\mathfrak{j}_{r1}$, $\mathfrak{j}_{r2}$, $\mathfrak{j}_{r3}$; hence from any ideal are obtained three adjacent ideals at the first step, 3^2 at the second step, and 3^s at the s-th step. Since the number of distinct reduced ideals is finite this process will involve repetitions, and we examine how far it need be carried. (1) If there are b semi-reduced ideals in all, the sequence

$$\mathfrak{j}, \mathfrak{j}_r, \mathfrak{j}_{rs}, \mathfrak{j}_{rst} \quad \text{to} \quad b+1 \text{ terms}$$

must recur and need not be continued after the first repetition. (2) If two ideals at the k-th step are identical, and do not appear in a step of lower order, only one need be continued. (3) If an ideal at the k-th step is identical with an ideal at the m-th step, where $k < m$, it is unnecessary to work out the continuation of this ideal beyond its first appearance at the m-th step. (4) It will be noticed that the initial member $\mathfrak{j}$ does not necessarily appear again in any of the sequences (*cf.* § 5 below).

4. *Equivalent semi-reduced ideals generate the same cluster.*

The cluster of semi-reduced ideals obtained in § 3 depends on the initial member $\mathfrak{j}$. It has already been noted (§ 2, ii) that there is some latitude in the choice of $\mathfrak{j}$, and the question arises whether every semi-reduced ideal in the class generates the same cluster. Here it will be proved that, if $\mathfrak{j}$ and $\mathfrak{k}$ are any two equivalent ideals, some member of the cluster generated by $\mathfrak{j}$ is also a member of the cluster which $\mathfrak{k}$ generates.

Take α to be any convergent to $\mathfrak{j}$, and α_1, α_2, α_3 the adjacent convergents at the first step, α_{11}, ..., those at the second step, and so on. As in § 3, there is in $\mathfrak{j}$ an integer γ such that $(\alpha_r, \alpha_{r1}, \gamma)$ is a basis of $\mathfrak{j}$. Putting

$$\alpha_r' \alpha_r''(\alpha_r, \alpha_{r1}, \gamma) = G_r(l_r, \gamma_r, \delta_r) = G_r \mathfrak{j}_r,$$

we see that the ideal $\mathfrak{j}_r$ is semi-reduced. There is a similar set of convergents and semi-reduced ideals arising from any convergent β to $\mathfrak{k}$.

Since $\mathfrak{j}$ and $\mathfrak{k}$ are equivalent,

$$h(l, \alpha, \beta) = h\mathfrak{j} = \mu\mathfrak{k} = \mu(k, \gamma, \delta).$$

A convergent to $h\mathfrak{j}$ is $hl = \theta$, one to $\mu\mathfrak{k}$ is $\mu k = \lambda$, and it is to be shown that some convergent of the ideal as first defined is also a convergent when the ideal is fixed by the second definition. It may be supposed that

$$|\theta| < |\lambda|, \quad |\theta'| > |\lambda'|, \quad |\theta''| > |\lambda''|,$$

and, on replacing θ by the next convergent θ_1, which lies within the region $(\lambda, \theta', \theta'')$, that

$$|\theta_1| < |\lambda|, \quad |\theta_1'| > |\lambda'|, \quad |\theta_1''| > |\lambda''|,$$

but that the next following θ-convergent θ_{11} is not such that

$$|\theta_{11}| < |\lambda|, \quad |\theta_{11}'| > |\lambda'|, \quad |\theta_{11}''| > |\lambda''|.$$

Since, necessarily, this convergent is such that $\theta_1 < \theta_{11} < \lambda$, for otherwise λ would come next, and since the inequalities $\theta_{11} < \lambda$, $\theta_{11}' < \lambda'$, $\theta_{11}'' < \lambda''$ are not possible together, the ones which must hold are

$$|\theta_{11}| < |\lambda|, \quad |\theta_{11}'| > |\lambda'|, \quad |\theta_{11}''| < |\lambda''|, \quad \text{or}$$

$$|\theta_{11}| < |\lambda|, \quad |\theta_{11}'| < |\lambda'|, \quad |\theta_{11}''| > |\lambda''|,$$

both of which lie in the region $(\lambda, \theta_1', \theta_1'')$.

Assuming that the first of these hold,

i.e. $$|\lambda'| < |\theta_{11}'|, \quad |\lambda| > |\theta_{11}|, \quad |\lambda''| > |\theta_{11}''|,$$

the convergent λ is to be replaced by λ_2, which lies in the region $(\lambda, \theta_{11}', \lambda'')$, and the process repeated. (If at any stage of the process, where $|\theta_r|$ is to be increased, θ_r is negative, θ_r is to be replaced by $-\theta_r$.) Since, if there is nothing better, λ_s always satisfies the inequalities, a next convergent always exists. At each step of the process when two of the quantities θ are greater than the corresponding two quantities λ the next convergent is a θ; when two λ's are greater than the corresponding two θ's the next convergent is a λ.

At every stage two of the three dimensions of the parallelepipedal box, within which the next convergent lies, are diminished, and there are only a finite number of convergents in the largest of these. Hence, after a finite number of the steps, a stage must be reached where a θ-convergent is identical with a λ-convergent, as was required to be proved.

5. *Reduced ideals.*

It has now been seen that the cluster of ideals generated by a reduced ideal $\mathfrak{j}$ contains some ideals in common with the cluster generated by an equivalent semi-reduced ideal $\mathfrak{k}$. The cluster generated by $\mathfrak{j}$ may or may not contain $\mathfrak{j}$ again. When $\mathfrak{j}$ lies in the cluster of ideals generated by every member of the cluster generated by $\mathfrak{j}$, $\mathfrak{j}$ will be said to be reduced.

Examples of clusters are

(I)
$$\begin{array}{ccc} & A & \\ A & B & A \\ & BBB & \end{array}$$

(II)
$$\begin{array}{ccc} & A & \\ B & C & D \\ DAB & ACB & DCB \end{array}$$

In (I) B is reduced but A is only semi-reduced. In (II) A, B, C, D are all reduced.

6. *Units.*

It has been shown in my previous paper that the field (θ) contains three units $\epsilon_1, \epsilon_2, \epsilon_3$ such that

$$\epsilon_1 > 1 \text{ and least}, \quad |\epsilon_1'| < 1, \quad |\epsilon_1''| < 1,$$

$$|\epsilon_2| < 1, \quad \epsilon_2' > 1 \text{ and least}, \quad |\epsilon_2''| < 1, \qquad \epsilon_1 \epsilon_2 \epsilon_3 = \pm 1,$$

$$|\epsilon_3| < 1, \quad |\epsilon_3'| < 1, \quad \epsilon_3'' > 1 \text{ and least},$$

and that every unit of the field is a product of positive integral powers of two of the ϵ's.

The sequence of ideals $\mathfrak{j}, \mathfrak{j}_1, \mathfrak{j}_{11}, \mathfrak{j}_{111}, \mathfrak{j}_{1111}, \ldots$ has been seen in § 3 to become periodic. If $\mathfrak{j} = \mathfrak{j}_{1111}$ the relations of § 3 show that

$$\frac{\phi_1}{l} \frac{\phi_{11}}{l_1} \frac{\phi_{111}}{l_{11}} \frac{\phi_{1111}}{l_{111}}$$

is a unit η_1 such that $\eta_1 > 1$, $|\eta_1'| < 1$, $|\eta_1''| < 1$, since each of the three factors of η_1 satisfies the same inequalities. It must not be inferred that $\eta_1 = \epsilon_1$ or one of its powers, since the conditions $\epsilon_1 > 1$, $|\epsilon_1'| < 1$, $|\epsilon_1''| < 1$ do not imply that ϵ_1 is a product of factors satisfying the same inequalities. The sequences

$$\mathfrak{j}, \mathfrak{j}_2, \mathfrak{j}_{22}, \mathfrak{j}_{222}, \ldots, \quad \mathfrak{i}, \mathfrak{j}_3, \mathfrak{j}_{33}, \mathfrak{j}_{333}, \ldots$$

similarly give units η_2, η_3 such that

$$|\eta_2| < 1, \eta_2' > 1, |\eta_2''| < 1, \quad |\eta_3| < 1, |\eta_3'| < 1, \eta''_3 > 1,$$

while from the sequences

$$\mathfrak{j},\ \mathfrak{j}_1,\ \mathfrak{j}_{12},\ \mathfrak{j}_{122}=\mathfrak{j},\quad \mathfrak{j}_1,\ \mathfrak{j},\ \mathfrak{j}_2,\ \mathfrak{j}_{23}=\mathfrak{j}_1$$

are derived the units

$$\frac{\phi_1}{l}\frac{\phi_{12}}{l_1}\frac{\phi_{122}}{l_{12}},\qquad \frac{l}{\phi_1}\frac{\phi_2}{l}\frac{\phi_{23}}{l_2}.$$

Only a finite number of units are obtained thus, and it can be shown that ϵ_1, ϵ_2, ϵ_3 are products of powers of the η's. For if

$$(l,\ \alpha,\ \beta)=(l\epsilon,\ \alpha\epsilon,\ \beta\epsilon)$$

is any reduced ideal, the argument of § 4 shows that $\mathfrak{j}$ and $\mathfrak{j}\epsilon$ are both linked with the same reduced ideal $\mathfrak{k}$. If the connecting relations are

$$\mathfrak{j}=\mu\mathfrak{k},\quad \mathfrak{j}\epsilon=\nu\mathfrak{k},\quad \text{and}\quad \epsilon=\mu/\nu,$$

the unit ϵ is obtained as above and is a product of the η's.

It remains to show how to find all the units from the set of η's. If η is any unit, the three units η_1, η_2, η are connected by a relation

$$\eta_1{}^f\eta_2{}^g\eta^h=\pm 1,$$

which can be found by solving the linear equations

$$-\log\eta=x\log\eta_1+y\log\eta_2,\quad -\log\eta'=x\log\eta_1'+y\log\eta_2'$$

to a sufficient degree of accuracy, whence $x=f/h$, $y=g/h$, where f, g, h are rational integers without common factor. It is then always possible to find rational integers p, etc., so that $|fqn|=1$. Putting

$$\zeta_1=\eta_1{}^p\eta_2{}^q\eta^r,\quad \zeta_2=\eta_1{}^l\eta_2{}^m\eta^n,$$

we have

$$\eta_1=\zeta_1^{mh-ng}\zeta_2^{nf-lh},\quad \eta_2=\zeta_1^{gr-qh}\zeta_2^{hp-fr},\quad \eta=\zeta_1^{qn-rm}\zeta_2^{rl-pn}.$$

Hence all units which can be built up as products of power of η_1, η_2, η can also be built up as products of powers of ζ_1, ζ_2. When we annex another unit η_0, the set ζ_1, ζ_2, η_0 can similarly be reduced to two and the process can be continued. Finally, after a finite number of such operations, all the units are products of powers of two units κ_1, κ_2.

To find ϵ_1, ϵ_2, ϵ_3, put

$$\log\epsilon=x\log\kappa_1+y\log\kappa_2,\quad \log\epsilon'=x\log\kappa_1'+y\log\kappa_2',$$

so that all the points $(\log\epsilon,\ \log\epsilon')$ for rational integral values of x, y form a lattice. The unit ϵ_1 is obtained, geometrically or otherwise, by taking

that point of the lattice for which

$$\log\epsilon > 0 \text{ and least, consistently with } \log\epsilon' < 0,\ \log\epsilon + \log\epsilon' > 0.$$

In the same way ϵ_2, ϵ_3 are found and the relation $\epsilon_1\epsilon_2\epsilon_3 = \pm 1$ provides a check.

7. *Three numerical examples.*

(I) $\theta^3 - 7\theta - 3 = 0$. $\qquad D(\theta) = D = 1129.$

$$\theta = 2{\cdot}839,\quad \theta' = -0{\cdot}441,\quad \theta'' = -2{\cdot}398,$$

$$\theta^2 = 8{\cdot}056,\quad \theta'^2 = 0{\cdot}195,\quad \theta''^2 = 5{\cdot}749.$$

We form the cluster of reduced ideals beginning with the integral basis

$$\mathfrak{j} = \begin{pmatrix} 1, & \theta, & \theta^2 \\ \theta^2+2\theta, & -2\theta^2+\theta+13, & 2\theta^2-5\theta-2 \end{pmatrix}$$

$$\mathfrak{j}_1 = (\theta^2-\theta-5)(\theta^2+2\theta,\ 1,\ \theta) = (3,\ \theta^2-\theta-5,\ \theta^2-2\theta-3)$$

$$= \begin{pmatrix} 3, & \theta+1, & \theta^2-1 \\ \theta+1, & -2\theta^2+\theta+12, & 3\theta^2-7\theta-4 \end{pmatrix}$$

$$\mathfrak{j}_2 = (\theta^2+\theta)(-2\theta^2+\theta+13,\ 1,\ \theta^2) = (3,\ \theta^2+\theta,\ 7\theta^2+10\theta+3)$$

$$= \begin{pmatrix} 3, & 3\theta, & \theta^2+\theta \\ \theta^2+4\theta+3, & -2\theta^2+\theta+12, & \theta^2-2\theta-3 \end{pmatrix}$$

$$\mathfrak{j}_3 = (\theta^2+2\theta-1)(2\theta^2-5\theta-2,\ 1,\ \theta,\ \theta^2)$$

$$= (1,\ \theta^2+2\theta-1,\ 2\theta^2+6\theta+3,\ 6\theta^2+17\theta+6) = \mathfrak{j}$$

$$\mathfrak{j}_{11} = \tfrac{1}{3}(\theta^2-\theta-6)(\theta+1,\ 3,\ \theta^2-\theta-5) = (3,\ \theta^2-\theta-6,\ \theta^2-8) = \mathfrak{j}$$

$$\mathfrak{j}_{12} = (\theta^2-1)(2\theta^2-\theta-12,\ 3,\ \theta^2-\theta-5) = (9,\ 3\theta^2-3,\ \theta^2-3\theta+2)$$

$$= \begin{pmatrix} 9, & 9\theta, & \theta^2-3\theta+2 \\ 2\theta^2+3\theta-5, & -3\theta^2+21, & \theta^2-3\theta+2 \end{pmatrix}$$

$$\mathfrak{j}_{13} = \tfrac{1}{3}(2\theta^2+\theta-9)(3\theta^2-7\theta-4,\ 3,\ \theta^2-2\theta-3)$$

$$= (1,\ 2\theta^2+\theta-9,\ \theta^2-6) = \mathfrak{j}$$

$$\mathfrak{j}_{21} = \tfrac{1}{3}(2\theta^2-3\theta-8)(\theta^2+4\theta+3,\ 3,\ \theta^2+\theta) = (3,\ 2\theta^2-3\theta-8,\ \theta^2-3\theta-1)$$

$$= \begin{pmatrix} 3, & 3\theta & \theta^2-1 \\ \theta^2+3\theta-1, & -\theta^2+7, & \theta^2-3\theta-1 \end{pmatrix}$$

$$\mathfrak{j}_{22} = \tfrac{1}{3}(\theta^2-1)(-2\theta^2+\theta+12,\ 3,\ \theta^2+\theta) = (3,\ \theta^2-1,\ 2\theta^2+3\theta+1) = \mathfrak{j}_{21}$$

$$\mathfrak{j}_{23} = \tfrac{1}{3}(\theta+2)(\theta^2-2\theta-3,\ 3,\ \theta^2+\theta) = (1,\ \theta+2,\ \theta^2+3\theta+1) = \mathfrak{j}$$

$$\mathfrak{j}_{121} = \tfrac{1}{9}(\theta^2-3\theta)\,(2\theta^2+3\theta-5,\ 9,\ \theta^2-3\theta+2) = (1,\ \theta^2-3\theta,\ 2\theta^2-5\theta-2) = \mathfrak{j}$$

$$\mathfrak{j}_{122} = \tfrac{1}{3}\theta(-3\theta^2+21,\ 9,\ \theta^2-3\theta+2) = (3,\ 3\theta,\ -\theta^2+3\theta+1) = \mathfrak{j}_{21}$$

$$\mathfrak{j}_{123} = \tfrac{1}{9}(\theta+3)\,(\theta^2-3\theta+2,\ 9,\ 3\theta^2-3) = (1,\ \theta+3,\ \theta^2+2\theta) = \mathfrak{j}$$

$$\mathfrak{j}_{211} = \tfrac{1}{3}(\theta^2+2\theta-12)\,(\theta^2+3\theta-1,\ 3,\ 3\theta) = (9,\ \theta^2+2\theta-12,\ 2\theta^2-5\theta+3)$$

$$= \begin{pmatrix} 9, & 9\theta, & \theta^2+2\theta-3 \\ \theta^2+2\theta-3, & 4\theta^2-\theta-30, & 4\theta^2-10\theta-3 \end{pmatrix}$$

$$\mathfrak{j}_{212} = \tfrac{1}{3}\theta(-\theta^2+7,\ 3,\ 3\theta) = (1,\ \theta,\ \theta^2) = \mathfrak{j}$$

$$\mathfrak{j}_{213} = \tfrac{1}{3}(\theta^2-6)\,(\theta^2-3\theta-1,\ 3,\ 3\theta) = (1,\ \theta^2-6,\ \theta+3) = \mathfrak{j}$$

$$\mathfrak{j}_{2111} = \tfrac{1}{9}(\theta-2)\,(\theta^2+2\theta-3,\ 9,\ 9\theta) = (1,\ \theta-2,\ \theta^2+2\theta) = \mathfrak{j}$$

$$\mathfrak{j}_{2112} = \tfrac{1}{9}(\theta^2+2\theta+1)\,(4\theta^2-\theta-30,\ 9,\ \theta^2+2\theta-3)$$

$$= (1,\ \theta^2+2\theta+1,\ \theta^2+3\theta+1) = \mathfrak{j}$$

$$\mathfrak{j}_{2113} = \tfrac{1}{9}(2\theta+5)\,(4\theta^2-10\theta-3,\ 9,\ 9\theta,\ \theta^2+2\theta-3)$$

$$= (1,\ 2\theta+5,\ 2\theta^2+5\theta,\ \theta^2+2\theta-1) = \mathfrak{j}.$$

There is no difficulty in seeing that, if any of the above ideals is taken as leader, all the other ideals above are members of its cluster. Hence all of them are reduced.

To obtain the unit ϵ_3 we note that $\mathfrak{j}_3 = \mathfrak{j}$, whence $\epsilon_3 = 2\theta^2-5\theta-2$. Similarly from the sequence $\mathfrak{j}$, $\mathfrak{j}_1$, $\mathfrak{j}_{11}$ we have

$$\epsilon_1 = \tfrac{1}{3}(\theta^2+2\theta)\,(\theta+1) = \theta^2+3\theta+1.$$

The unit ϵ_2 can be found directly as $1/\epsilon_1\epsilon_3$, otherwise from the sequence

$$\mathfrak{j}_{21},\ \mathfrak{j}_{212} = \mathfrak{j},\ \mathfrak{j}_1,\ \mathfrak{j}_{12},\ \mathfrak{j}_{122} = \mathfrak{j}_{21},$$

and is

$$\tfrac{1}{2\,7}(-\theta^2+7)\,(\theta^2+2\theta)\,(2\theta^2-\theta-12)\,(-\theta^2+7) = -2\theta^2+\theta+14.$$

By multiplication, $\epsilon_1\epsilon_2\epsilon_3 = -1$.

It is readily seen that every ideal in the field is equivalent to one of the ideals in the above cluster.

(II) $\theta^3-12\theta-1 = 0.$ $D(\theta) = D = 3^4.5.17.$

$$\theta = 3{\cdot}505,\quad \theta' = -0{\cdot}083,\quad \theta'' = -3{\cdot}422,$$

$$\theta^2 = 12{\cdot}285,\quad \theta'^2 = 0{\cdot}007,\quad \theta''^2 = 11{\cdot}708.$$

In this field there are three classes of ideals; we first form the cluster of principal reduced ideals, beginning with

$$\mathfrak{i} = \begin{pmatrix} 1, & \theta, & \theta^2 \\ \theta^2+4\theta+1, & 12-\theta^2, & -1-3\theta+\theta^2 \end{pmatrix}$$

$\mathfrak{i}_1 = (-\theta^2+\theta+9)(\theta^2+4\theta+1,\ 1,\ \theta) = (6,\ 2\theta-2,\ \theta^2-\theta-3)$

$\mathfrak{i}_2 = \theta(12-\theta^2,\ 1,\ \theta) = (1,\ \theta,\ \theta^2) = \mathfrak{i}$

$\mathfrak{i}_3 = (\theta^2+\theta-8)(\theta^2-3\theta-1,\ 1,\ \theta) = (6,\ 3\theta-3,\ \theta^2+\theta-8)$

$\mathfrak{i}_{11} = \frac{1}{2}(\theta^2-\theta-6)(\theta^2+\theta-5,\ 6,\ 2\theta-2) = (15,\ 15\theta,\ \theta^2+4\theta+4)$

$\mathfrak{i}_{12} = \frac{1}{2}(\theta-1)(-\theta^2-\theta+11,\ 6,\ 2\theta-2) = (6,\ 3\theta-3,\ \theta^2-2\theta+1) = \mathfrak{i}_3$

$\mathfrak{i}_{13} = \frac{1}{3}(\theta^2+\theta-11)(2\theta-2,\ 6,\ \theta^2-\theta-3) = (8,\ 8\theta,\ \theta^2-3\theta-3)$

$\mathfrak{i}_{31} = \frac{1}{6}(-\theta^2+\theta+9)(\theta^2+4\theta+1,\ 6,\ 3\theta-3) = \mathfrak{i}$

$\mathfrak{i}_{32} = \frac{1}{6}(\theta^2+5\theta-3)(-\theta^2-\theta+14,\ 6,\ 3\theta-3) = (8,\ 8\theta,\ \theta^2-3\theta-3) = \mathfrak{i}_{13}$

$\mathfrak{i}_{33} = \frac{1}{6}(\theta^2+3\theta-1)(\theta^2-2\theta-5,\ 6,\ 3\theta-3) = (1,\ \theta^2+3\theta-1,\ \theta^2+4\theta+1) = \mathfrak{i}$

$\mathfrak{i}_{111} = \frac{1}{15}(\theta^2+3\theta-13)(\theta^2+4\theta-11,\ 15,\ 3\theta^2-3\theta-18)$

$= (10,\ 10\theta,\ \theta^2+3\theta-13)$

$\mathfrak{i}_{112} = \frac{1}{5}(\theta+1)(-3\theta^2+3\theta+33,\ 15,\ 2\theta^2-7\theta-7)$

$= (6,\ 3\theta+3,\ -\theta^2+2\theta-1) = \mathfrak{i}_3$

$\mathfrak{i}_{113} = \frac{1}{15}(\theta^2-3\theta-19)(2\theta^2-7\theta-7,\ 15,\ 3\theta^2-3\theta-18)$

$= (8,\ \theta^2-3\theta-19,\ -2\theta^2-2\theta+22) = \mathfrak{i}_{13}$

$\mathfrak{i}_{131} = \frac{1}{8}(\theta^2-3\theta-2)(\theta^2+5\theta+5,\ 8,\ 2\theta^2-2\theta-22)$

$= (1,\ \theta^2-3\theta-2,\ -\theta^2+2\theta+5) = \mathfrak{i}$

$\mathfrak{i}_{132} = \frac{1}{8}(\theta^2+3\theta)(-3\theta^2+\theta+33,\ 8,\ 2\theta^2+2\theta-22)$

$= (1,\ \theta^2+3\theta,\ \theta^2+4\theta+1) = \mathfrak{i}$

$\mathfrak{i}_{133} = \frac{1}{8}(\theta+3)(\theta^2-3\theta-3,\ 8,\ 2\theta^2+2\theta-22) = (1,\ \theta+3,\ \theta^2+\theta-8) = \mathfrak{i}$

$\mathfrak{i}_{1111} = \frac{1}{10}(\theta-3)(\theta^2+3\theta-3,\ 10,\ -2\theta^2+4\theta+16) = (1,\ \theta-3,\ \theta^2-2\theta-5) = \mathfrak{i}$

$\mathfrak{i}_{1112} = \frac{1}{10}(-\theta^2-\theta+2)(-3\theta^2+\theta+29,\ 10,\ \theta^2+3\theta-13)$

$= (6,\ -\theta^2-\theta+2,\ 3\theta+3) = \mathfrak{i}_3$

$\mathfrak{i}_{1113} = \frac{1}{5}(-\theta^2-\theta+7)(2\theta^2-4\theta-6,\ 10,\ \theta^2+3\theta-13)$

$= (8,\ -2\theta^2-2\theta+14,\ \theta^2-3\theta-19) = \mathfrak{i}_{13}.$

Before finding the units we form the clusters of non-principal reduced ideals, viz.:

$$\mathfrak{h} = (4,\ 4\theta,\ \theta^2+\theta+1)$$

$$\mathfrak{h}_1 = \tfrac{1}{2}(-\theta^2+2\theta+5)\,(2\theta^2+6\theta-2,\ 4,\ \theta^2+\theta+1) = (6,\ 6\theta,\ \theta^2+\theta+1)$$

$$\mathfrak{h}_2 = \tfrac{1}{4}(\theta^2-7\theta-2)\,(-3\theta^2+\theta+37,\ 4,\ \theta^2+\theta+1) = (13,\ 13\theta,\ \theta^2-7\theta-2)$$

$$\mathfrak{h}_3 = \tfrac{1}{4}(-\theta^2+\theta+16)\,(\theta^2-3\theta+1,\ 4,\ 4\theta) = (5,\ 5\theta,\ \theta^2-\theta-1)$$

$$\mathfrak{h}_{11} = \tfrac{1}{6}(-\theta^2+\theta+6)\,(\theta^2+\theta-5,\ 6,\ 6\theta) = (5,\ -\theta^2+\theta+6,\ \theta^2-6\theta-1) = \mathfrak{h}_3$$

$$\mathfrak{h}_{12} = \tfrac{1}{6}(\theta-1)\,(\theta^2+\theta-11,\ 6,\ 6\theta) = (2,\ \theta-1,\ \theta^2-\theta)$$

$$\mathfrak{h}_{13} = \tfrac{1}{6}(2\theta^2+\theta-21)\,(\theta^2-5\theta+1,\ 6,\ 6\theta) = (5,\ 5\theta,\ \theta^2+3\theta+2)$$

$$\mathfrak{h}_{21} = \tfrac{1}{13}(2\theta^2+\theta-26)\,(\theta^2+6\theta-2,\ 13,\ 13\theta)$$
$$= (5,\ 2\theta^2+\theta-26,\ \theta^2-2\theta+2) = \mathfrak{h}_{13}$$

$$\mathfrak{h}_{22} = \tfrac{1}{13}(-\theta^2-\theta+3)\,(-2\theta^2+\theta+17,\ 13,\ \theta^2-7\theta-2)$$
$$= (4,\ -\theta^2-\theta+3,\ 4\theta) = \mathfrak{h}$$

$$\mathfrak{h}_{23} = \tfrac{1}{13}(\theta^2+3\theta-2)\,(\theta^2-7\theta+11,\ 13,\ 13\theta)$$
$$= (2,\ \theta^2+3\theta-2,\ 3\theta^2+10\theta+1) = \mathfrak{h}_{12}$$

$$\mathfrak{h}_{31} = \tfrac{1}{5}(\theta^2+\theta-15)\,(\theta^2+4\theta-1,\ 5,\ 5\theta) = (4,\ \theta^2+\theta-15,\ \theta^2-3\theta+1) = \mathfrak{h}$$

$$\mathfrak{h}_{32} = \theta(6\theta-5\theta^2,\ 5,\ \theta^2-\theta-1) = (5,\ 5\theta,\ -\theta^2+11\theta+1) = \mathfrak{h}_3$$

$$\mathfrak{h}_{33} = \tfrac{1}{5}(\theta^2-2\theta-18)\,(2\theta^2-7\theta-2,\ 5,\ \theta^2-\theta-1)$$
$$= (5,\ \theta^2-2\theta-18,\ -\theta^2-3\theta+3) = \mathfrak{h}_{13}$$

$$\mathfrak{h}_{121} = \tfrac{1}{2}(-3\theta^2+\theta+33)\,(\theta^2+3\theta,\ 2,\ \theta-1)$$
$$= (4,\ -3\theta^2+\theta+33,\ 2\theta^2-2\theta-18) = \mathfrak{h}$$

$$\mathfrak{h}_{122} = \tfrac{1}{2}(-\theta^2+\theta+1)\,(-\theta^2+11,\ 2,\ \theta-1) = (5,\ -\theta^2+\theta+1,\ \theta^2-6\theta-1) = \mathfrak{h}_3$$

$$\mathfrak{h}_{123} = \tfrac{1}{2}(-\theta^2-5\theta-5)\,(\theta^2-3\theta-2,\ 2,\ \theta-1)$$
$$= (4,\ \theta^2+5\theta+5,\ 2\theta^2+6\theta-2) = \mathfrak{h}$$

$$\mathfrak{h}_{131} = \tfrac{1}{5}(\theta-3)\,(\theta^2+3\theta-3,\ 5,\ 5\theta) = (2,\ \theta-3,\ \theta^2-3\theta) = \mathfrak{h}_{12}$$

$$\mathfrak{h}_{132} = \tfrac{1}{5}(\theta^2+5\theta+1)\,(-3\theta^2+\theta+34,\ 5,\ \theta^2-2\theta-3) = (4,\ \theta^2+5\theta+1,\ 4\theta) = \mathfrak{h}$$

$$\mathfrak{h}_{133} = \tfrac{1}{5}(\theta^2+\theta-7)\,(\theta^2-2\theta-3,\ 5,\ 5\theta) = (4,\ \theta^2+\theta-7,\ \theta^2+5\theta+1) = \mathfrak{h};$$

and $\mathfrak{k} = (2, 2\theta, \theta^2+\theta+1)$

$\mathfrak{k}_1 = \frac{1}{2}(-\theta^2+2\theta+5)\,(\theta^2+3\theta-1, 2, 2\theta) = (3, 3\theta, \theta^2+\theta+1)$

$\mathfrak{k}_2 = \theta(24-2\theta^2, 2, \theta^2+\theta+1) = (2, 2\theta, \theta^2+13\theta+1) = \mathfrak{k}$

$\mathfrak{k}_3 = \frac{1}{2}(-\theta^2-\theta+8)\,(\theta^2-3\theta-1, 2, 2\theta) = (3, -\theta^2-\theta+8, \theta^2+4\theta+1) = \mathfrak{k}_1$

$\mathfrak{k}_{11} = (\theta^2-\theta-9)\,(\theta^2+4\theta+1, 3, 3\theta) = (2, \theta^2-\theta-9, -\theta^2+3\theta+1) = \mathfrak{k}$

$\mathfrak{k}_{12} = \theta(36-3\theta^2, 3, \theta^2+\theta+1) = (3, 3\theta, \theta^2+13\theta+1) = \mathfrak{k}_1$

$\mathfrak{k}_{13} = (\theta^2-\theta-15)\,(2\theta^2-7\theta-1, 3, \theta^2-2\theta+1) = (2, \theta^2-\theta-15, 2\theta+6) = \mathfrak{k}.$

There is no difficulty in seeing that these are the only clusters. The unit $\epsilon_2 = -\theta^2+12$ is found from any of the sequences $(\mathfrak{k}_1, \mathfrak{k}_{12})$, $(\mathfrak{h}_3, \mathfrak{h}_{32})$, $(\mathfrak{j}, \mathfrak{j}_2)$. From the sequence $(\mathfrak{k}, \mathfrak{k}_1, \mathfrak{k}_{11})$ we have similarly

$$\epsilon_1 = \tfrac{1}{6}(\theta^2+3\theta-1)\,(\theta^2+4\theta+1) = 4\theta^2+14\theta+1;$$

whereas ϵ_3 is given by the sequence $(\mathfrak{j}, \mathfrak{j}_3, \mathfrak{j}_{33})$, namely

$$\epsilon_3 = \tfrac{1}{6}(\theta^2-3\theta-1)\,(\theta^2-2\theta-5) = 2\theta^2-7\theta.$$

By multiplication, $\epsilon_1\,\epsilon_2\,\epsilon_3 = -1$, as noted in my previous paper.

It is not to be inferred that every semi-reduced ideal is necessarily reduced, an example being

$$\mathfrak{h} = (3, \theta-1, \theta^2+\theta+1).$$

(III) $\eta^3+\eta^2-42\eta+80 = 0.$ $D(\eta) = 4.127^2,$ $D = 127^2.$

This cubic equation is normal, being the cyclotomic equation of three periods for $p = 127$. The integral basis is $(1, \eta, \lambda)$, where $\lambda = \frac{1}{2}(\eta^2+\eta)$ and the conjugate roots are

$\eta = 4{\cdot}410,$ $\eta' = \frac{1}{2}(\eta^2+3\eta-28) = 2{\cdot}340,$ $\eta'' = \frac{1}{2}(-\eta^2-5\eta+26) = -7{\cdot}750,$

$\lambda = 11{\cdot}930,$ $\lambda' = 41-3\eta-2\lambda = 3{\cdot}908,$ $\lambda'' = 1+3\eta+\lambda = 26{\cdot}158.$

All the sixty-one reduced ideals are products of powers of ideal factors of 2, 5, 19, viz.:

$\mathfrak{p} = (2, \eta, \lambda),$ $\mathfrak{q} = (2, \eta-1, \lambda-1),$ $\mathfrak{r} = (2, \eta, \lambda-1),$

$\mathfrak{f} = (5, \eta, \lambda),$ $\mathfrak{g} = (5, \eta-1, \lambda-1),$ $\mathfrak{h} = (5, \eta-3, \lambda-1),$

$\mathfrak{x} = (19, \eta-10, \lambda-17),$ $\mathfrak{y} = (19, \eta-13, \lambda-15),$ $\mathfrak{z} = (19, \eta-14, \lambda-10).$

Beginning as before,

$$\mathfrak{j} = \begin{pmatrix} 1, & \eta, & \lambda \\ -46+11\eta+5\lambda, & 102-7\eta-6\lambda, & 5-4\eta+\lambda \end{pmatrix}$$

$\mathfrak{j}_1 = (-25+3\eta+\lambda)(-46+11\eta+5\lambda,\ 1,\ -9+2\eta+\lambda)$

$= (10,\ 5\eta,\ 5+3\eta+\lambda) = \mathfrak{r}\mathfrak{h}\mathfrak{f}$

$\mathfrak{j}_2 = (15-3\eta-2\lambda)(102-7\eta-6\lambda,\ 1,\ 17-\eta-\lambda)$

$= (10,\ 5\eta-5,\ \eta-\lambda) = \mathfrak{q}\mathfrak{f}\mathfrak{g}$

$\mathfrak{j}_3 = (-26+\lambda)(5-4\eta+\lambda,\ 1,\ \eta-1) = (10,\ 5\eta,\ 4+\lambda) = \mathfrak{p}\mathfrak{g}\mathfrak{h}$

$\mathfrak{j}_{11} = \frac{1}{10}(10+\eta-\lambda)\mathfrak{j}_1 = (5,\ 10+\eta-\lambda,\ 15-6\eta+\lambda) = \mathfrak{f}\mathfrak{g}$

$\mathfrak{j}_{22} = \frac{1}{10}(22-5\eta-2\lambda)\mathfrak{j}_2 = (5,\ 22-5\eta-2\lambda,\ 23-5\eta-3\lambda) = \mathfrak{g}\mathfrak{h}$

$\mathfrak{j}_{33} = \frac{1}{10}(-45+4\eta+3\lambda)\mathfrak{j}_3 = (5,\ -45+4\eta+3\lambda,\ -145+15\eta+10\lambda) = \mathfrak{h}\mathfrak{f}$

$\mathfrak{j}_{12} = \frac{1}{5}(-8+2\eta+\lambda)\mathfrak{j}_1 = (4,\ -16+4\eta+2\lambda,\ 4-\eta) = \mathfrak{p}^2\mathfrak{r}$

$\mathfrak{j}_{23} = \frac{1}{5}(19-\eta-\lambda)\mathfrak{j}_2 = (4,\ 38-2\eta-2\lambda,\ 9-2\eta-\lambda) = \mathfrak{r}^2\mathfrak{q}$

$\mathfrak{j}_{31} = \frac{1}{5}(5-\eta)\mathfrak{j}_3 = (4,\ 10-2\eta,\ 18-\eta-\lambda) = \mathfrak{q}^2\mathfrak{p}$

$\mathfrak{j}_{13} = \frac{1}{10}(-4+3\eta+\lambda)\mathfrak{j}_1 = (4,\ -4+3\eta+\lambda,\ -16+\eta+\lambda) = \mathfrak{p}^2\mathfrak{q}$

$\mathfrak{j}_{21} = \frac{1}{10}(36-3\eta-2\lambda)\mathfrak{j}_2 = (4,\ 36-3\eta-2\lambda,\ 2-\eta) = \mathfrak{r}^2\mathfrak{p}$

$\mathfrak{j}_{32} = \frac{1}{10}(-5+\lambda)\mathfrak{j}_3 = (4,\ 5-\lambda,\ 11-2\eta-\lambda) = \mathfrak{q}^2\mathfrak{r}$

$\mathfrak{j}_{111} = \frac{1}{5}(-8-\eta+\lambda)\mathfrak{j}_{11} = (4,\ -8-\eta+\lambda,\ -28+3\eta+\lambda) = \mathfrak{p}^2\mathfrak{q}^2$

$\mathfrak{j}_{222} = \frac{1}{5}(-20+5\eta+2\lambda)\mathfrak{j}_{22} = (4,\ -20+5\eta+2\lambda,\ -44+12\eta+4\lambda) = \mathfrak{r}^2\mathfrak{p}^2$

$\mathfrak{j}_{333} = \frac{1}{5}(47-4\eta-3\lambda)\mathfrak{j}_{33} = (4,\ 47-4\eta-3\lambda,\ 72-4\eta-4\lambda) = \mathfrak{q}^2\mathfrak{r}^2$

$\mathfrak{j}_{112} = \frac{1}{5}(13-4\eta-\lambda)\mathfrak{j}_{11} = (4,\ 13-4\eta-\lambda,\ 5-\lambda) = \mathfrak{j}_{333}$

$\mathfrak{j}_{223} = \frac{1}{5}(-40+5\eta+3\lambda)\mathfrak{j}_{22} = (4,\ -40+5\eta+3\lambda,\ -4+8\eta+4\lambda) = \mathfrak{j}_{111}$

$\mathfrak{j}_{331} = \frac{1}{5}(28-\eta-2\lambda)\mathfrak{j}_{33} = (4,\ 28-\eta-2\lambda,\ 20+\eta-2\lambda) = \mathfrak{j}_{222}$

$\mathfrak{j}_{113} = \frac{1}{5}(38-5\eta-3\lambda)\mathfrak{j}_{11} = (10,\ 38-5\eta-3\lambda,\ 120-20\eta-10\lambda) = \mathfrak{p}\mathfrak{q}\mathfrak{g}\mathfrak{h}$

$\mathfrak{j}_{221} = \frac{1}{5}(-30+\eta+2\lambda)\mathfrak{j}_{22} = (10,\ 30-\eta-2\lambda,\ 10-5\eta) = \mathfrak{r}\mathfrak{p}\mathfrak{h}\mathfrak{f}$

$\mathfrak{j}_{332} = \frac{1}{5}(-15+4\eta+\lambda)\mathfrak{j}_{33} = (10,\ -15+4\eta+\lambda,\ -45+18\eta+7\lambda) = \mathfrak{q}\mathfrak{r}\mathfrak{f}\mathfrak{g}$

$\mathfrak{j}_{121} = \frac{1}{2}(11-\lambda)\mathfrak{j}_{12} = (10,\ 2\lambda-22,\ 80-5\eta+2\lambda) = \mathfrak{r}\mathfrak{p}\mathfrak{g}\mathfrak{h}$

$\mathfrak{j}_{232} = \frac{1}{2}(10-3\eta-\lambda)\mathfrak{j}_{23} = (10,\ 20-6\eta-2\lambda,\ 65-16\eta-7\lambda) = \mathfrak{q}\mathfrak{r}\mathfrak{h}\mathfrak{f}$

$\mathfrak{j}_{313} = \frac{1}{2}(-30+3\eta+2\lambda)\mathfrak{j}_{31} = (10,\ -60+6\eta+4\lambda,\ 150-11\eta-9\lambda) = \mathfrak{p}\mathfrak{q}\mathfrak{f}\mathfrak{g}$

$j_{122} = (2-\eta)\, j_{12} = (8,\ 8-4\eta,\ 80-39\eta+2\lambda) = \mathfrak{r}^3\mathfrak{p}^3$

$j_{233} = (-11+2\eta+\lambda)\, j_{23} = (8,\ -44+8\eta+4\lambda,\ -81+20\eta+9\lambda) = \mathfrak{q}^3\mathfrak{r}^3$

$j_{311} = (16-\eta-\lambda)\, j_{31} = (8,\ 64-4\eta-4\lambda,\ -188+13\eta+11\lambda) = \mathfrak{p}^3\mathfrak{q}^3$

$j_{123} = \frac{1}{4}(-11+2\eta+\lambda)\, j_{12} = (1,\ -11+2\eta+\lambda,\ -50+11\eta+5\lambda) = j$

$j_{231} = \frac{1}{4}(16-\eta-\lambda)\, j_{23} = (1,\ 16-\eta-\lambda,\ 10-\lambda) = j$

$j_{312} = \frac{1}{4}(-\eta+2)\, j_{31} = (1,\ 2-\eta,\ 4-4\eta+\lambda) = j$

$j_{131} = \frac{1}{2}(-11+\lambda)\, j_{13} = (10,\ 22-2\lambda,\ 70-5\eta-5\lambda) = j_{113}$

$j_{212} = \frac{1}{2}(-10+3\eta+\lambda)\, j_{21} = (10,\ -20+6\eta+2\lambda,\ -20+\eta+2\lambda) = j_{221}$

$j_{323} = \frac{1}{2}(30-3\eta-2\lambda)\, j_{32} = (10,\ -60+6\eta+4\lambda,\ 5-4\eta-\lambda) = j_{332}$

$j_{132} = \frac{1}{4}(9-2\eta-\lambda)\, j_{13} = (1,\ 9-2\eta-\lambda,\ 46-11\eta-5\lambda) = j$

$j_{213} = \frac{1}{4}(-18+\eta+\lambda)\, j_{21} = (1,\ -18+\eta+\lambda,\ -102+7\eta+6\lambda) = j$

$j_{321} = \frac{1}{4}(-4+\eta)\, j_{32} = (1,\ 4-\eta,\ 5-4\eta+\lambda) = j$

$j_{133} = \frac{1}{2}(-11+2\eta+\lambda)\, j_{13} = (4,\ -22+4\eta+2\lambda,\ -70+15\eta+7\lambda) = j_{31}$

$j_{211} = \frac{1}{2}(16-\eta-\lambda)\, j_{21} = (4,\ 32-2\eta-2\lambda,\ 168-11\eta-10\lambda) = j_{12}$

$j_{322} = \frac{1}{2}(2-\eta)\, j_{32} = (4,\ 4-2\eta,\ 15-8\eta+\lambda) = j_{23}$

$j_{1111} = \frac{1}{4}(-65+7\eta+3\lambda)\, j_{111} = (25,\ 25\eta,\ 5+6\eta-\lambda) = \mathfrak{f}^2\mathfrak{g}^2$

$j_{2222} = \frac{1}{4}(29-5\eta-4\lambda)\, j_{222} = (25,\ 5-5\lambda,\ 1+5\eta-\lambda) = \mathfrak{g}^2\mathfrak{h}^2$

$j_{3333} = \frac{1}{4}(-40-2\eta+\lambda)\, j_{222} = (25,\ 25\eta,\ 10-2\eta+\lambda) = \mathfrak{h}^2\mathfrak{f}^2$

$j_{1112} = \frac{1}{4}(2-\eta)\, j_{111} = (2,\ \eta,\ \lambda) = \mathfrak{p}$

$j_{2223} = \frac{1}{4}(-11+2\eta+\lambda)\, j_{222} = (2,\ \eta,\ 1+\lambda) = \mathfrak{r}$

$j_{3331} = \frac{1}{4}(16-\eta-\lambda)\, j_{333} = (2,\ 1+\eta,\ 1+\lambda) = \mathfrak{q}$

$j_{1113} = \frac{1}{4}(28-3\eta-2\lambda)\, j_{111} = (8,\ 2\eta-4,\ 2\lambda+\eta) = \mathfrak{p}^2\mathfrak{r}^3$

$j_{2221} = \frac{1}{4}(-13+\lambda)\, j_{222} = (8,\ 4+4\eta,\ 3+\lambda) = \mathfrak{r}^2\mathfrak{q}^3$

$j_{3332} = \frac{1}{4}(-12+3\eta+\lambda)\, j_{333} = (8,\ 4\eta,\ -4+3\eta+\lambda) = \mathfrak{q}^2\mathfrak{p}^3$

$j_{1131} = \frac{1}{10}(-30+\eta+2\lambda)\, j_{113} = (10,\ 5\lambda,\ \eta+2\lambda) = \mathfrak{p}\mathfrak{h}\mathfrak{f}$

$j_{2212} = \frac{1}{10}(-15+4\eta+\lambda)\, j_{221} = (10,\ 5\eta,\ 5+4\eta+\lambda) = \mathfrak{r}\mathfrak{f}\mathfrak{g}$

$j_{3323} = \frac{1}{10}(38-5\eta-3\lambda)\, j_{332} = (10,\ 5+5\eta,\ \lambda-1) = \mathfrak{q}\mathfrak{g}\mathfrak{h}$

$$\mathfrak{j}_{1132} = \tfrac{1}{10}(-15+3\eta+\lambda)\,\mathfrak{j}_{113} = (10,\ 5\eta,\ 5+3\eta+\lambda) = \mathfrak{j}_1$$

$$\mathfrak{j}_{2213} = \tfrac{1}{10}(25-3\eta-2\lambda)\,\mathfrak{j}_{221} = (10,\ 5+5\eta,\ \eta-\lambda) = \mathfrak{j}_2$$

$$\mathfrak{j}_{3321} = \tfrac{1}{10}(-16+\lambda)\,\mathfrak{j}_{332} = (10,\ 5\eta,\ 4+\lambda) = \mathfrak{j}_3$$

$$\mathfrak{j}_{1133} = \tfrac{1}{10}(-25+\lambda)\,\mathfrak{j}_{113} = (8,\ 4\eta,\ -1+\lambda) = \mathfrak{q}^2\mathfrak{r}^3$$

$$\mathfrak{j}_{2211} = \tfrac{1}{10}(-24+3\eta+\lambda)\,\mathfrak{j}_{221} = (8,\ 4+4\eta,\ \lambda+3\eta) = \mathfrak{p}^2\mathfrak{q}^3$$

$$\mathfrak{j}_{3322} = \tfrac{1}{10}(16-3\eta-2\lambda)\,\mathfrak{j}_{332} = (8,\ 4\lambda,\ \eta+2\lambda) = \mathfrak{r}^2\mathfrak{p}^3$$

$$\mathfrak{j}_{1211} = \tfrac{1}{10}(-35+3\eta+2\lambda)\,\mathfrak{j}_{121} = (5,\ 5\eta,\ \eta-\lambda) = \mathfrak{j}_{11}$$

$$\mathfrak{j}_{2322} = \tfrac{1}{10}(6-\lambda)\,\mathfrak{j}_{232} = (5,\ 5\eta,\ \lambda-1) = \mathfrak{j}_{22}$$

$$\mathfrak{j}_{3133} = \tfrac{1}{10}(5-3\eta-\lambda)\,\mathfrak{j}_{313} = (5,\ 5\eta,\ \lambda+3\eta) = \mathfrak{j}_{33}$$

$$\mathfrak{j}_{1212} = \tfrac{1}{10}(5-\lambda)\,\mathfrak{j}_{121} = (4,\ 4\eta,\ 3+\lambda) = \mathfrak{j}_{112}$$

$$\mathfrak{j}_{2323} = \tfrac{1}{10}(4-3\eta-\lambda)\,\mathfrak{j}_{232} = (4,\ 4\eta,\ \eta-\lambda) = \mathfrak{j}_{223}$$

$$\mathfrak{j}_{3131} = \tfrac{1}{10}(-36+3\eta+2\lambda)\,\mathfrak{j}_{313} = (4,\ 4\lambda,\ \eta+2\lambda) = \mathfrak{j}_{331}$$

$$\mathfrak{j}_{1213} = \tfrac{1}{2}(-11+2\eta+\lambda)\,\mathfrak{j}_{121} = (10,\ 10\eta,\ -1+\lambda) = \mathfrak{q}\mathfrak{r}\mathfrak{g}\mathfrak{h}$$

$$\mathfrak{j}_{2321} = \tfrac{1}{2}(16-\eta-\lambda)\,\mathfrak{j}_{232} = (10,\ 5\eta+5\lambda,\ 2\eta+4\lambda) = \mathfrak{p}\mathfrak{q}\mathfrak{h}\mathfrak{f}$$

$$\mathfrak{j}_{3132} = \tfrac{1}{2}(2-\eta)\,\mathfrak{j}_{313} = (10,\ 5\eta,\ 2\eta-2\lambda) = \mathfrak{r}\mathfrak{p}\mathfrak{f}\mathfrak{g}$$

$$\mathfrak{j}_{1221} = \tfrac{1}{8}(-13+\lambda)\,\mathfrak{j}_{122} = (4,\ 4\eta,\ 3+\lambda) = \mathfrak{j}_{333}$$

$$\mathfrak{j}_{2332} = \tfrac{1}{8}(-12+3\eta+\lambda)\,\mathfrak{j}_{233} = (4,\ 4\eta,\ \eta-\lambda) = \mathfrak{j}_{111}$$

$$\mathfrak{j}_{3113} = \tfrac{1}{8}(28-3\eta-2\lambda)\,\mathfrak{j}_{311} = (4,\ \eta+2\lambda,\ \eta-2\lambda) = \mathfrak{j}_{222}$$

$$\mathfrak{j}_{1222} = \tfrac{1}{8}(39-6\eta-5\lambda)\,\mathfrak{j}_{122} = (38,\ 38\eta,\ 15-14\eta+\lambda) = \mathfrak{q}\mathfrak{r}\mathfrak{y}\mathfrak{z}$$

$$\mathfrak{j}_{2333} = \tfrac{1}{8}(-44-3\eta+\lambda)\,\mathfrak{j}_{233} = (38,\ 38\eta,\ 6+3\eta-\lambda) = \mathfrak{p}\mathfrak{q}\mathfrak{x}\mathfrak{z}$$

$$\mathfrak{j}_{3111} = \tfrac{1}{8}(-82+9\eta+4\lambda)\,\mathfrak{j}_{311} = (38,\ 38\lambda,\ 12+\eta-8\lambda) = \mathfrak{r}\mathfrak{p}\mathfrak{x}\mathfrak{y}$$

$$\mathfrak{j}_{1223} = (11-2\eta-\lambda)\,\mathfrak{j}_{122} = (4,\ \eta+2,\ 2\lambda+2) = \mathfrak{j}_{21}$$

$$\mathfrak{j}_{2331} = (-16+\eta+\lambda)\,\mathfrak{j}_{233} = (4,\ 2+2\eta,\ 1-\lambda) = \mathfrak{j}_{32}$$

$$\mathfrak{j}_{3112} = (-2+\eta)\,\mathfrak{j}_{311} = (4,\ 2\eta,\ \eta-\lambda) = \mathfrak{j}_{13}$$

$$\mathfrak{j}_{11111} = \tfrac{1}{25}(-17+2\eta+\lambda)\,\mathfrak{j}_{1111} = (4,\ 2\eta,\ \lambda-1) = \mathfrak{j}_{23}$$

$$\mathfrak{j}_{22222} = \tfrac{1}{25}(10-\eta-\lambda)\,\mathfrak{j}_{2222} = (4,\ 2+2\eta,\ 2+2\lambda,\ 2+\eta+\lambda) = \mathfrak{j}_{31}$$

$$\mathfrak{j}_{33333} = \tfrac{1}{25}(-4-\eta)\,\mathfrak{j}_{3333} = (4,\ \eta,\ 2\lambda) = \mathfrak{j}_{12}$$

$j_{11112} = \frac{1}{25}(25-4\eta-3\lambda)\, j_{1111} = (10,\ 10\eta,\ 5-2\eta+\lambda) = j_{232}$

$j_{22223} = \frac{1}{25}(-30-\eta+\lambda)\, j_{2222} = (10,\ 5\eta+5\lambda,\ \eta-\lambda) = j_{313}$

$j_{33331} = \frac{1}{25}(-42+5\eta+2\lambda)\, j_{3333} = (10,\ 5\eta,\ -2+2\lambda) = j_{121}$

$j_{11113} = \frac{1}{25}(-28-\eta+\lambda)\, j_{1111} = (16,\ 8\eta,\ 4-\eta+\lambda) = \mathfrak{p}^4\mathfrak{q}^3$

$j_{22221} = \frac{1}{25}(-40+5\eta+2\lambda)\, j_{2222} = (16,\ 8-4\eta,\ \eta+2\lambda) = \mathfrak{r}^4\mathfrak{p}^3$

$j_{33332} = \frac{1}{25}(27-4\eta-3\lambda)\, j_{3333} = (16,\ 8+8\eta,\ 1-4\eta-\lambda) = \mathfrak{q}^4\mathfrak{r}^3$

$j_{11121} = \frac{1}{2}(-11+\lambda)\,(-10+2\eta+\lambda,\ 2,\ \eta) = (5,\ 5\eta,\ -1+\lambda) = j_{22}$

$j_{22232} = \frac{1}{2}(-10+3\eta+\lambda)\,(17-\eta-\lambda,\ 2,\ \eta) = (5,\ 5\eta,\ 3\eta+\lambda) = j_{33}$

$j_{33313} = \frac{1}{2}(30-3\eta-2\lambda)\,(3-\eta,\ 2,\ 1+\lambda) = (5,\ 5\eta,\ \eta-\lambda) = j_{11}$

$j_{11122} = (2-\eta)\,(18-\eta-\lambda,\ 2,\ \eta) = (4,\ 4\lambda,\ \eta+2\lambda) = j_{222}$

$j_{22233} = (-11+2\eta+\lambda)\,(4-\eta,\ 2,\ 1+\lambda) = (4,\ 4\eta,\ 1-\lambda) = j_{333}$

$j_{33311} = (16-\eta-\lambda)\,(-9+2\eta+\lambda,\ 2,\ 1+\eta) = (4,\ 2\eta+2\lambda,\ \eta-\lambda) = j_{111}$

$j_{11123} = \frac{1}{2}(-11+2\eta+\lambda)\,(4-\eta,\ 2,\ \lambda) = (2,\ 1+\eta,\ 1+\lambda) = j_{3331}$

$j_{22231} = \frac{1}{2}(16-\eta-\lambda)\,(-9+2\eta+\lambda,\ 2,\ \eta) = (2,\ \eta,\ \lambda) = j_{1112}$

$j_{33312} = \frac{1}{2}(2-\eta)\,(18-\eta-\lambda,\ 2,\ 1+\eta) = (2,\ \eta,\ 1+\lambda) = j_{2223}$

$j_{11131} = \frac{1}{4}(-13+\lambda)\, j_{1113} = (8,\ 8\eta,\ 1-4\eta-\lambda) = j_{233}$

$j_{22212} = \frac{1}{4}(-12+3\eta+\lambda)\, j_{2221} = (8,\ 8\eta,\ 4-\eta+\lambda) = j_{311}$

$j_{33323} = \frac{1}{4}(28-3\eta-2\lambda)\, j_{3332} = (8,\ 4\eta,\ \eta+2\lambda) = j_{122}$

$j_{11132} = \frac{1}{8}(-30+7\eta+3\lambda)\, j_{1113} = (5,\ 5\eta,\ \eta-\lambda) = j_{11}$

$j_{22213} = \frac{1}{8}(64-5\eta-4\lambda)\, j_{2221} = (5,\ 5\eta,\ 1-\lambda) = j_{22}$

$j_{33321} = \frac{1}{8}(-5-2\eta+\lambda)\, j_{3332} = (5,\ 2\eta-\lambda,\ \eta+2\lambda) = j_{33}$

$j_{11133} = \frac{1}{4}(-18+\eta+\lambda)\, j_{1113} = (2,\ \eta,\ 2\lambda) = \mathfrak{r}\mathfrak{p}$

$j_{22211} = \frac{1}{4}(-4+\eta)\, j_{2221} = (2,\ 2\eta,\ 1+\lambda) = \mathfrak{q}\mathfrak{r}$

$j_{33322} = \frac{1}{4}(9-2\eta-\lambda)\, j_{3332} = (2,\ 2\eta,\ \eta+\lambda) = \mathfrak{p}\mathfrak{q}$

$j_{11311} = \frac{1}{5}(-17+\eta+\lambda)\, j_{1131} = (2,\ \eta,\ 2\lambda) = j_{11133}$

$j_{22122} = \frac{1}{5}(-3+\eta)\, j_{2212} = (2,\ 2\eta,\ 1+\lambda) = j_{22211}$

$j_{33233} = \frac{1}{5}(10-2\eta-\lambda)\, j_{3323} = (2,\ 2\eta,\ \eta+\lambda) = j_{33322}$

$j_{11312} = \frac{1}{10}(-15+4\eta+\lambda)\, j_{1131} = (5,\ 5\eta,\ \eta-\lambda) = j_{11}$

$j_{22123} = \frac{1}{10}(38-5\eta-3\lambda)\, j_{2212} = (5,\ 5\eta,\ 1-\lambda) = j_{22}$

$j_{33231} = \frac{1}{10}(-30+\eta+2\lambda)\, j_{3323} = (5,\ \eta+2\lambda,\ 2\eta-\lambda) = j_{33}$

$j_{11313} = \frac{1}{10}(13-2\eta-\lambda)\, j_{1131} = (4,\ 2+\eta,\ 3+\lambda) = \mathfrak{r}^2$

$j_{22121} = \frac{1}{10}(-14+\eta+\lambda)\, j_{2212} = (4,\ 1-\eta,\ 1-\lambda) = \mathfrak{q}^2$

$j_{33232} = \frac{1}{10}\eta\, j_{3323} = (4,\ \eta,\ \lambda) = \mathfrak{p}^2$

$j_{11331} = \frac{1}{4}(16-\eta-\lambda)\, j_{1133} = (2,\ 2\eta,\ 1+\lambda) = j_{22211}$

$j_{22112} = \frac{1}{4}(2-\eta)\, j_{2211} = (2,\ 2\eta,\ \eta+\lambda) = j_{33322}$

$j_{33223} = \frac{1}{4}(-11+2\eta+\lambda)\, j_{3322} = (2,\ \eta,\ 2\lambda) = j_{11133}$

$j_{11332} = \frac{1}{8}(-12+3\eta+\lambda)\, j_{1133} = (4,\ 2\eta,\ \eta+\lambda) = j_{13}$

$j_{22113} = \frac{1}{8}(28-3\eta-2\lambda)\, j_{2211} = (4,\ 2-\eta,\ 2-2\lambda) = j_{21}$

$j_{33221} = \frac{1}{8}(-13+\lambda)\, j_{3322} = (4,\ 2+2\eta,\ 1-\lambda) = j_{32}$

$j_{11333} = \frac{1}{8}(36-5\eta-3\lambda)\, j_{1133} = (8,\ 8\eta,\ 4+\eta-\lambda) = j_{311}$

$j_{22111} = \frac{1}{8}(-32+\eta+2\lambda)\, j_{2211} = (8,\ 8\lambda,\ \eta+2\lambda) = j_{122}$

$j_{33222} = \frac{1}{8}(-17+4\eta+\lambda)\, j_{3322} = (8,\ 8\eta,\ 1+4\eta-\lambda) = j_{233}$

$j_{12131} = \frac{1}{10}(-30+\eta+2\lambda)\, j_{1213} = (10,\ 5-5\lambda,\ \eta+2\lambda) = j_{1}$

$j_{23212} = \frac{1}{10}(-15+4\eta+\lambda)\, j_{2321} = (10,\ 5-5\eta,\ \eta-\lambda) = j_{2}$

$j_{31323} = \frac{1}{10}(38-5\eta-3\lambda)\, j_{3132} = (10,\ 5\eta,\ 4+\lambda) = j_{3}$

$j_{12132} = \frac{1}{10}\eta(21-\lambda,\ 10,\ -55+10\eta+5\lambda) = (4,\ \eta,\ 2\lambda) = j_{12}$

$j_{23213} = \frac{1}{10}(13-2\eta-\lambda)(20-3\eta-\lambda,\ 10,\ 5\eta+5\lambda) = (4,\ 2\eta,\ \lambda-1) = j_{23}$

$j_{31321} = \frac{1}{10}(-14+\eta+\lambda)(-20+3\eta+2\lambda,\ 10,\ 2\eta-2\lambda)$

$= (4,\ 2+2\eta,\ \eta-\lambda) = j_{31}$

$j_{12133} = \frac{1}{10}(10-2\eta-\lambda)\, j_{1213} = (1,\ \eta,\ \lambda) = \mathfrak{j}$

$j_{23211} = \frac{1}{10}(-17+\eta+\lambda)\, j_{2321} = (1,\ \eta,\ \lambda) = \mathfrak{j}$

$j_{31322} = \frac{1}{10}(-3+\eta)\, j_{3132} = (1,\ \eta,\ \lambda) = \mathfrak{j}$

$j_{12221} = \frac{1}{38}(18-2\eta-\lambda)\, j_{1222} = (5,\ 1-\eta,\ 1-\lambda) = \mathfrak{g}$

$j_{23332} = \frac{1}{38}(-9+\eta+\lambda)\, j_{2333} = (5,\ 2+\eta,\ 4+\lambda) = \mathfrak{h}$

$j_{31113} = \frac{1}{38}(5+\eta)\, j_{3111} = (5,\ \eta,\ \lambda) = \mathfrak{f}$

$j_{12222} = \frac{1}{38}(8-\eta-\lambda)\, j_{1222} = (2,\ 2\eta,\ \eta+\lambda) = j_{33322}$

$j_{23333} = \frac{1}{38}(-6-\eta)\, j_{2333} = (2,\ \eta,\ 2\lambda) = j_{11133}$

$j_{31111} = \frac{1}{38}(-19+2\eta+\lambda)\, j_{3111} = (2,\ 2\eta,\ 1+\lambda) = j_{22211}$

$j_{12223} = \frac{1}{38}(44-5\eta-3\lambda)\, j_{1222} = (8,\ 4-4\eta,\ 3+\lambda) = j_{2211}$

$j_{23331} = \frac{1}{38}(-24+\eta+2\lambda)\, j_{2333} = (8,\ 2\eta,\ \eta+2\lambda) = j_{3322}$

$j_{31112} = \frac{1}{38}(-9+4\eta+\lambda)\, j_{3111} = (8,\ 4\eta,\ 1-\lambda) = j_{1133}$

$j_{111131} = \frac{1}{16}(-9+\lambda)\, j_{11113} = (4,\ 2\eta,\ 1-\lambda) = j_{23}$

$j_{222212} = \frac{1}{16}(-8+3\eta+\lambda)\, j_{22221} = (4,\ 2-2\eta,\ 2-\eta-\lambda) = j_{31}$

$j_{333323} = \frac{1}{16}(32-3\eta-2\lambda)\, j_{33332} = (4,\ \eta,\ 2\lambda) = j_{12}$

$j_{111132} = \frac{1}{8}(-24+5\eta+2\lambda)\, j_{11113} = (16,\ 16\lambda,\ \eta+2\lambda) = \mathfrak{r}^4\mathfrak{p}^4$

$j_{222213} = \frac{1}{8}(43-4\eta-3\lambda)\, j_{22221} = (16,\ 16\eta,\ -1+4\eta+\lambda) = \mathfrak{q}^4\mathfrak{r}^4$

$j_{333321} = \frac{1}{8}(-12-\eta+\lambda)\, j_{33332} = (16,\ 16\eta,\ 4+7\eta+\lambda) = \mathfrak{p}^4\mathfrak{q}^4$

$j_{111133} = \frac{1}{16}(-1+4\eta+\lambda)\, j_{11113} = (8,\ 8\eta,\ -1+4\eta+\lambda) = j_{233}$

$j_{222211} = \frac{1}{16}(52-5\eta-3\lambda)\, j_{22221} = (8,\ 8\eta,\ 4-\eta+\lambda) = j_{311}$

$j_{333322} = \frac{1}{16}(-16+\eta+2\lambda)\, j_{33332} = (8,\ 4\eta,\ \eta+2\lambda) = j_{122}$

$j_{111331} = \frac{1}{2}(-45+2\eta+3\lambda)\, j_{11133} = (10,\ 10\eta,\ 5+4\eta+\lambda) = j_{323}$

$j_{222112} = \frac{1}{2}(-16+5\eta+\lambda)\, j_{22211} = (10,\ 10\eta,\ 4+5\eta+\lambda) = j_{131}$

$j_{333223} = \frac{1}{2}(50-7\eta-4\lambda)\, j_{33322} = (10,\ 5\eta,\ \eta+2\lambda) = j_{212}$

$j_{111332} = \frac{1}{2}(15-3\eta-2\lambda)\, j_{11133} = (5,\ 5\eta,\ \eta-\lambda) = j_{11}$

$j_{222113} = \frac{1}{2}(-26+\lambda)\, j_{22211} = (5,\ 5\eta,\ 1-\lambda) = j_{22}$

$j_{333221} = \frac{1}{2}(-25+3\eta+\lambda)\, j_{33322} = (5,\ 5\eta,\ 3\eta+\lambda) = j_{33}$

$j_{111333} = \frac{1}{2}(-11+2\eta+\lambda)\, j_{11133} = (2,\ 2\eta,\ 1-\lambda) = j_{22211}$

$j_{222111} = \frac{1}{2}(16-\eta-\lambda)\, j_{22211} = (2,\ 2\eta,\ \eta+\lambda) = j_{33322}$

$j_{333222} = \frac{1}{2}(2-\eta)\, j_{33322} = (2,\ \eta,\ 2\lambda) = j_{11133}$

$j_{113131} = \frac{1}{2}(16-\eta-\lambda)(-9+2\eta+\lambda,\ 4,\ 2-\eta) = (2,\ \eta,\ 2\lambda) = j_{11133}$

$j_{221212} = \frac{1}{2}(2-\eta)(18-\eta-\lambda,\ 4,\ 1-\lambda) = (2,\ 2\eta,\ 1-\lambda) = j_{22211}$

$j_{332323} = \frac{1}{2}(-11+2\eta+\lambda)(4-\eta,\ 4,\ \lambda) = (2,\ 2\eta,\ \eta+\lambda) = j_{33322}$

$j_{113132} = \frac{1}{4}(28-5\eta-3\lambda)\, j_{11313} = (10,\ 5\eta+5\lambda,\ 2-2\lambda) = j_{131}$

$j_{221213} = \frac{1}{4}(-40+\eta+2\lambda)\, j_{22121} = (10,\ 5\eta,\ \eta+2\lambda) = j_{212}$

$j_{332321} = \frac{1}{4}(-25+4\eta+\lambda)\, j_{33232} = (10,\ 5+5\lambda,\ 2\eta-2\lambda) = j_{323}$

$j_{113133} = \frac{1}{2}(-18+\eta+\lambda)\, j_{11313} = (2,\ 2\eta,\ 1+\lambda) = j_{22211}$

$j_{221211} = \frac{1}{2}(-4+\eta)\, j_{22121} = (2,\ 2\eta,\ \eta+\lambda) = j_{33322}$

$j_{332322} = \frac{1}{2}(9-2\eta-\lambda)\, j_{33232} = (2,\ \eta,\ 2\lambda) = j_{11133}$

$j_{122211} = \frac{1}{5}(-20+3\eta+\lambda)(-13+2\eta+\lambda,\ 5,\ 1-\eta) = (8,\ 4\eta,\ 4+\eta-\lambda) = j_{3332}$

$j_{233322} = \frac{1}{5}(20-3\eta-2\lambda)(14-\eta-\lambda,\ 5,\ 2+\eta) = (8,\ 4+2\eta,\ \eta+2\lambda) = j_{1113}$

$j_{311133} = \frac{1}{5}(-21+\lambda)(\eta,\ 5,\ \lambda) = (8,\ 4+4\eta,\ 3+\lambda) = j_{2221}$

$j_{122212} = \frac{1}{5}(-10+3\eta+\lambda)\, j_{12221} = (2,\ 2\eta,\ \eta+\lambda) = j_{33322}$

$j_{233323} = \frac{1}{5}(30-3\eta-2\lambda)\, j_{23332} = (2,\ \eta,\ 2\lambda) = j_{11133}$

$j_{311131} = \frac{1}{5}(-11+\lambda)\, j_{31113} = (2,\ 2\eta,\ \lambda+1) = j_{22211}$

$j_{122213} = \frac{1}{5}(-40+\eta+2\lambda)\, j_{12221} = (8,\ 4\eta,\ \eta+2\lambda) = j_{122}$

$j_{233321} = \frac{1}{5}(-25+4\eta+\lambda)\, j_{23332} = (8,\ 8\eta,\ 1+4\eta-\lambda) = j_{233}$

$j_{311132} = \frac{1}{5}(28-5\eta-3\lambda)\, j_{31113} = (8,\ 8\eta,\ 4-\eta+\lambda) = j_{311}$

$j_{1111321} = \frac{1}{16}(-45+\eta+4\lambda)\, j_{111132} = (25,\ 25\eta,\ 5+6\eta-\lambda) = j_{1111}$

$j_{2222132} = \frac{1}{16}(-28+10\eta+3\lambda)\, j_{222213} = (25,\ 25\eta,\ 1+5\eta-\lambda) = j_{2222}$

$j_{3333213} = \frac{1}{16}(105-11\eta-7\lambda)\, j_{333321} = (25,\ 25\eta,\ 10-2\eta+\lambda) = j_{3333}$

$j_{1111322} = \frac{1}{16}(-8+3\eta+\lambda)\, j_{111132} = (4,\ 4\eta,\ \eta-\lambda) = j_{111}$

$j_{2222133} = \frac{1}{16}(32-3\eta-2\lambda)\, j_{222213} = (4,\ 2\eta,\ \eta+2\lambda) = j_{222}$

$j_{3333211} = \frac{1}{16}(-9+\lambda)\, j_{333321} = (4,\ 4\eta,\ 1-\lambda) = j_{333}$

$j_{1111323} = \frac{1}{16}(43-4\eta-3\lambda)\, j_{111132} = (8,\ 8\eta,\ 1+4\eta-\lambda) = j_{233}$

$j_{2222131} = \frac{1}{16}(-12-\eta+\lambda)\, j_{222213} = (8,\ 8\eta,\ 4+\eta-\lambda) = j_{311}$

$j_{3333212} = \frac{1}{16}(-24+5\eta+2\lambda)\, j_{333321} = (8,\ 4\eta,\ \eta+2\lambda) = j_{122}.$

The unit ϵ_1, that for which $\epsilon > 1$ and least, subject to $|\epsilon'| < 1$, $|\epsilon''| < 1$, appears to be found from the sequence of ideals

$$j,\ j_1,\ j_{11},\ j_{111},\ j_{1111},\ j_{11111} = j_{23},\ j_{231} = j,$$

viz.:

$$\frac{1}{\epsilon_1} = \frac{-25+3\eta+\lambda}{1}\,\frac{10+\eta-\lambda}{10}\,\frac{-8-\eta+\lambda}{5}$$

$$\times\frac{-65+7\eta+3\lambda}{4}\,\frac{-17+2\eta+\lambda}{25}\,\frac{16-\eta-\lambda}{4}$$

$$= 153+14\eta-18\lambda,$$

whence $\epsilon_1 = -35463+8622\eta+3910\lambda.$

The sequence

$$j_{22211},\quad j_{222111}=j_{33322},\quad j_{333221}=j_{33},\quad j_{331}=j_{222},\quad j_{2221},\quad j_{22211}$$

gives the same unit. As in any normal field ϵ_2 and ϵ_3 are the conjugates of ϵ_1, namely

$$\epsilon_3 = -35463+8622\eta'+3910\lambda' = 4139-3108\eta+802\lambda,$$

$$\epsilon_2 = -35463+8622\eta''+3910\lambda'' = 80533-5514\eta-4712\lambda.$$

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ON SOME RELATIONS BETWEEN THE ABEL, BOREL, AND CESÀRO METHODS OF SUMMATION

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It is well known that a series summable by the method of Abel or by that of Borel will be summable C only if subject to some further condition, which is usually given the form of an o or O restriction on the partial Cesàro sums. In certain cases generalizations with O_L conditions have been proved*, and this paper shows how such results may be used to obtain further theorems of the same character. In particular, Theorem 1 is an extension of Andersen's theorem (**1**) that a *series summable A is summable $(C, \alpha' > \alpha)$ if it is bounded (C, α)*, while Theorem 5 is an extension of the theorem due to Hardy and Littlewood, proved when p is an integer and stated for the non-integral case (**5**), that *a series summable B is summable $(C, \alpha+2p)$ if its n-th partial (C, α) sum is $o(n^p)$*. Theorems 3 and 5 provide conditions each sufficient for the Cesàro summability of a series Σa_n whose partial (C, α) sums are $O_L(n^p)$. These conditions are (i) that the associated function $\Sigma a_n z^n$, or any one of its Cesàro means of positive order, tends to a limit as $z \to 1$ by all paths lying within the unit circle; (ii) that the Borel transform, or any one of its Cesàro means of positive order, tends to a limit.

1. *Definitions.*

The (C, α)-, A-, and B-transforms of $\{s_n\}$, the sequence of partial sums of Σa_n†, are respectively $\{s_n{}^\alpha\}$, $f(s_n, x)$, $h(s_n, u)$, defined, whenever the

* *E.g.* Hardy and Littlewood (**6**). We say that $f = O_L(\phi)$, ϕ being positive, if a positive constant K exists such that $f > -K\phi$, and that $f = O_R(\phi)$, if $f < K\phi$. In particular, f is bounded below if $f = O_L(1)$. In what follows K will be used to denote a positive constant, not necessarily the same at each occurrence.

† Here, and elsewhere, Σ denotes $\sum_0^\infty$.

infinite sums exist, by the following relations:

$$s_n^{\alpha} = S_n^{\alpha}/A_n^{\alpha}, \quad s_n^{0} = S_n^{0} = s_n = \sum_0^n a_\nu,$$

where $\alpha \neq -1, -2, -3, \ldots$ (actually we restrict α to be real and greater than -1), and

$$A_n^{\alpha} = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\,\Gamma(n+1)} \sim \frac{n^{\alpha}}{\Gamma(\alpha+1)};$$

$$S_n^{\alpha} = \sum_{\nu=0}^{n} s_\nu A_{n-\nu}^{\alpha-1};$$

$$f(s_n, x) = f(x) = \Sigma a_n x^n;$$

$$h(s_n, u) = h(u) = e^{-u} \Sigma s_n \frac{u^n}{n!}.$$

If $f(x)$ exists for $|x| < 1$, then

$$f(x) = (1-x)\Sigma s_n x^n = (1-x)^{\alpha+1} \Sigma S_n^{\alpha} x^n.$$

We are concerned with the behaviour of s_n^{α} as $n \to \infty$, of $f(x)$ as $x \to 1-0$, and of $h(u)$ as $u \to +\infty$; it will be convenient to use n, x, u in this way, while a statement such as $h(u) = o(u^p)$ will mean "$h(u)$ is defined for all values of u and is $o(u^p)$ as $u \to \infty$ by positive real values". Greater generality is obtained by using combinations of these methods: putting

$$f_\alpha(x) = f(s_n^{\alpha}, x),$$

$$h_\alpha(u) = h(s_n^{\alpha}, u),$$

we say that Σa_n is summable $(C, \alpha;\ A)$ to s if $f_\alpha(x) \to s$ as $x \to 1-0$, and that it is summable $(C, \alpha;\ B)$ to s if $h_\alpha(u) \to s$ as $u \to +\infty$.

2. *Some lemmas.*

2.1. We use the following lemmas.

LEMMA 1. *If $\alpha > -1$, $p \geqslant 0$, and $s_n = O_L(n^p)(C, \alpha)$, i.e. $s_n^{\alpha} > -Kn^p$, then $s_n = O_L(n^p)$ $(C, \alpha' > \alpha)$. O or o can replace O_L.*

This is a simple consequence of the definitions, while the next lemma is due to Zygmund (see Kogbetliantz, **10**, 23).

LEMMA 2. *If $\alpha > -1, \beta > -1, \beta-\alpha > -1, p \geqslant 0$, and $s_n = o(n^p)(C, \beta)$, then $s_n^{\alpha} = o(n^p)$ $(C, \beta-\alpha)$, and conversely.*

The next two lemmas are easy extensions of the well-known theorem to the effect that a series summable C is necessarily summable A, while Lemma 5 follows immediately by combining Lemmas 2 and 3.

Lemma 3. *If* $\alpha > -1$, $p \geqslant 0$, *and* $s_n{}^\alpha \sim \lambda n^p$, *then* $f(x) \sim \mu(1-x)^{-p}$, *where* $\lambda\Gamma(\alpha+p+1) = \mu\Gamma(\alpha+1)$, *while, if* $s_n = o(n^p)$, *then* $f(x) = o\{(1-x)^{-p}\}$.

Lemma 4. *If* $p > -1$, $g(u)$ *is defined for all real values of* u, *and* $g(u) \sim \lambda u^p$, *then* $\int_0^\infty e^{-st} g(t)\,dt \sim \lambda\Gamma(p+1)\,s^{-p-1}$ *as* $s \to +0$, *this integral existing for all real* $s > 0$.

Lemma 5. *If* $\alpha > -1$, $\beta > -1$, $p \geqslant 0$, *and* $s_n{}^\alpha = o(n^p)$, *then*

$$f(s_n{}^\beta, x) = o\{(1-x)^{-p}\}.$$

In particular: a series summable (C, α) *is summable* $(C, \beta;\ A)$.

2.2. Hardy and Littlewood (**5**, Lemma 2) have shown that

$$h_\alpha(u) = e^{-u} \Sigma s_n{}^\alpha \frac{u^n}{n!} = \frac{\alpha}{u^\alpha} \int_0^u (u-t)^{\alpha-1} h(t)\,dt,$$

so that $h_\alpha(u)$ is the (C, α)-transform of $h(u)$. Similarly (Kogbetliantz, **10**, 37-38),

$$f_\alpha(x) = (1-x) \Sigma s_n{}^\alpha x^n = \frac{\alpha}{x^\alpha} (1-x) \int_0^x \left(\frac{x-t}{1-t}\right)^{\alpha-1} \frac{f(t)}{(1-t)^2}\,dt.$$

Making the transformation $1-x = u^{-1}$, $1-t = \omega^{-1}$, $f(t) = \phi(\omega)$, and omitting the term arising from the inessential factor $x^{-\alpha}$, we obtain the right-hand side in the more familiar form

$$\frac{\alpha}{u^\alpha} \int_1^u (u-\omega)^{\alpha-1} \phi(\omega)\,d\omega.$$

Consequent, then, on the corresponding theorems for the Cesàro limit of a function, we have the consistency theorems:

Lemma 6. *If* $p \geqslant 0$, $\beta \geqslant \alpha \geqslant 0$, *and* $h_\alpha(u) = o(u^p)$, *then* $h_\beta(u) = o(u^p)$.

Lemma 7. *If* $p \geqslant 0$, $\beta \geqslant \alpha \geqslant 0$, *and* $f_\alpha(x) = o\{(1-x)^{-p}\}$, *then*

$$f_\beta(x) = o\{(1-x)^{-p}\}.$$

2.3. It is easy to verify the following identities (**10**, 23, 30), which account for the importance of the rôle played by the sequence $\{t_n\}$, where $t_n = na_n$.

$$t_n^{\alpha} = n(s_n^{\alpha} - s_{n-1}^{\alpha}) \qquad (\alpha \neq -1, -2, \ldots) \tag{1}$$

$$t_n^{\alpha} = \alpha(s_n^{\alpha-1} - s_n^{\alpha}) \qquad (\alpha \neq 0, -1, -2, \ldots). \tag{2}$$

3. *Relations between the methods of Abel and Cesàro.*

3.1. We first prove the following two theorems.

THEOREM 1. *If* $q \geqslant 0$, $p \geqslant 0$, $\alpha > -1$, *and if*

$$f_q(x) = o\{(1-x)^{-p}\}, \tag{3}$$

$$s_n^{\alpha} = O_L(v_n), \quad \textit{where} \quad v_n^{1} \sim n^p, \tag{4}$$

then

$$s_n^{\alpha+1} = o(n^p).$$

THEOREM 2. *If* $q \geqslant 0$, $p \geqslant 0$, $\alpha > -1$, *and* (3) *holds, and if also*

$$t_n^{\alpha} = O_L(v_n), \quad \textit{where} \quad v_n^{1} \sim n^p, \tag{5}$$

then

$$s_n^{\alpha} = o(n^p).$$

These theorems are deductions from the following well-known theorem of Hardy and Littlewood (**6**; see also Karamata, **8**).

THEOREM A. *If* $r \geqslant 0$, $\Sigma b_n x^n \sim B(1-x)^{-r}$ *as* $x \to 1-0$, *and if* $b_n = O_L(n^{r-1})$, *then* $\sum_0^n b_\nu \sim Bn^r/\Gamma(r+1)$.

In fact, taking $b_n = S_n^{\alpha}$, $r = \alpha+1$, and remembering that

$$(1-x)^{-\alpha-1} f(x) = \Sigma S_n^{\alpha} x^n,$$

we immediately find that *a series summable A and unilaterally bounded* (C, α) *is summable* $(C, \alpha+1)$. This is a special case of Theorem 1, and other special cases have been given by various writers. Both theorems are given by Kogbetliantz (**10**, 39, 40) with $p = 0$, $\alpha \geqslant 0$, $v_n = 1$; the restriction on α is removed by use of Lemma 5. Karamata (**9**) has proved a theorem that is more general in some respects, and he employed the auxiliary sequence $\{v_n\}$ of condition (4). This is chiefly of advantage in

obtaining the generalization to series of complex terms. For suppose Σa_n to have complex terms and to be subject to the conditions of Theorem 1, with (4) replaced by

$$|\arg(s_n^{\alpha}+v_n)| \leqslant \theta < \tfrac{1}{2}\pi, \quad v_n^{1} \sim K n^{p}. \tag{4.1}$$

Put $a_n = \alpha_n + i\beta_n$, $s_n = \sigma_n + i\tau_n$, $v_n = u_n + iw_n$, $K = K_1 + iK_2$, where $\alpha_n, \beta_n, \sigma_n, \tau_n, u_n, w_n, K_1, K_2$ are real. Then $s_n^{r} = \sigma_n^{r} + i\tau_n^{r}$, for any real value of r. In virtue of (3), $f(\sigma_n^{q}, x)$ and $f(\tau_n^{q}, x)$ both exist for $|x| < 1$ and are each $o\{(1-x)^{-p}\}$ as $x \to 1-0$, while, by (4.1),

$$\sigma_n^{\alpha} + u_n \geqslant 0,$$

and

$$|\tau_n^{\alpha} + w_n| < (\sigma_n^{\alpha} + u_n)\tan\theta,$$

where $u_n^{1} \sim K_1 n^{p}$ and $w_n^{1} \sim K_2 n^{p}$.

Theorem 1, applied to $\Sigma \alpha_n$, shows that $\sigma_n^{\alpha+1} = o(n^{p})$, and hence, by the O form of Lemma 1,

$$\tau_n^{\alpha} = O(\sigma_n^{\alpha}) + O(u_n) + O(w_n) = O(y_n),$$

where $y_n^{1} \sim K_3 n^{p}$. Then, by Theorem 1 applied to $\Sigma \beta_n$, we have $\tau_n^{\alpha+1} = o(n^{p})$, so that $s_n^{\alpha+1} = \sigma_n^{\alpha+1} + i\tau_n^{\alpha+1} = o(n^{p})$.

Theorem 2 can be extended in a similar manner. By employing a different method, Neder (**11**) has made the extension when $p = q = \alpha = 0$.

3.2. We may notice that, if, in Theorem 1, (4) is replaced by $s_n^{\alpha} = O(n^{p})$, then $s_n^{\beta} = o(n^{p})$, whenever $\beta > \alpha$. This follows immediately by using Theorem 1 and the following lemma, proved for $p = 0$ by Andersen (**1**), and for $p > 0$ by the same method.

LEMMA 8. *If* $\alpha > -1$, $p \geqslant 0$, $s_n^{\alpha+1} = o(n^{p})$, *and* $s_n^{\alpha} = O(n^{p})$, *then* $s_n^{\beta} = o(n^{p})$, *whenever* $\beta > \alpha$.

3.3. *Proof of Theorem* 1. We distinguish the possibilities $\alpha = q$, $\alpha > q$, $\alpha < q$.

(i) *Case* $\alpha = q$. $s_n^{\alpha} > -Kv_n$, so that $b_n = s_n^{\alpha} + Kv_n \geqslant 0$. Then

$$f(b_n, x) = f(s_n^{\alpha}, x) + Kf(v_n, x)$$

exists for $|x|<1$, by (3) and (4), with an application of Lemma 3, and

$$f(b_n, x)\sim K\Gamma(p+2)(1-x)^{-p},$$

i.e.
$$\Sigma b_n x^n \sim K\Gamma(p+2)(1-x)^{-p-1}.$$

Applying Theorem A, we have

$$\sum_0^n s_\nu^\alpha + K\sum_0^n v_\nu = \sum_0^n b_\nu \sim Kn^{p+1}.$$

But
$$\sum_0^n v_\nu \sim n^{p+1},$$

so that
$$\sum_0^n s_\nu^\alpha = o(n^{p+1}),$$

i.e., by Lemma 2, $s_n^{\alpha+1}=o(n^p)$.

(ii) *Case* $q<\alpha$. By (3) and Lemma 7, $f_\alpha(x)=o\{(1-x)^{-p}\}$, and the theorem reduces to the case proved.

(iii) *Case* $q>\alpha$. We have $s_n^\alpha+Kv_n\geqslant 0$,

i.e.
$$S_n^\alpha + KA_n^\alpha v_n \geqslant 0,$$

so that
$$A_n^{\alpha+1}s_n^{\alpha+1} \geqslant -K\sum_0^n A_\nu^\alpha v_\nu,$$

since $S_n^{\alpha+1}=\sum_0^n S_\nu^\alpha$. Now

$$\begin{aligned}\sum_0^n A_\nu^\alpha v_\nu &= \sum_0^n A_\nu^\alpha (V_\nu^1 - V_{\nu-1}^1) \quad (V_{-1}^1=0)\\ &= A_n^\alpha V_n^1 + \sum_0^{n-1} V_\nu^1 (A_\nu^\alpha - A_{\nu+1}^\alpha)\\ &\cdots A_n^\alpha V_n^1 - \sum_0^{n-1} V_\nu^1 A_{\nu+1}^{\alpha-1}\\ &\sim Kn^{\alpha+p+1},\end{aligned}$$

since $V_n^1\sim n^{p+1}$ and $\alpha+p>-1$. Hence $s_n^{\alpha+1}=O_L(n^p)$.

By Lemma 1, $s_n^\beta=O_L(n^p)$, provided that $\beta\geqslant\alpha+1$. If also $\beta>q$, an application of (ii) shows that $s_n^\beta=o(n^p)$. By Lemma 5, $f_\alpha(x)=o\{(1-x)^{-p}\}$, and (i) can be used to complete the proof of the theorem.

3.4. *Proof of Theorem* 2. If the theorem is proved for $q = \alpha$, it can be extended to cover the cases $q < \alpha$ and $q > \alpha$ by the methods used in the proof of Theorem 1. Suppose, then, that $q = \alpha$.

(i) *Case* $p > 0$. The identity (1) enables us to write (5) in the form

$$s_n^{\alpha} - s_{n-1}^{\alpha} = O_L(w_n),$$

where $v_n = nw_n$. From $v_n^1 \sim n^p$ it follows (when, and only when, $p > 0$) that $w_n^1 \sim \lambda n^{p-1}$, where $\lambda = \lambda(p)$ is some positive constant. For

$$W_n^1 = \sum_1^n w_\nu = \sum_1^n \frac{v_\nu}{\nu} = \sum_1^n \frac{1}{\nu}(V_\nu^1 - V_{\nu-1}^1)$$

$$= \frac{1}{n} V_n^1 + \sum_1^n \frac{V_\nu^1}{\nu(\nu+1)}.$$

Thus there is a positive constant K such that

$$b_n = (s_n^{\alpha} - s_{n-1}^{\alpha}) + Kw_n \geqslant 0,$$

where $w_n^1 \sim n^{p-1}$. Also

$$\Sigma b_n x^n = \Sigma(s_n^{\alpha} - s_{n-1}^{\alpha}) x^n + K \Sigma w_n x^n$$

$$= f_{\alpha}(x) + K \Sigma w_n x^n$$

$$\sim K\Gamma(p+2)(1-x)^{-p}.$$

Applying Theorem A, we find that $s_n^{\alpha} = o(n^p)$.

(ii) *Case* $p = 0$. We use the method by which Hardy and Littlewood (6) prove Theorem A when $r = 0$. They prove

LEMMA 9. *If* $g(x)$ *is real and twice differentiable in* (0, 1), *and if, as* $x \to 1-0$,

$$g(x) = o(1), \text{ and } (1-x)^2 g''(x) = O_L(1),$$

then

$$(1-x)g'(x) = o(1).$$

Take

$$g(x) = f_{\alpha}(x) = \Sigma(s_n^{\alpha} - s_{n-1}^{\alpha}) x^n.$$

Then

$$g'(x) = \sum_1^{\infty} n(s_n^{\alpha} - s_{n-1}^{\alpha}) x^{n-1} = \sum_1^{\infty} t_n^{\alpha} x^{n-1},$$

and

$$g''(x) = \sum_2^{\infty} (n-1) t_n^{\alpha} x^{n-2}.$$

Let $y_n = (n-1)\,v_n$, when $n \geqslant 2$, and $y_n = 0$, otherwise. Then

$$Y_n^{1} = \sum_{0}^{n} y_\nu = \sum_{0}^{n} (\nu-1)(V_\nu^{1} - V_{\nu-1}^{1})$$

$$= (n-1)\,V_n^{1} - \sum_{1}^{n} V_\nu^{1}$$

$$= (n-1)\,V_n^{1} - V_{n-1}^{2} + V_0^{1}.$$

Now, $V_n^{1} \sim n$, $V_n^{2} \sim \frac{1}{2}n^2$ (by the consistency theorem), so that $y_n^{1} \sim \frac{1}{2}n$, and, by Lemma 3,

$$\sum_{2}^{\infty} (n-1)\,v_n x^n \sim (1-x)^{-2}.$$

Thus
$$g''(x) > -K \sum_{2}^{\infty} (n-1)\,v_n x^n \sim K(1-x)^{-2}$$

i.e.
$$(1-x)^2 g''(x) = O_L(1).$$

By Lemma 9,

$$(1-x) \sum_{1}^{\infty} t_n^{\alpha} x^{n-1} = o(1).$$

Then $(1-x)\,\Sigma\,(t_n^{\alpha} + Kv_n)\,x^n \to K$, and, since the coefficients are positive,

$$t_n^{\alpha} + Kv_n \to K \ (C,\ 1).$$

Hence $t_n^{\alpha} = o(1)$ $(C,\ 1)$, *i.e.*

$$n(s_n^{\alpha} - s_{n-1}^{\alpha}) = o(1) \ (C,\ 1).$$

Since $\Sigma(s_n^{\alpha} - s_{n-1}^{\alpha})\ x^n = o(1)$, Tauber's theorem shows $\Sigma(s_n^{\alpha} - s_{n-1}^{\alpha}) = o(1)$, *i.e.* $s_n^{\alpha} = o(1)$.

4. *A theorem on the summability of power series.*

Given a series such that $s_n^{\alpha} = O_L(n^p)$ or $t_n^{\alpha+1} = O_L(n^p)$, special cases of Theorems 1 and 2 show that $s_n^{\alpha+1} = o(n^p)$ if also the series is summable $(C, q;\ A)$. To proceed further and infer the convergence of the Cesàro means of some definite order, we need to strengthen the hypothesis of summability $(C, q;\ A)$; this may be done by supposing that $f(s_n^{q}, z)$ tends to a limit as $z \to 1$ by *any* path lying within the unit circle. That Σa_n is summable $(C, \gamma \geqslant 1)$, when $a_n = o(n^\gamma)$ and $\Sigma a_n z^n$ tends to a limit as $z \to 1$

by any path within the unit circle, is a well-known theorem due to Dienes, and more recently Bosanquet (**2**) has proved

THEOREM B. *If* $p > 0$, $q \geqslant 0$, $a_n = o(n^p)$, *and* $f(z) \to A(C, q)$ *as* $z \to 1$ *in* $|z| < 1$, *then* Σa_n *is summable* (C, β) *to the sum* A, *provided that* $\beta > q$, $\beta \geqslant p$.

We say that $f(z) \to A(C, q)$ as $z \to 1$ in $|z| < 1$ if

$$\frac{q}{(1-z)^q} \int_z^1 (t-z)^{q-1} f(t)\, dt \to A,$$

the integral $\int_z^1$ being taken in the sense of Cauchy, *i.e.* as $\lim_{\zeta \to 1} \int_z^\zeta$, where $\zeta \to 1$ along the straight line $(z, 1)$. In particular, if this limit exists, then $f(x) \to A(C, q)$, as $x \to 1-0$, and then (see Kogbetliantz, **10**, 21, 37–38, and Hardy and Littlewood, **7**) Σa_n is summable $(C, q;\ A)$. If, in addition, we suppose that

$$|\arg (t_n + K v_n)| \leqslant \theta < \tfrac{1}{2}\pi, \quad v_n^{-1} \sim n^p, \tag{6}$$

the complex version of Theorem 2 (with $\alpha = 0$) shows that $s_n = o(n^p)$, *i.e.* $a_n = o(n^p)$. Applying Theorem B, we obtain

THEOREM 3. *If* $p > 0$, $q \geqslant 0$, *if* (6) *holds, and if*

$$f(z) \to A \ (C, q) \ \textit{as} \ z \to 1 \ \textit{in} \ |z| < 1, \tag{7}$$

then Σa_n *is summable* (C, β) *to the sum* A, *provided that* $\beta > q$, $\beta \geqslant p$.

When $p = 0$ this theorem is still true: the case $p = 0$ of the complex version of Theorem 2 is, in fact, a much better result. This overlapping of Theorems 2 and 3 provides a link between Dienes' theorem, quoted above, and Theorem A of Hardy and Littlewood.

Suppose that, instead of (6), either

$$|\arg (s_n^{\alpha} + K v_n)| \leqslant \theta < \tfrac{1}{2}\pi,$$

or

$$|\arg (t_n^{\alpha+1} + K v_n)| \leqslant \theta < \tfrac{1}{2}\pi,$$

where $v_n^{-1} \sim n^p$, $p \geqslant 0$, $\alpha > -1$. Then, by Theorem 1 or Theorem 2, $s_n^{\alpha+1} = o(n^p)$, whence $a_n = o(n^{p+\alpha+1})$, and, by Theorem B, Σa_n is summable (C, β), provided that $\beta > q$, $\beta \geqslant p+\alpha+1$.

5. *Relations involving Borel summability.*

5.1. Another way of strengthening the hypothesis of summability $(C, q;\ A)$ is to suppose the series summable B, though it is not true that

a series summable B is always summable $(C, q;\ A)$. As a first step we obtain the following analogue of Theorem 1.

THEOREM 4. *If* $q \geqslant 0$, $p \geqslant 0$, $\alpha > -1$, *and if*

$$h_q(u) = o(u^p), \tag{8}$$

$$|\arg(s_n{}^\alpha + v_n)| \leqslant \theta < \tfrac{1}{2}\pi, \tag{4.1}$$

where $v_n{}^1 \sim Kn^p$, $h(v_n, u) \sim Ku^p$, *e.g.* $v_n = n^p$, *then* $s_n^{\alpha+1} = o(n^p)$.

If the series is summable B, the o theorem of Hardy and Littlewood [**5**, Theorem 5: summability B, coupled with the condition $s_n{}^\alpha = o(n^p)$ implies summability $(C, \alpha+2p)$] now shows that the series is summable (C, β), where $\beta = \alpha+2p+1$. When $\alpha = -1$*, $p = \frac{1}{2}$, and a_n is real, this gives summability $(C, 1)$ as a consequence of summability B and the condition $a_n\sqrt{n} > -K$, but it is known (see Theorem C below) that the series is then convergent. This suggests that we may take $\beta = \alpha+2p$, and we can, in fact, prove

THEOREM 5. *If* $0 \leqslant q \leqslant \alpha+1$, $\alpha \geqslant -1$, $p \geqslant \frac{1}{2}$, *and*

$$\Sigma a_n \text{ is summable } (C, q;\ B), \tag{9}$$

$$s_n{}^\alpha = O_L(n^p), \tag{10}$$

then the series is summable $(C, \alpha+2p)$ *to the same sum.*

The proof uses Theorem 4 to show that $t_n^{\alpha+1} = O_L(n^p)$, and the conclusion of the theorem still holds if (10) is replaced by $t_n^{\alpha+1} = O_L(n^p)$. We then have a theorem bearing much the same relation to Theorem 5 as Theorem 2 bears to Theorem 1.

5.2. *Proof of Theorem* 4. Suppose that $\alpha = q$, that a_n is real, and that $s_n{}^\alpha \geqslant -Kv_n$, where $v_n{}^1 \sim n^p$ and $h(v_n, u) \sim u^p$; the more general result

* We define s_n^{-1} as equal to $s_n + t_n$, and say, in order that certain consistency theorems may remain true, that $\Sigma a_n = o(n^p)\ (C, -1)$ if both s_n and t_n are $o(n^p)$. If $p > 0$ and $t_n = o(n^p)$ or $O_L(n^p)$, then also s_n is $o(n^p)$ or $O_L(n^p)$, and thus the series is $o(n^p)\ (C, -1)$, or $O_L(n^p)\ (C, -1)$. But if, say, $t_n = o(1)$, we can only infer that $s_n = o(\log n)$; and some further condition is necessary to ensure that $s_n = o(1)$; *e.g.*, by Tauber's theorem, $s_n = o(1)$ if $t_n = o(1)$ and if, in addition, the series is summable A to zero. Theorem 1, and hence Theorem 4, is true when $\alpha = -1$: the case $p > 0$, $\alpha = -1$ is equivalent to the case $\alpha = 0$ of Theorem 2, while the case $p = 0$, $\alpha = -1$ is the case $r = 0$ of Theorem A.

follows from arguments similar to those which prove Theorem 1. Let

$$b_n = s_n^{\alpha} + Kv_n \geqslant 0.$$

Making the substitution $x = (1+s)^{-1}$, so that $s \to +0$ as $x \to 1-0$, we obtain formally (*cf.* Doetsch, **4**):

$$\begin{aligned} f(b_n, x) &= s \Sigma \frac{b_n}{(s+1)^{n+1}} \\ &= s \Sigma \frac{b_n}{n!} \int_0^{\infty} e^{-(1+s)t} t^n \, dt \\ &= s \int_0^{\infty} e^{-(1+s)t} \left\{ \Sigma \frac{b_n}{n!} t^n \right\} dt \\ &= s \int_0^{\infty} e^{-st} h(b_n, t) \, dt = \phi(s), \text{ say.} \end{aligned}$$

Since $h(b_n, t)$ is a series of positive terms, the existence of $\phi(s)$ will be sufficient to justify the term-by-term integration. For all values of u, the function

$$h(b_n, u) = h(s_n^{\alpha}, u) + Kh(v_n, u)$$

exists and $h(b_n, u) \sim Ku^p$, so that, by Lemma 4, $\phi(s)$ exists for all real $s > 0$; this justifies the equation

$$f(b_n, x) = \phi(s).$$

Again, by Lemma 4, $\phi(s) \sim K\Gamma(p+1)s^{-p}$ as $s \to +0$.

Hence $$f(b_n, x) \sim K\Gamma(p+1)(1-x)^{-p}.$$

But $$f(v_n, x) \sim \Gamma(p+1)(1-x)^{-p},$$

since $v_n^{1} \sim n^p$, so that

$$f(s_n^{\alpha}, x) = o\{(1-x)^{-p}\},$$

and Theorem 1 can now be used to complete the proof.

5.3. *Proof of Theorem* 5. Assume for the moment the following lemmas.

Lemma 10. *If* $\alpha \geqslant -1$, $p \geqslant \frac{1}{2}$, $s_n^{\alpha} = O_L(n^p)$, *and* $h_{\alpha+1}(u) = O(u^{p-\frac{1}{2}})$, *then* $s_n^{\alpha} = O(n^{p-\frac{1}{2}})$.

LEMMA 11. *If $V(n)$, $W(n)$ are non-decreasing functions of n such that*

$$s_n^{\alpha} = O_L\{V(n)\}, \quad s_n^{\alpha+1} = O\{W(n)\},$$

then

$$s_n^{\alpha+\delta} = O_L\{V^{1-\delta} W^{\delta}\},$$

where $0 < \delta < 1$.

One application of Lemma 10 shows that $s_n^{\alpha+1} = O(n^{p-\frac{1}{2}})$, and an application of Lemma 11 with $V(n) = n^p$, $W(n) = n^{p-\frac{1}{2}}$, $\delta = 2p - [2p]$ yields $s_n^{\alpha+\delta} = O_L(n^{p-\frac{1}{2}\delta}) = O_L(n^{\frac{1}{2}[2p]})$.

If $p \geqslant 1$, $[2p]-1$ applications of Lemma 10 now show that $s_n^{\beta-1} = O(n^{\frac{1}{2}})$, where $\beta = \alpha + 2p$. From Theorem 4, $s_n{}^{\beta} = o(n^{\frac{1}{2}})$. Hence

$$t_n{}^{\beta} = \beta(s_n^{\beta-1} - s_n{}^{\beta}) = O(n^{\frac{1}{2}}).$$

If $\frac{1}{2} \leqslant p < 1$, then $[2p] = 1$, $\beta - 1 = \alpha + \delta$ and $s_n^{\beta-1} = O_L(n^{\frac{1}{2}})$; by Theorem 4, $s_n{}^{\beta} = o(n^{\frac{1}{2}})$, whence $t_n{}^{\beta} = O_L(n^{\frac{1}{2}})$.

Since, by Lemma 6, the series is summable $(C, \beta;\ B)$, and since also $t_n{}^{\beta} = n(s_n{}^{\beta} - s_{n-1}^{\beta})$, summability (C, β) is now an immediate consequence of the following theorem (proved, for example, by Vijayaraghavan, **12**).

THEOREM C. *If Σa_n is summable B and $t_n = O_L(\sqrt{n})$, then the series is convergent.*

5.4. *Proof of Lemma* 10. Theorem 4 shows that $s_n^{\alpha+1} = o(n^p)$, so that

$$t_n^{\alpha+1} = (\alpha+1)(s_n{}^{\alpha} - s_n^{\alpha+1}) = O_L(n^p),$$

and, for $m > n$,

$$s_m^{\alpha+1} - s_n^{\alpha+1} = \sum_{n+1}^{m} \frac{1}{\nu}\, t_{\nu}^{\alpha+1} > -K \sum_{n+1}^{m} \nu^{p-1} > -K(m^p - n^p),$$

i.e.

$$K(m^p - n^p) > s_n^{\alpha+1} - s_m^{\alpha+1}. \tag{11}$$

Write $s_n^{\alpha+1} = \sigma_n n^{p-\frac{1}{2}}$ and let

$$\sigma_1(n) = \max_{\nu \leqslant n} \sigma_{\nu}, \quad \sigma_2(n) = \max_{\nu \leqslant n} (-\sigma_{\nu}).$$

Then either

(i) $\sigma_1(n) \geqslant \sigma_2(n)$ for an infinity of values of n,

or (ii) $\sigma_1(n) < \sigma_2(n)$ for all sufficiently large values of n.

We prove that the assumption that σ_n is unbounded leads to a contradiction with the hypothesis $h_{\alpha+1}(u) = O(u^{p-\frac{1}{2}})$. Only the details for

case (i) need be given; the proof is on the lines of that given by Vijayaraghavan for $\alpha = -1$, $p = \frac{1}{2}$ (**12**, Lemma 1), and the slight modification which he makes to deal with case (ii) is immediately applicable here.

If (i) holds and σ_n is unbounded, then, given $H > 0$, there exist integers $N = N(H)$ such that

$$\sigma_1(N) \geqslant \sigma_2(N), \quad \sigma_N = \sigma_1(N) > 2H.$$

Take the least value of N and the least integer $M = M(H) > N$ such that $\sigma_N \geqslant 2\sigma_M$. The hypothesis $h_{\alpha+1} = O(u^{p-\frac{1}{2}})$ rules out the possibility that $\sigma_N \to +\infty$, so that M exists if H is sufficiently large. Let

$$u = u(H) = \tfrac{1}{2}(M+N).$$

Plainly, $u \to \infty$ as $H \to \infty$.

If $M/N \to 1$ as $H \to \infty$, take $m = M$, $n = N$ in (11). Then

$$K(M^p - N^p) > S_N^{\alpha+1} - S_M^{\alpha+1} > H(2N^{p-\frac{1}{2}} - M^{p-\frac{1}{2}}).$$

We easily see from this last inequality that

$$\sqrt{M} - \sqrt{N} > K \,.\, H,$$

so that

$$M - N = (\sqrt{M} - \sqrt{N})(\sqrt{M} + \sqrt{N}) > K \,.\, H\sqrt{u}.$$

If it is not true that $M/N \to 1$, there exists a sequence of values of H, tending to $+\infty$, and a positive constant λ, such that

$$M - N > \lambda M > \lambda \sqrt{M} \,.\, \sqrt{u}.$$

Thus there is always a sequence $\{H\}$ of values of H such that

$$M - N = L\sqrt{u},$$

where $L \to \infty$ as H runs through the values $\{H\}$, and we then have

$$M = u + \tfrac{1}{2}(M-N) = u + L\sqrt{u},$$

$$N = u - \tfrac{1}{2}(M-N) = u - L\sqrt{u}.$$

Also for $n > M$, $H > H_0$, by (11),

$$s_n^{\alpha+1} - s_M^{\alpha+1} > -K(n^p - M^p),$$

while

$$s_M^{\alpha+1} = s_{M-1}^{\alpha+1} + \frac{1}{M}\, t_M^{\alpha+1}$$

$$> H(M-1)^{p-\frac{1}{2}} - KM^{p-1} > 0,$$

so that

$$s_n^{\alpha+1} > -K(n^p - M^p).$$

We now write

$$h_{\alpha+1}(u) = e^{-u}\Sigma s_n^{\alpha+1}\frac{u^n}{n!} = \sum_0^{N-1} + \sum_N^{M-1} + \sum_M^{\infty},$$

and let H run through the sequence $\{H\}$. Following Vijayaraghavan, we can easily show (as below) that, uniformly in u,

$$e^{-u}\sum_0^{N-1} n^{p-\frac{1}{2}}\frac{u^n}{n!} = o(N^{p-\frac{1}{2}}) = o(u^{p-\frac{1}{2}}), \tag{12}$$

$$e^{-u}\sum_N^{M-1} n^{p-\frac{1}{2}}\frac{u^n}{n!} \sim u^{p-\frac{1}{2}}, \tag{13}$$

$$e^{-u}\sum_M^{\infty}(n^p - M^p)\frac{u^n}{n!} = o(u^{p-\frac{1}{2}}), \tag{14}$$

and then, writing δ_H for a number which tends to zero uniformly in u as $H\to\infty$, we have, bearing in mind the definitions of N and M,

$$\sum_0^{N-1} \geqslant -2He^{-u}\sum_0^{N-1} n^{p-\frac{1}{2}}\frac{u^n}{n!} > -2H\delta_H u^{p-\frac{1}{2}},$$

$$\sum_N^{M-1} > He^{-u}\sum_N^{M-1} n^{p-\frac{1}{2}}\frac{u^n}{n!} \sim Hu^{p-\frac{1}{2}},$$

$$\sum_N^{\infty} > -Ke^{-u}\sum_N^{\infty}(n^p - M^p)\frac{u^n}{n!} > -\delta_H u^{p-\frac{1}{2}}.$$

Thus $h_{\alpha+1}(u) > (1-\delta_H)Hu^{p-\frac{1}{2}}$, which provides our contradiction.

To prove (14) we use the inequality (Vijayaraghavan, **12**, Lemma α)

$$\frac{u^n}{n!} < \frac{u^u}{u!}e^{-m}, \tag{15}$$

where $n = [u+m\sqrt{u}]$ and, say, $m > 4$, $u > 25$.

Put $M_0 = M = u + L\sqrt{u}$, $M_r = [u+(L+r)\sqrt{u}]$, where $r = 1, 2, 3, \ldots$, and take $n = M_r$, $m = L+r$ in (15). Then

$$\sum_{M_r}^{M_{r+1}}\frac{u^n}{n!}(n^p - M^p) < \frac{u^{M_r}}{M_r!}(M_{r+1}-M_r)(M_{r+1}^p - M^p)$$

$$< e^{-L-r}\frac{u^u}{u!}(\sqrt{u}+2)(M_{r+1}^p - M^p)$$

$$< Ke^{-L-r}(M_{r+1}^p - M^p),$$

by Stirling's theorem. Hence

$$e^{-u}\sum_{M}^{\infty}\frac{u^n}{n!}(n^p-M^p)=e^{-u}\left(\sum_{M}^{M_1}+\sum_{M_1}^{M_2}+\dots\right)$$

$$<Ke^{-L}\sum_{r=0}^{\infty}e^{-r}(M_{r+1}^{p}-M^p)$$

$$=e^{-L}O(M_1{}^p-M^p)$$

$$=o(M^{p-\frac{1}{2}}),$$

uniformly in M, as $L\to\infty$. (12) and (13) are proved similarly.

5.5. *Proof of Lemma* 11. This lemma is equivalent to the following (*cf.* Dixon and Ferrar, 3): *if* $V(n)$, $W(n)$ *are positive non-decreasing functions of* n *such that*

$$S_n>-V(n),\quad |S_n{}^1|<W(n),$$

then, if $0<\delta<1$,

$$S_n{}^{\delta}>-K\{V(n)\}^{1-\delta}\{W(n)\}^{\delta}.$$

If $nV(n)<W(n)$, the lemma is a special case of Lemma 1, the hypothesis on $S_n{}^1$ being superfluous, while if $W(n)<V(n)$, the hypothesis on S_n is superfluous, since $|S_n{}^{\alpha}|<W(n)$ implies $|S_n{}^{\beta}|<W(n)$, if $\beta<\alpha$. We may therefore suppose that $nW(n)\geqslant nV(n)\geqslant W(n)$. We observe that

$$S_\nu=S_\nu{}^1-S_{\nu-1}^1,\quad A_\nu^{\delta-2}=A_\nu^{\delta-1}-A_{\nu-1}^{\delta-1},$$

that, if $0<\delta<1$, then $A_\nu^{\delta-1}>0$ while $A_\nu^{\delta-2}<0$, and that $A_\nu{}^{\alpha}\sim\nu^{\alpha}/\Gamma(\alpha+1)$. Then

$$S_n{}^{\delta}=\sum_{\nu=0}^{p-1}S_{n-\nu}A_\nu^{\delta-1}+\sum_{\nu=p}^{n}S_{n-\nu}A_\nu^{\delta-1}$$

$$=\sum_{0}^{p-1}S_{n-\nu}A_\nu^{\delta-1}+\sum_{p}^{n}S_{n-\nu}^1A_\nu^{\delta-2}+S_{n-p}^1A_{p-1}^{\delta-1}$$

$$>-V(n)\sum_{0}^{p-1}A_\nu^{\delta-1}-W(n)\left|\sum_{p}^{n}A_\nu^{\delta-2}\right|-W(n)A_{p-1}^{\delta-1}$$

$$=-V(n)A_{p-1}^{\delta}-W(n)|A_n^{\delta-1}-A_{p-1}^{\delta-1}|-W(n)A_{p-1}^{\delta-1}$$

$$>-V(n)A_{p-1}^{\delta}-2W(n)A_{p-1}^{\delta-1}.$$

Taking $p-1=W(n)/V(n)$, so that $0<p-1<n$, the lemma is proved.

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THE ASYMPTOTIC EXPANSION OF THE GENERALIZED BESSEL FUNCTION

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1.1. In a recent paper*, I sketched a method by which it is possible to determine the asymptotic expansion of the n-th coefficient of certain power series. The method involved the use of the integral function

$$\phi(z) = \phi(\rho, \beta;\ z) = \sum_{l=0}^{\infty} \frac{z^l}{\Gamma(l+1)\,\Gamma(\rho l+\beta)},$$

where ρ is real and positive and β is any number, real or complex. By this means we can find, for example, the asymptotic expansion of $p_k(n)$, the number of partitions of n into perfect k-th powers†.

An essential part of the method is the determination of the asymptotic expansion of $\phi(z)$ for large values of z. This expansion was given in the former paper without proof and my object here is to supply a proof.

1.2. We use ϵ to denote a positive number (to be thought of as small); M is a positive integer or zero; K is a positive number, not always the same at each occurrence, depending at most on ρ, β, M, and ϵ. We shall use K_1, K_2, ... to denote fixed numbers of the K type; thus K_1, for example, always denotes the same positive function of ρ, β, M, and ϵ. The constant implied in the $O(\ )$ notation is of the type K.

We write

$$g(v) = \left\{1 + \frac{\rho+2}{3}\,v + \frac{(\rho+2)(\rho+3)}{3\,.\,4}\,v^2 + \ldots\right\}^{\frac{1}{2}}, \qquad (1.21)$$

that square root being taken which reduces to 1 when $v = 0$. Then we

* Wright, 1.

† Wright, 2.

define a_m as the coefficient of v^{2m} in the expansion of

$$\frac{\Gamma(m+\frac{1}{2})}{2\pi}\left(\frac{2}{\rho+1}\right)^{m+\frac{1}{2}}(1-v)^{-\beta}\{g(v)\}^{-2m-1}$$

in ascending powers of v. It is clear that a_m can be calculated for a given value of m; for example,

$$a_0 = \{2\pi(\rho+1)\}^{-\frac{1}{2}},$$

$$a_1 = \frac{12\beta^2-12\beta(\rho+1)+(\rho+2)(2\rho+1)}{24(\rho+1)\{2\pi(\rho+1)\}^{\frac{1}{2}}}.$$

With these definitions, the asymptotic expansion of $\phi(z)$ is completely described as follows.

THEOREM 1. *If* $\arg(-z)=\xi$, $|\xi|\leqslant\pi$, *and*

$$Z_1 = (\rho|z|)^{1/(\rho+1)}e^{i(\xi+\pi)/(\rho+1)}, \quad Z_2 = (\rho|z|)^{1/(\rho+1)}e^{i(\xi-\pi)/(\rho+1)},$$

then we have $$\phi(z) = H(Z_1)+H(Z_2),$$

where $$H(Z) = Z^{\frac{1}{2}-\beta}e^{\{1+(1/\rho)\}Z}\left\{\sum_{m=0}^{M}\frac{(-1)^m a_m}{Z^m}+O\left(\frac{1}{|Z|^{M+1}}\right)\right\}.$$

From this we can deduce a simpler but less complete theorem.

THEOREM 2. *If* $\arg z=\vartheta$, $\epsilon>0$, $|\vartheta|\leqslant\pi-\epsilon$, *and*

$$Z = (\rho|z|)^{1/(\rho+1)}e^{i\vartheta/(\rho+1)},$$

then we have $$\phi(z) = H(Z).$$

1.3. We see that

$$\log\phi(z)\sim\left(1+\frac{1}{\rho}\right)Z,$$

except along the line $\arg z=\pi$. Hence our asymptotic expansion is exponentially large when $\cos\arg Z>0$, that is, in either of the two cases

(i) $$\rho>1, \quad |\arg z|<\pi,$$

(ii) $$0<\rho<1, \quad |\arg z|<\tfrac{1}{2}\pi(1+\rho).$$

When this is so, a method due to Watson* can be applied to determine the

* Watson, 1. The coefficients a_0, a_1, ... are defined differently by Watson, and it does not seem easy to prove directly that the definition is equivalent to mine. His method, of course, is applicable to a much wider class of functions than mine.

asymptotic expansion of $\phi(z)$. On the other hand, when $0<\rho<1$ and $|\arg(-z)|<\frac{1}{2}\pi(1-\rho)$, $\cos\arg Z$ is negative and our expansion is exponentially small. In this case, Watson's method is inapplicable and it is apparently necessary to use the method of "steepest descent"*. Since this method gives our result in every case, we shall use it here and make no appeal to Watson's results.

The function $\phi(z)$ is a particular case of the generalized hypergeometric function, whose asymptotic expansion has been investigated by Fox†. His method is applicable only to the case ρ rational, and, like Watson's, does not give a result when the expansion is exponentially small.

If $\rho=m/n$, where m and n are positive integers, there is a discrepancy between Fox's result on the one hand and Watson's and mine on the other, except for the range $|\arg z|\leqslant\pi/n$. Outside this range, wherever all three methods apply, Fox's expansion appears to be of too high an order. Presumably the coefficients in his expansion all vanish in this case, so that there is no actual contradiction.

Our knowledge of the asymptotic expansion of the generalized hypergeometric function is incomplete, in spite of the attention devoted to the problem by Barnes and others. By combining the methods of Watson, Fox, and myself, it is possible to find the asymptotic expansion for all values of $\arg z$. For the present, however, I postpone discussion of this question, only remarking that the methods and results of the present paper appear to be an essential preliminary to a full solution of the more general problem.

1.4. We remark that, when $\rho=1$, we have

$$J_{\beta-1}(t)=(\tfrac{1}{2}t)^{\beta-1}\phi(-\tfrac{1}{4}t^2),$$

where $J_\nu(t)$ is the ordinary Bessel function. Theorems 1 and 2 then give the asymptotic expansion of the Bessel function (in an unusual form). Hence we need prove the theorems only when $0<\rho<1$ and when $\rho>1$.

We first express $\phi(z)$ as a contour integral. Let the u-plane be cut along the line $\arg u=\pi$ and let $|\arg u|\leqslant\pi$. Then we have‡

$$\phi(z)=\frac{1}{2\pi i}\int_C u^{-\beta}\exp\left(u+\frac{z}{u^\rho}\right)du,$$

where C is any contour starting from $-\infty$ on the real axis, passing round

* See Watson, 2, 117, for a list of memoirs describing the method.

† Fox, 1.

‡ Wright, 1, Theorem 3.

the origin in a counterclockwise direction and returning to $-\infty$, thus enclosing the cut. If there is a number K such that

$$-\tfrac{3}{2}\pi+K<\theta'<-\tfrac{1}{2}\pi-K, \quad \tfrac{1}{2}\pi+K<\theta''<\tfrac{3}{2}\pi-K,$$

we may dispense with the cut and take C to be any contour passing round the origin once counterclockwise with extremities at infinity in the directions $\arg u=\theta'$ and $\arg u=\theta''$.

The saddle points are the points where

$$\frac{d}{du}\left(u+\frac{z}{u^\rho}\right)=0,$$

that is, the points

$$u=(\rho|z|)^{1/(\rho+1)}e^{i(\vartheta+2c\pi)/(\rho+1)}=Ze^{2c\pi i/(\rho+1)},$$

where c takes all integral values, positive, zero, or negative. It will be found that the saddle points with which we are concerned are those for which $c=-1$, 0, or 1.

If we write $u=|Z|v$, we have

$$\phi(z)=\frac{|Z|^{1-\beta}}{2\pi i}\int_{C'}v^{-\beta}\exp\left\{|Z|\left(v+\frac{e^{i\vartheta}}{\rho v^\rho}\right)\right\}dv, \tag{1.41}$$

where C' is a contour of the same type as C. The saddle points now lie on the circumference of the unit circle.

2.1. We first define certain contours in the t-plane from which by a simple rotation we can find the "steepest descent" curves in the v-plane. If we write

$$f(t)=t+\frac{1}{\rho t^\rho}-1-\frac{1}{\rho},$$

taking $t^\rho=1$ at $t=1$, the contours in question are branches of the curve on which

$$\Im\{f(t)\}=0, \tag{2.11}$$

or of the curve on which

$$\Re\{f(t)\}=0. \tag{2.12}$$

Since $f(1)=0$, $f'(1)=0$, and $f''(1)\neq 0$, each of these curves passes through $t=1$ and two branches of the same curve cut each other at right angles at that point.

If $t = re^{i\gamma}$, then on the curve (2.11) we have

$$\rho r^{1+\rho} \sin\theta = \sin\rho\theta.$$

One branch of (2.11) is the positive half of the real axis. If this is described from 0 to ∞ it will be denoted by Γ_1, while Γ_2 will be the same contour described in the reverse direction. Γ_3 is the other branch of (2.11) through $t = 1$. If $0 < \rho < 1$, then Γ_3 starts from $-\infty$ on the real axis, encircles the origin once counter-clockwise, and returns to $-\infty$. If $\rho > 1$, then Γ_3 starts from the origin, where it touches the straight line $\theta = -\pi/\rho$, passes through $t = 1$ in an upward direction and returns to the origin, touching the straight line $\theta = \pi/\rho$.

On the curve (2.12) we have

$$r\cos\theta + \frac{1}{\rho r^\rho}\cos\rho\theta = 1 + \frac{1}{\rho}.$$

This curve has two branches Γ_4 and Γ_5, cutting one another at $t = 1$. Of these, Γ_4 starts from the origin, where it touches the straight line $\theta = -\pi/(2\rho)$, goes through $t = 1$, making an angle $\frac{1}{4}\pi$ with the positive direction of the real axis, and passes to infinity in the upper half of the plane, having the straight line

$$r\cos\theta = 1 + \frac{1}{\rho}$$

as asymptote. The contour Γ_5 is the reflexion of Γ_4 in the real axis, but is traversed from ∞ to the origin. It has the same asymptote, and the tangent $\theta = \pi/(2\rho)$ at the origin.

2.2. We write

$$I_d(Y) = \frac{Y^{1-\beta}}{2\pi i}\int_{\Gamma_d} t^{-\beta}\exp\left\{Y\left(t + \frac{1}{\rho t^\rho}\right)\right\} dt$$

$$= \frac{Y^{1-\beta}e^{\{1+(1/\rho)\}Y}}{2\pi i}\int_{\Gamma_d} t^{-\beta}e^{Yf(t)}dt \quad (d = 1, 2, 3, 4, 5), \qquad (2.21)$$

provided that the integral converges. The number Y is of the Z type, and we always assume that $|Y| > K$. We write $\eta = \arg Y$.

Our first object is to determine the asymptotic expansion of $I_d(Y)$ for large values of Y. To do this we apply the transformation

$$w = \{f(t)\}^{\frac{1}{2}} \qquad (2.22)$$

to Γ_d. We require two lemmas to make this transformation precise.

LEMMA 1. *If t lies on Γ_d and if $|t-1|>K$, then $|r-1|>K$.*

For $dr/d\theta$ does not vanish or change sign on any Γ_d except Γ_3, and on Γ_3 only at $t=1$; also the form of Γ_d depends only on the parameter ρ.

LEMMA 2. *Except at $t=1$, there is no zero of $f(t)$ or of $f'(t)$ on Γ_d.*

Since $f'(t)=1-t^{-1-\rho}$, the zeros of $f'(t)$ lie on the unit circle. By Lemma 1, none of these except the one at $t=1$ can lie on Γ_d.

Next, let us suppose that $d=1$, 2, or 3, so that Γ_d is part of the curve

$$\Im f(t)=0.$$

Let s be the length of arc of Γ_d measured from $t=1$. Then, if t_0 is a zero of $f(t)$ on Γ_d and $t_0\neq 1$, we have $\Re\{f(t_0)\}=0$, and so $d\Re\{f(t)\}/ds$ must vanish at some point on Γ_d between 1 and t_0. But, on Γ_d, we have

$$\frac{d}{ds}\Re\{f(t)\}=\frac{d}{ds}f(t)=f'(t)\frac{dt}{ds}=e^{i\alpha}f'(t),$$

where α is the angle which the tangent at t makes with the real axis. But $f'(t)$ cannot vanish on Γ_d except at $t=1$, so we have a contradiction.

If $d=4$ or 5, so that Γ_d is part of the curve

$$\Re\{f(t)\}=0,$$

a similar contradiction follows from the assumption that $f(t)$ has a zero on Γ_d other than the zero at $t=1$. Hence $f(t)$ has no zeros on Γ_d, when $d=1, 2, 3, 4,$ or 5, except at $t=1$.

2.3. Near $t=1$, we have

$$f(t)=\frac{1}{\rho}\{1+(t-1)\}^{-\rho}-\frac{1}{\rho}+(t-1)$$

$$=\frac{\rho+1}{2}(t-1)^2\{g(1-t)\}^2.$$

At $t=1$, the function $g(1-t)$ is regular and equal to 1. At any point on Γ_d except 0, 1, or ∞, $f(t)$ is regular and one-valued and, by Lemma 2, it does not vanish. Consequently $g(1-t)$ is regular and one-valued on Γ_d except at 0 and ∞. Hence we may make (2.22) precise by writing

$$w=\left(\frac{\rho+1}{2}\right)^{\frac{1}{2}}(t-1)g(1-t). \tag{2.31}$$

This transformation is equivalent to

$$w^2=f(t),$$

and it transforms Γ_d into an infinite straight line through the origin in the w-plane. In fact, Γ_1 and Γ_2 each become the complete real axis, described in opposite directions, Γ_3 becomes the imaginary axis, and Γ_4 and Γ_5 become straight lines through the origin inclined at angles $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$ respectively to the positive direction of the real axis.

Let Γ_d' be the transform of Γ_d and let us write

$$\mu(w) = \frac{t^{-\beta}}{2\pi i}\frac{dt}{dw}.$$

Then
$$I_d(Y) = Y^{1-\beta} e^{\{1+(1/\rho)\}Y} \int_{\Gamma_d'} \mu(w)\, e^{Yw^\rho}\, dw. \tag{2.32}$$

3.1. We now prove five lemmas preliminary to finding the asymptotic expansion of $I_d(Y)$.

LEMMA 3. *If* $|w| > K$, *then* $|t-1| > K$ *and*

$$|t|^{-\Re(\beta)} < K|w|^\nu,$$

where
$$\nu = \max\left(-2\Re(\beta), \frac{2\Re(\beta)}{\rho}\right).$$

Suppose that $|w| > K_1$; then $|f(t)| > K_1^{\frac{1}{2}} = K_2$. Let $0 \leqslant R < 1$, and let
$$D(R) = \max_{|t-1|\leqslant R} |f(t)|.$$

We know that, for this range of values of R, $D(R)$ is a continuous increasing function of R, depending on R and ρ only and tending to infinity as $R \to 1$. Also, $D(0) = 0$. Hence there is a number K_3 $(0 < K_3 < 1)$ such that $D(K_3) = K_2$. Since $|f(t)| > K_2$, we must have $|t-1| > K_3$.

To prove the second part of the lemma, we note that

$$w^2 = t + \frac{1}{\rho t^\rho} - 1 - \frac{1}{\rho}.$$

Then, if $|t| \leqslant 1$ and $|w| > K$, we have

$$|t|^{-\rho} \leqslant \rho|w|^2 + |t| + 1 + \frac{1}{\rho} < K|w|^2.$$

If $|t| \geqslant 1$ and $|w| > K$, then

$$|t| \leqslant |w|^2 + \frac{1}{\rho|t|^\rho} + 1 + \frac{1}{\rho} < K|w|^2.$$

Hence
$$|t|^{-\Re(\beta)} < K|w|^\nu.$$

LEMMA 4. *If w lies on Γ_d' and if $|w|>K$, then*

$$|\mu(w)|<K|w|^{1+\nu}.$$

When w satisfies the conditions of this lemma, we have $|t-1|>K$, by Lemma 3. Hence, by Lemma 1,

$$|t^{1+\rho}-1|\geqslant|r^{1+\rho}-1|\geqslant|r-1|>K,$$

and so

$$\left|\frac{t^{1+\rho}}{t^{1+\rho}-1}\right|<K.$$

Also, with the same restrictions on w, we have $|\arg t|<K$ and

$$|t^{-\beta}|=|t|^{-\Re(\beta)}e^{\Im(\beta)\arg t}<K|w|^{\nu},$$

by Lemma 3. Combining these results, we have

$$\begin{aligned}|\mu(w)|&=\left|\frac{t^{-\beta}w}{\pi f'(t)}\right|\\&=\frac{1}{\pi}\left|\frac{wt^{1+\rho-\beta}}{t^{1+\rho}-1}\right|<K|w|^{1+\nu}.\end{aligned}$$

LEMMA 5. *There exist numbers*

$$b_l=b(\rho,\beta,l)$$

such that, when $M\geqslant\frac{1}{2}\nu$,

$$\left|\mu(w)-\sum_{l=0}^{2M+1}b_lw^l\right|<K|w|^{2M+2}$$

on Γ_d'.

From (1.21) and (2.31) we see that, near $w=0$,

$$w=\sum_{l=1}^{\infty}g_l(t-1)^l,$$

where

$$g_1=\left(\frac{\rho+1}{2}\right)^{\frac{1}{\rho}}\neq0.$$

Hence we may revert the series and, near $t=1$, we have

$$t=1+\sum_{l=1}^{\infty}G_lw^l,$$

$$\frac{dt}{dw}=\sum_{l=1}^{\infty}lG_lw^{l-1},$$

and

$$\mu(w)=\sum_{l=1}^{\infty}b_lw^l.$$

Now $\mu(w)$ depends only on ρ, β, and w, so that the latter expansion will be valid for $|w| \leqslant K_4$ for some K_4. Then

$$\left| \mu(w) - \sum_{l=0}^{2M+1} b_l w^l \right| < K|w|^{2M+2},$$

when $|w| \leqslant K_4$.

It only remains to prove the lemma when $|w| > K_4$. But, in this case, by Lemma 4,

$$\left| \mu(w) - \sum_{l=0}^{2M+1} b_l w^l \right| \leqslant |\mu(w)| + \sum_{l=0}^{2M+1} |b_l w^l| < K|w|^{2M+1} < K|w|^{2M+2}.$$

LEMMA 6. *If b_l is defined as in Lemma 5, then $ib_{2m}\Gamma(m+\frac{1}{2}) = a_m$.*

We have

$$b_{2m} = \frac{1}{2\pi i}\int_{0+} \frac{\mu(w)}{w^{2m+1}}\,dw = -\frac{1}{4\pi^2}\int_{1+} \frac{t^{-\beta}}{w^{2m+1}}\,dt,$$

and

$$\frac{a_m}{i\Gamma(m+\frac{1}{2})} = -\frac{1}{4\pi^2}\left(\frac{2}{\rho+1}\right)^{m+\frac{1}{2}}\int_{0+} \frac{(1-v)^{-\beta}\{g(v)\}^{-2m-1}}{v^{2m+1}}\,dv.$$

If we put $v = 1-t$, the lemma follows at once.

LEMMA 7. *If $\tau = \arg T$ and if $|\tau| \leqslant \frac{1}{2}\pi - \epsilon$, then*

$$\int_{-\infty}^{\infty} x^{2m} e^{-Tx^2}\,dx = T^{-m-\frac{1}{2}}\Gamma(m+\tfrac{1}{2}) = |T|^{-m-\frac{1}{2}} e^{-\tau i(m+\frac{1}{2})}\Gamma(m+\tfrac{1}{2}).$$

This is a well-known result.

3.2. The asymptotic expansion of $I_d(Y)$ is contained in the following lemma.

LEMMA 8. *If $M \geqslant 0$, and if*

(i) $d = 1$, $\quad |\eta - \pi| \leqslant \frac{1}{2}\pi - \epsilon$,

(ii) $d = 2$, $\quad |\eta + \pi| \leqslant \frac{1}{2}\pi - \epsilon$,

(iii) $d = 3$, $\quad |\eta| \leqslant \frac{1}{2}\pi - \epsilon$,

(iv) $d = 4$, $\quad |\eta - \frac{1}{2}\pi| \leqslant \frac{1}{2}\pi - \epsilon$,

or (v) $d = 5$, $\quad |\eta + \frac{1}{2}\pi| \leqslant \frac{1}{2}\pi - \epsilon$,

then the integral $I_d(Y)$ is convergent and $I_d(Y) = H(Y)$.

We first take $M \geqslant \frac{1}{2}\nu$. Then, by Lemma 5,

$$\int_{\Gamma_1'} \mu(w)\, e^{Yw^2}\, dw = \sum_{l=1}^{2M+1} b_l \int_{\Gamma_1'} w^l e^{Yw^2} + R,$$

where $$|R| < K \int_{\Gamma_1'} w^{2M+2}\, e^{\Re(Y)w^2}\, dw.$$

If l is odd, $$\int_{\Gamma_1'} w^l e^{Yw^2}\, dw = 0.$$

If $l = 2m$, we put $T = Ye^{-\pi i}$, so that $|T| = |Y|$ and $\tau = \eta - \pi$. We then have $|\tau| \leqslant \frac{1}{2}\pi - \epsilon$ and the conditions of Lemma 7 are satisfied. Hence

$$\int_{\Gamma_1'} w^{2m}\, e^{Yw^2}\, dw = |Y|^{-m-\frac{1}{2}}\, e^{-(m+\frac{1}{2})(\eta-\pi)i}\, \Gamma(m+\tfrac{1}{2}) = (-1)^m i\, Y^{-m-\frac{1}{2}}\, \Gamma(m+\tfrac{1}{2}).$$

Also, $$\Re(Y) = |Y| \cos\eta \leqslant -|Y| \sin\epsilon < -K|Y|.$$

Hence $$|R| < K \int_{-\infty}^{\infty} w^{2M+2}\, e^{-K|Y|w^2}\, dw = K\, |Y|^{-M-\frac{3}{2}},$$

and so

$$\int_{\Gamma_1'} \mu(w) e^{Yw^2}\, dw = \sum_{m=0}^{M} (-1)^m a_m\, Y^{-m-\frac{1}{2}} + O(|Y|^{-M-\frac{3}{2}}).$$

Since this is true for all $M \geqslant \frac{1}{2}\nu$, it is also true for all $M \geqslant 0$.

Then by (2.32) the lemma is proved when condition (i) is satisfied.

The proof of the lemma under the other conditions proceeds on the same lines. In each case we put $|T| = |Y|$ and

(ii) $\tau = \eta + \pi$, $x = w$,

(iii) $\tau = \eta$, $x = -iw$,

(iv) $\tau = \eta - \frac{1}{2}\pi$, $x = we^{-\frac{1}{4}\pi i}$,

or (v) $\tau = \eta + \frac{1}{2}\pi$, $x = we^{\frac{1}{4}\pi i}$.

The conditions of Lemma 7 are then satisfied and the result of that lemma may be used as before.

4. *Proof of Theorems* 1 *and* 2. If we write

$$v = te^{i\psi},$$

the contour Γ_d transforms into a corresponding contour in the v-plane, which we shall denote by $\Gamma_d(\psi)$. We thus get a whole family of contours

by varying the value of ψ. In fact, we always choose ψ so that the point $t=1$ transforms into one of the saddle points in the v-plane. We write arg $v=\theta$.

We now show that C' in (1.41) can always be replaced by suitable combinations of these contours. We have to distinguish between several cases.

(i) $0<\rho<1$, $|\vartheta|\leqslant\frac{1}{2}\pi$. The extremities of the contour $\Gamma_3\{\vartheta/(\rho+1)\}$ go to infinity in the directions

$$\theta'=\frac{\vartheta}{\rho+1}-\pi, \quad \theta''=\frac{\vartheta}{\rho+1}+\pi.$$

Hence, if there is a number K such that

$$-\tfrac{3}{2}\pi+K\leqslant\theta'\leqslant\tfrac{1}{2}\pi-K,$$

and

$$\tfrac{1}{2}\pi+K\leqslant\theta''\leqslant\tfrac{3}{2}\pi-K,$$

we may take $\Gamma_3\{\vartheta/(\rho+1)\}$ for our C' in (1.41). These are equivalent to the one condition

$$\left|\frac{\vartheta}{\rho+1}\right|\leqslant\tfrac{1}{2}\pi-K,$$

and so are satisfied for $K=\frac{1}{2}\pi\rho/(\rho+1)$.

If we put $v=t\exp\{i\vartheta/(\rho+1)\}$ we have

$$\phi(z)=I_3(Z)=H(Z).$$

If $0\leqslant\vartheta\leqslant\frac{1}{2}\pi$, we have $\xi=\arg(-z)=\vartheta-\pi$, $Z_1=Z$, and

$$\cos\arg Z_1-\cos\arg Z_2=\cos\frac{\vartheta}{\rho+1}-\cos\frac{\vartheta-2\pi}{\rho+1}$$

$$=2\sin\frac{\pi}{\rho+1}\sin\frac{\pi-\vartheta}{\rho+1}>K.$$

Hence $H(Z_2)$ is negligible compared with the error term in $H(Z_1)$, and we have

$$\phi(z)=H(Z)=H(Z_1)=H(Z_1)+H(Z_2).$$

If $-\frac{1}{2}\pi\leqslant\vartheta\leqslant0$, then $Z_2=Z$ and $H(Z_1)$ is negligible compared with the error term in $H(Z_2)$. Hence, as before,

$$\phi(z)=H(Z_1)+H(Z_2).$$

(ii) $0<\rho<1$, $\frac{1}{2}\pi\leqslant\vartheta\leqslant\pi$. In this case we take the two contours

$$\Gamma_2\Big(\frac{\vartheta-2\pi}{\rho+1}\Big),\quad \Gamma_4\Big(\frac{\vartheta}{\rho+1}\Big).$$

These two contours have extremities at the origin, and to avoid the singularity of the integrand there we take an arc of the small circle $|v|=\delta$ to join them. If $v=\delta e^{i\theta}$, we have θ lying between

$$\frac{\vartheta-2\pi}{\rho+1}\quad\text{and}\quad\frac{\vartheta}{\rho+1}-\frac{\pi}{2\rho}+\gamma \tag{4.01}$$

on this arc, where $\gamma\to 0$ as $\delta\to 0$. This combination of contours can replace C', provided that there is a number K such that

$$-\tfrac{3}{2}\pi+K\leqslant\frac{\vartheta-2\pi}{\rho+1}\leqslant-\tfrac{1}{2}\pi-K,$$

$$\tfrac{1}{2}\pi+K\leqslant\frac{\vartheta}{\rho+1}+\tfrac{1}{2}\pi\leqslant\tfrac{3}{2}\pi-K,$$

for then the extremities go to infinity in the required directions. If we take

$$K=\min\Big(\frac{\pi\rho}{\rho+1},\quad\frac{\pi(1-\rho)}{2(\rho+1)}\Big),$$

these inequalities are satisfied when $\frac{1}{2}\pi\leqslant\vartheta\leqslant\pi$.

Near the origin, the integrand in (1.41) is dominated by the factor

$$\exp\Big(\frac{|Z|e^{i\vartheta}}{\rho v^\rho}\Big)=\exp\Big(\frac{|Z|}{\rho|v|^\rho}e^{i(\vartheta-\rho\theta)}\Big).$$

Now, on the arc of the small circle, $\vartheta-\rho\theta$ lies between

$$\frac{\vartheta+2\pi\rho}{\rho+1},\quad\frac{\vartheta}{\rho+1}+\tfrac{1}{2}\pi-\rho\gamma$$

by (4.01). When $\frac{1}{2}\pi\leqslant\vartheta\leqslant\pi$ and γ is sufficiently small, these lie between $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$, so that $\cos(\vartheta-\rho\theta)<-K$ on the small arc, and the integral along the arc tends to zero with δ. Hence C' may be replaced by

$$\Gamma_2\Big(\frac{\vartheta-2\pi}{\rho+1}\Big),\quad\Gamma_4\Big(\frac{\vartheta}{\rho+1}\Big).$$

We put $\qquad v=t\exp\Big(\frac{i(\vartheta-2\pi)}{\rho+1}\Big),\quad v=t\exp\Big(\frac{i\vartheta}{\rho+1}\Big),$

respectively in the integrals along these contours and we have

$$\phi(z)=I_2(Z_2)+I_4(Z_1)=H(Z_1)+H(Z_2).$$

(iii) $0<\rho<1$, $-\pi\leqslant\vartheta\leqslant-\frac{1}{2}\pi$. Here we take the two contours

$$\Gamma_5\Big(\frac{\vartheta}{\rho+1}\Big),\quad \Gamma_1\Big(\frac{\vartheta+2\pi}{\rho+1}\Big),$$

and an arc of a small circle. As before, we have

$$\phi(z)=H(Z_1)+H(Z_2).$$

(iv) $\rho>1$. In this case, we replace C' by three contours joined by arcs of a small circle. When $\rho\geqslant\frac{5}{2}$ and $|\vartheta|\leqslant\pi$, we take

$$\Gamma_5\Big(\frac{\vartheta-2\pi}{\rho+1}\Big),\quad \Gamma_3\Big(\frac{\vartheta}{\rho+1}\Big),\quad \Gamma_4\Big(\frac{\vartheta+2\pi}{\rho+1}\Big).$$

When $1<\rho\leqslant\frac{5}{2}$ and $0\leqslant\vartheta\leqslant\pi$, we take

$$\Gamma_5\Big(\frac{\vartheta-2\pi}{\rho+1}\Big),\quad \Gamma_3\Big(\frac{\vartheta}{\rho+1}\Big),\quad \Gamma_1\Big(\frac{\vartheta+2\pi}{\rho+1}\Big),$$

and, when $-\pi\leqslant\vartheta\leqslant0$,

$$\Gamma_2\Big(\frac{\vartheta-2\pi}{\rho+1}\Big),\quad \Gamma_3\Big(\frac{\vartheta}{\rho+1}\Big),\quad \Gamma_4\Big(\frac{\vartheta+2\pi}{\rho+1}\Big).$$

We can then show that the integrals along the arcs of the small circle tend to zero with the radius of the circle. Hence we have

$$\phi(z)=H(Z')+H(Z'')+H(Z'''),$$

where

$$|Z'|=|Z''|=|Z'''|=|Z|,$$

$$\arg Z'=\frac{\vartheta-2\pi}{\rho+1},\quad \arg Z''=\frac{\vartheta}{\rho+1},\quad \arg Z'''=\frac{\vartheta+2\pi}{\rho+1}.$$

If $0\leqslant\vartheta\leqslant\pi$, we have $\xi=\vartheta-\pi$, $Z_1=Z''$, and $Z_2=Z'$, and we can show that $H(Z''')$ is negligible compared with the error term in $H(Z'')$. Hence

$$\phi(z)=H(Z_1)+H(Z_2).$$

When $-\pi\leqslant\vartheta\leqslant0$, the same result is true.

This complete the proof of Theorem 1. The deduction of Theorem 2 is easy. If $0\leqslant\vartheta\leqslant\pi-\epsilon$, we have $\vartheta=\xi+\pi$, $Z=Z_1$, and

$$\cos\arg Z_1-\cos\arg Z_2=\cos\frac{\vartheta}{\rho+1}-\cos\frac{\vartheta-2\pi}{\rho+1}$$

$$=2\sin\frac{\pi-\vartheta}{\rho+1}\sin\frac{\pi}{\rho+1}<K,$$

and so $H(Z_2)$ may be neglected.

On the other hand, if $-\pi+\epsilon \leqslant \vartheta \leqslant 0$, then $\vartheta = \xi - \pi$, $Z = Z_2$,

$$\cos \arg Z_2 - \cos \arg Z_1 > K,$$

and $H(Z_1)$ is negligible compared with the error term in $H(Z_2)$. In either case, we have

$$\phi(z) = H(Z).$$

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A NOTE ON THE COMPLETE SYSTEM OF THE BINARY (2, 1, 1) FORM

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1. In a previous communication* the complete system of the binary (2, 1, 1) form has been obtained. It has since been found that four of the retained forms are reducible, the reductions being given below. The notation is that of the paper cited.

2. *Reduction of* $(fJ_2, Q_4)^3$.

$$\begin{aligned}(fJ_2, Q_4)^3 &\equiv \{A_x^2 A_y'^2 a_x^2 a_y', (B'C')^2 B_x^2 C_x^2\}^3\\ &= (AC)(Ba)^2(B'C')^2 A_y'^2 a_y' A_x C_x\\ &\qquad +\lambda\{(Aa), (B'C')^2\}^2+\mu(Aa).\Delta_4\\ &= (AC)(Ba)^2(B'C')^2 A_y'^2 a_y' A_x C_x+\lambda(T_2, Q_4)^2\\ &= \lambda(T_2, Q_4)^2+\{(Ba)^2 B_y'^2 a_y', (AC) A_y'^2 C_y'^2\}^2\\ &\qquad +\nu(AC)(A'C').(Ba)^2(B'a')\\ &\equiv \lambda(T_2, Q_4)^2+\nu(J_2, J_1)^2 U, \quad \text{since} \quad (Ba)^2 = 0 = (AC).\end{aligned}$$

Thus $(fJ_2, Q_4)^3$ is expressed in terms of other members of the system.

Similarly

$$(fJ_3, Q_4)^3 \equiv \lambda'(T_1, Q_4)^2+\nu'(J_3, J_1)^2 U \equiv 0.$$

3. *Reduction of* $(fJ_2, P_6)^4$.

$$\begin{aligned}(J_2, P_6)^2 &\equiv \{A_x^2, (\xi B)(B'C')^2 \xi_x^3 B_x C_x^2)^2\\ &\equiv (\xi A)^2(\xi B)(B'C')^2+\lambda\{A_x^2, (\xi B)^2(B'C')^2\}^1\\ &\qquad +\mu J_2(\xi B)^2(\xi C)(B'C')^2.\end{aligned}$$

Now $$(\xi B)^2(B'C')^2 \equiv \{\xi_x^4, (B'C')^2\}^2 = (\xi_x^4, Q_4)^2 = (J_1, Q_4)^2.$$

But Q_4 is the Hessian of J_1, so $(J_1, Q_4)^4 \equiv 0 \pmod{\Delta_4}$.

* C. W. Gilham, *Proc. London Math. Soc.* (2), 32 (1931), 259–272.

Similarly

$$(\xi B)^2(\xi C)(B'C')^2 \equiv (\xi_x{}^4,\ Q_4)^3 = (J_1,\ Q_4)^3 \equiv 0 \pmod{\Delta_4}.$$

Hence

$$\begin{aligned}(J_2,\ P_6)^2 &\equiv (\xi A)^2(\xi B)(B'C')^2 = \{(\xi A)^2,\ (B'C')^2\}^1 \\ &\equiv \{(AD)(A'D'),\ (B'C')^2\}^1 = (AB)(AD)(A'D')(B'C')^2 \\ &= (AB)^2(A'D')(B'C')^2.\end{aligned}$$

$$\begin{aligned}(fJ_2,\ P_6)^4 &= \{f,\ (J_2P_6)^2\}^2 \equiv \{a_x{}^2,\ (AB)^2(A'D')(B'C')^2\}^2 \\ &= (AB)^2(aD)^2(A'D')(B'C')^2 + \lambda\{a_x{}^2,\ (AB)^2(CD)(A'D')(B'C')^2\}^1 \\ &\qquad + \nu a_x{}^2(AB)^2(CD)^2(A'D')(B'C')^2 \\ &= \{(AB)^2(B'C')^2,\ (aD)^2\}^{0.1} + \rho U(AB)^2(B'C')^2 \\ &\qquad + \lambda\{a_x{}^2,\ (AB)^2(CD)(A'D')(B'C')^2\}^1 \\ &\equiv \lambda\{a_x{}^2,\ (AB)^2(CD)(A'D')(B'C')^2\}^1, \quad \text{since} \quad (aD)^2 \equiv 0.\end{aligned}$$

Now

$$\begin{aligned}&(AB)^2(CD)(A'D')(B'C')^2 \\ &= (AB)(AC)(CD)(A'D')(B'C')^2 - (AB)(BC)(CD)(A'D')(B'C')^2 \\ &= (AB)(AC)(BD)(A'D')(B'C')^2 - (AB)(AC)(BC)(A'D')(B'C')^2 \\ &\qquad - (AB)(BC)(CD)(A'D')(B'C')^2 \\ &= \tfrac{1}{2}(AB)(AD)(BD)(A'D')(B'C')^2 - (AB)(BC)(AD)(A'D')(B'C')^2 \\ &= \tfrac{1}{2}\{(AB)(AD)(BD)(A'D'),\ J_2\}^{0.2} \\ &\qquad + \lambda\{(AB)(AD)(BD)(A'D')(A'B'),\ J_2\}^{0.1} \\ &\qquad + \mu J_2 . (AB)(AD)(BD)(A'B')(A'D')(B'D') \\ &\qquad + \tfrac{1}{2}(AD)(A'D') . (BC)^2(B'C')^2 \\ &\equiv \mu J_2\Delta_6 + \tfrac{1}{2}(J_2,\ J_1)^{2.0}\Delta_4 \equiv 0.\end{aligned}$$

Similarly $$(fJ_3,\ P_6)^4 \equiv 0.$$

Thus the four concomitants

$$(fJ_2,\ Q_4)^3, \quad (fJ_3,\ Q_4)^3, \quad (fJ_2,\ P_6)^4, \quad (fJ_3,\ P_6)^4$$

are reducible, and the complete system of the binary (2, 1, 1) form contains not more than sixty-one concomitants.

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THE RELATION BETWEEN RIEMANN AND CESÀRO SUMMABILITY

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We shall say that a series $\sum_{n=1}^{\infty} a_n$ is summable (R, k) to s if

$$\lim_{h\to 0} \sum_{n=1}^{\infty} a_n \left(\frac{\sin nh}{nh}\right)^k = s,$$

where k is a positive integer, and the series

$$\sum_{n=1}^{\infty} a_n \left(\frac{\sin nh}{nh}\right)^k$$

is supposed convergent for all values of h in some neighbourhood of the origin. This is a generalization of Riemann summability, which is, under this definition, summability $(R, 2)$. This method of summability has been considered by Verblunsky*, who has shown that, if a series is summable $(C, k-\delta)$ to s, where $\delta > 0$, then it is summable $(R, k+1)$ to s This result had previously been stated without proof by Kogbetliantz†, and some other results of this nature had been obtained by Rajchmann and Zygmund‡.

In this paper we consider the converse problem; that is to say we consider whether it is possible to deduce Cesàro summability from Riemann summability. In this direction Gronwall§ has proved that, if $\sum_{n=1}^{\infty} a_n$ is summable $(R, 2)$, then it is summable $(C, 3)$ to the same sum,

* Verblunsky, **8**.

† Kogbetliantz, **5**.

‡ Rajchmann and Zygmund, **6**.

§ Gronwall, **1**. Gronwall proves that, if $f(x)$ possesses a generalized second differential coefficient $f^{(2)}(x)$ at any point, then the series obtained by twice differentiating the Fourier series of $f(x)$ at that point is summable $(C, 3)$ to $f^{(2)}(x)$. The result stated evidently follows from this.

provided that $\sum_{n=1}^{\infty} \frac{a_n}{n^2} \cos nh$

is a Fourier series. This result has been improved by Zygmund*, who has shown that, under the same conditions, the series $\sum_{n=1}^{\infty} a_n$ is summable $(C, 2+\delta)$ for $\delta > 0$, and also that it is summable $(\log n, 2)$, *i.e.* by Riesz's logarithmic means† of order 2. Another proof of this result has been given by Jacob‡. The relation between summability $(R, 2)$ and Abel summability for series satisfying this condition has been investigated by Rajchmann and Zygmund§, who have obtained inequalities between the upper and lower Riemann and Abel sums.

The chief object of this paper is to remove the restriction that

$$\sum_{n=1}^{\infty} \frac{a_n}{n^2} \cos nh$$

should be a Fourier series; that is to say, to prove the following theorem.

If a series $\sum_{n=1}^{\infty} a_n$ *is summable* $(R, 2)$ *to* s, *then it is summable* $(C, 2+\delta)$ *to* s, *where* $\delta > 0$. *Further, the series is summable* $(\log n, 2)$ *to* s.

It should be stated here that Verblunsky‖ has considered a generalization of summability (R, k), in which, instead of

$$\sum_{n=1}^{\infty} a_n \left(\frac{\sin nh}{nh}\right)^k$$

being supposed convergent, it is supposed to be Abel summable. He has proved that, if

$$\sum_{n=1}^{\infty} a_n \left(\frac{\sin nh}{nh}\right)^2$$

is Abel summable to $\phi(h)$ for all values of h in some neighbourhood of the origin, and if $\phi(h) \to s$ as $h \to 0$, then $\sum_{n=1}^{\infty} a_n$ is summable $(C, 4+\delta)$ to s, provided only that $a_n = o(n^3)$. In a later paper¶ he has improved this

* Zygmund, **11**.

† For the definition and properties of Riesz's means, see Hardy and Riesz, **2**.

‡ Jacob, **4**, 62.

§ Rajchmann and Zygmund, **7**.

‖ Verblunsky, **9**.

¶ Verblunsky, **10**.

result by replacing summability $(C, 4+\delta)$ by summability $(C, 3)$ in the conclusion. Verblunsky's hypotheses are much more general than those of this paper, since the hypothesis $a_n = o(n^3)$ is implied by the convergence of

$$\sum_{n=1}^{\infty} a_n \left(\frac{\sin nh}{nh}\right)^2,$$

which, indeed, implies the stronger result $a_n = o(n^2)$. On the other hand, his conclusion is weaker: of course, since he supposes only that $a_n = o(n^3)$, summability $(C, 3)$ is the most which could be expected. The methods employed by Verblunsky are quite different from those of this paper.

It can be proved by methods similar to those of this paper that, if a series is summable $(R, 1)$, then it is summable $(C, 1+\delta)$ to the same sum. Further, it is summable $(\log n, 1)$. These results have been obtained by Zygmund*.

No such result holds for other orders of Riemann summability, and we conclude this paper with an example of a series which is summable $(R, 3)$, but which is not even Abel summable. This difference arises from the following reasons. If we put

$$F(h) = \sum_{n=1}^{\infty} \frac{a_n}{n^2} \cos nh$$

(assuming for the moment that the sum exists), then, if $\sum_{n=1}^{\infty} a_n$ is summable $(R, 2)$, it follows that

$$\frac{F(h) - F(0)}{h^2}$$

tends to a limit as $h \to 0$, and thus $F(h)$ must be fairly regular in the neighbourhood of the origin. Now, when we are dealing with summability $(R, 3)$, the corresponding definition of $F(h)$ is

$$F(h) = \sum_{n=1}^{\infty} \frac{a_n}{n^3} \sin nh,$$

and the corresponding result is that

$$\frac{F(3h) - 3F(h)}{h^3}$$

tends to a limit as $h \to 0$. Thus the behaviour of $F(h)$ in the neighbourhood of the origin may be quite irregular, provided that, roughly speaking, the irregular part of $e^{-x} F(e^x)$ is periodic, with period $\log 3$.

* Zygmund, 12.

Similar considerations evidently hold for Riemann summability of higher orders.

Considerable use is made in this paper of the theorems on the formal multiplication of trigonometrical series obtained by Rajchmann and by Zygmund*. With the aid of these theorems it is possible to deduce our result in the general case when once it has been proved in two particular cases. The first of these is the case already referred to, in which

$$\sum_{n=1}^{\infty} \frac{a_n}{n^2} \cos nh$$

is a Fourier series, and the second the case in which

$$\sum_{n=1}^{\infty} \frac{a_n}{n^2} \cos nh$$

converges uniformly to 0 for all values of h in some neighbourhood of the origin. This case is dealt with in Lemma 4.

I am indebted to a Referee for drawing my attention to the papers connected with this subject published in the Bulletin de l'Académie Polonaise, and also for pointing out the possibility of proving summability $(\log n, 2)$: my paper originally contained only the proof of summability $(C, 2+\delta)$.

LEMMA 1. *If* $b_\nu \sin^2 \nu h \to 0$ *as* $\nu \to \infty$ *at all points of a set* G *of positive measure, then* $b_\nu \to 0$ *as* $\nu \to \infty$.

The proof is similar to that of the known theorem† in which it is assumed that $b_\nu \sin \nu h \to 0$.

LEMMA 2. *If* $\sum_{\nu=1}^{\infty} b_\nu \sin^2 \nu h$ *converges for all* h *in some interval* (α, β), *then* $\sum_{\nu=1}^{\infty} b_\nu$ *is convergent.*

By Lemma 1, $b_\nu = o(1)$, and hence $\sum_{\nu=1}^{\infty} b_\nu \nu^{-2}$ and $\sum_{\nu=1}^{\infty} b_\nu \nu^{-3}$ are absolutely convergent.

Suppose $\sum_{\nu=1}^{\infty} b_\nu$ to be divergent. Then we can find a number $\delta > 0$ such that, for an infinite set of values of n, m,

$$\left| \sum_{\nu=n}^{m} b_\nu \right| > \delta.$$

* Zygmund, 13.

† See, *e.g.*, Hobson, 3, 662–663.

We can thus choose n_1, m_1 such that

$$\left| \sum_{\nu=n_1}^{m_1} b_\nu \right| > \delta,$$

$$\sum_{\nu=n_1}^{m_1} |b_\nu \nu^{-2}| < \tfrac{1}{12}\delta(\beta-\alpha)^2,$$

$$\sum_{\nu=n_1}^{m_1} |b_\nu \nu^{-3}| < \tfrac{1}{12}\delta(\beta-\alpha)^3.$$

Then

$$\left| \int_\alpha^\beta \left\{ \sum_{\nu=n_1}^{m_1} b_\nu \sin^2 \nu h \right\} (\beta-h)(h-\alpha)\, dh \right|$$

$$= \left| \tfrac{1}{12}(\beta-\alpha)^3 \sum_{\nu=n_1}^{m_1} b_\nu + \tfrac{1}{8}(\beta-\alpha) \sum_{\nu=n_1}^{m_1} b_\nu \nu^{-2}(\cos 2\nu\beta + \cos 2\nu\alpha) \right.$$

$$\left. - \tfrac{1}{8} \sum_{\nu=n_1}^{m_1} b_\nu \nu^{-3}(\sin 2\nu\beta - \sin 2\nu\alpha) \right|$$

$$> {}_2\tfrac{1}{4}(\beta-\alpha)^3 \delta.$$

Hence, at all points of some closed interval (α_1, β_1) contained in (α, β),

$$\left| \sum_{\nu=n_1}^{m_1} b_\nu \sin^2 \nu h \right| (\beta-h)(h-\alpha) \geqslant \tfrac{1}{24}(\beta-\alpha)^2 \delta,$$

and hence

$$\left| \sum_{\nu=n_1}^{m_1} b_\nu \sin^2 \nu h \right| \geqslant \tfrac{1}{6}\delta.$$

We may now choose n_2, m_2 such that

$$n_2 > n_1,$$

$$\left| \sum_{\nu=n_2}^{m_2} b_\nu \right| > \delta,$$

$$\sum_{\nu=n_2}^{m_2} |b_\nu \nu^{-2}| < \tfrac{1}{12}\delta(\beta_1-\alpha_1)^2,$$

$$\sum_{\nu=n_2}^{m_2} |b_\nu \nu^{-3}| < \tfrac{1}{12}\delta(\beta_1-\alpha_1)^3.$$

We then prove as before that, at all points of some closed interval (α_2, β_2) contained in (α_1, β_1),

$$\left| \sum_{\nu=n_2}^{m_2} b_\nu \sin^2 \nu h \right| \geqslant \tfrac{1}{6}\delta.$$

We may now choose n_3, m_3 $(n_3 > n_2)$ such that, at all points of some closed interval (α_3, β_3) contained in (α_2, β_2),

$$\left| \sum_{\nu=n_3}^{m_3} b_\nu \sin^2 \nu h \right| \geqslant \tfrac{1}{6}\delta,$$

and so on.

There must be at least one point h contained in all the intervals (α_r, β_r); at this point we have, for all r,

$$\left| \sum_{\nu=n_r}^{m_r} b_\nu \sin^2 \nu h \right| \geqslant \tfrac{1}{6}\delta,$$

and, since $n_{r+1} > n_r$, this contradicts the convergence of $\sum_{\nu=1}^{\infty} b_\nu \sin^2 \nu h$.

The lemma follows.

LEMMA 3. *Let* $C_3(t) = t^3\gamma_3(t) = t - \sin t$, *and let* $\gamma_3''(t) = d^2\gamma_3(t)/dt^2$. *Then, for fixed* η *and large* ω *and* n,

$$\int_0^\eta \left\{\omega^3\gamma_3''(\omega t) - \frac{6}{\omega\eta^4} - \frac{\sin\omega t}{\eta^3} - \frac{6\cos\omega t}{\omega\eta^4} + \frac{12\sin\omega t}{\omega^2\eta^5}\right\}\cos nt\,dt$$
$$= -n^2 G(n, \omega) + R(n, \omega),$$

where
$$G(n, \omega) = \begin{cases} \tfrac{1}{4}\pi\left(1 - \dfrac{n}{\omega}\right)^2 & \text{if } n \leqslant \omega, \\ 0 & \text{if } n \geqslant \omega. \end{cases}$$

and where (i)

$$R(n, \omega) = \frac{1}{\omega^2 - n^2}\left(-\frac{\omega}{\eta^3} + \frac{12}{\omega\eta^5}\right) + O\left(\frac{1}{\omega n^2}\right) + O\left(\frac{1}{(\omega - n)^2}\right).$$

(ii)
$$R(n, \omega) = O(1).$$

It is known* that

$$\int_0^\infty \gamma_3(t)\cos\frac{n}{\omega}t\,dt = G(n, \omega).$$

Hence

$$\tfrac{1}{2}\omega\int_{-\infty}^{\infty}\gamma_3\{\omega(t-x)\}\cos nt\,dt = \tfrac{1}{2}\omega\int_0^\infty \gamma_3(\omega t)\{\cos n(t+x) + \cos n(t-x)\}\,dt$$
$$= G(n, \omega)\cos nx.$$

* Hobson, **3**, 567.

Differentiating this equation twice with respect to x, and then putting $x = 0$, we obtain

$$(1) \qquad \omega^3 \int_0^\infty \gamma_3''(\omega t) \cos nt\, dt = -n^2 G(n, \omega).$$

Again,

$$(2) \quad \omega^3 \int_\eta^\infty \gamma_3''(\omega t) \cos nt\, dt$$

$$= \frac{6}{\omega} \int_\eta^\infty \frac{\cos nt}{t^4}\, dt + \int_\eta^\infty \frac{\sin \omega t \cos nt}{t^3}\, dt$$

$$+ \frac{6}{\omega} \int_\eta^\infty \frac{\cos \omega t \cos nt}{t^4}\, dt - \frac{12}{\omega^2} \int_\eta^\infty \frac{\sin \omega t \cos nt}{t^5}\, dt$$

$$= -\frac{6 \sin n\eta}{\omega n \eta^4} + \frac{\cos (\omega+n)\eta}{2(\omega+n)\eta^3} + \frac{\cos (\omega-n)\eta}{2(\omega-n)\eta^3} - \frac{3 \sin (\omega+n)\eta}{\omega(\omega+n)\eta^4}$$

$$- \frac{3 \sin (\omega-n)\eta}{\omega(\omega-n)\eta^4} - \frac{6 \cos (\omega+n)\eta}{\omega^2(\omega+n)\eta^5} - \frac{6 \cos (\omega-n)\eta}{\omega^2(\omega-n)\eta^5}$$

$$+ O\left(\frac{1}{\omega n^2}\right) + O\left(\frac{1}{(\omega-n)^2}\right),$$

and it is evident that

$$(3) \qquad \omega^3 \int_\eta^\infty \gamma_3''(\omega t) \cos nt\, dt = O(1).$$

Finally,

$$(4) \quad \int_0^\eta \left\{ -\frac{6}{\omega\eta^4} - \frac{\sin \omega t}{\eta^3} - \frac{6 \cos \omega t}{\omega \eta^4} + \frac{12 \sin \omega t}{\omega^2 \eta^5} \right\} \cos nt\, dt$$

$$= -\frac{6 \sin n\eta}{\omega n \eta^4} + \frac{\cos (\omega+n)\eta}{2(\omega+n)\eta^3} + \frac{\cos (\omega-n)\eta}{2(\omega-n)\eta^3} - \frac{3 \sin (\omega+n)\eta}{\omega(\omega+n)\eta^4}$$

$$- \frac{3 \sin (\omega-n)\eta}{\omega(\omega-n)\eta^4} - \frac{6 \cos (\omega+n)\eta}{\omega^2(\omega+n)\eta^5} - \frac{6 \cos (\omega-n)\eta}{\omega^2(\omega-n)\eta^5}$$

$$+ \frac{1}{\omega^2 - n^2} \left(-\frac{\omega}{\eta^3} + \frac{12}{\omega \eta^5} \right)$$

and

$$(5) \qquad \int_0^\eta \left\{ -\frac{6}{\omega\eta^4} - \frac{\sin \omega t}{\eta^3} - \frac{6 \cos \omega t}{\omega \eta^4} + \frac{12 \sin \omega t}{\omega^2 \eta^5} \right\} \cos nt\, dt = O(1).$$

From these equations the lemma follows, the result (i) following from equations (1), (2), and (4), while (ii) follows from equations (1), (3), and (5).

LEMMA 4. *If* $\sum_{n=1}^{\infty} c_n \cos nh$ *converges to* 0 *uniformly for* h *in* $(-\eta, \eta)$, *then* $\sum_{n=1}^{\infty} n^2 c_n$ *is summable* $(C, 2)$ *to sum* 0.

We multiply the equation

$$\sum_{n=1}^{\infty} c_n \cos nh = 0$$

by $$\omega^3 \gamma_3''(\omega h) - \frac{6}{\omega\eta^4} - \frac{\sin \omega h}{\eta^3} - \frac{6\cos \omega h}{\omega\eta^4} + \frac{12 \sin \omega h}{\omega^2 \eta^5},$$

and integrate with respect to h from 0 to η. Since the series is uniformly convergent, we may integrate term by term. We use Lemma 3 to evaluate the integrals occurring, using the result (i) when $|\omega - n| > 1$, and (ii) when $|\omega - n| \leqslant 1$. We obtain

$$(1)\quad \tfrac{1}{4}\pi \sum_{n<\omega} n^2 c_n \left(1 - \frac{n}{\omega}\right)^2 = \left(-\frac{\omega}{\eta^3} + \frac{12}{\omega\eta^5}\right) \sum_{|\omega - n|>1} \frac{c_n}{\omega^2 - n^2} + O\left(\frac{1}{\omega} \sum_{n=1}^{\infty} \frac{|c_n|}{n^2}\right) + O\left(\sum_{|\omega-n|>1} \frac{|c_n|}{(\omega-n)^2}\right) + O\left(\sum_{|\omega-n|\leqslant 1} |c_n|\right).$$

Since Σc_n converges, it follows by partial summation that

$$\left(-\frac{\omega}{\eta^3} + \frac{12}{\omega\eta^5}\right) \sum_{|\omega-n|>1} \frac{c_n}{\omega^2 - n^2} \to 0 \quad \text{as} \quad \omega \to \infty.$$

Further, $c_n = o(1)$, whence it follows that the remaining terms on the right of equation (1) tend to 0 as $\omega \to \infty$. Thus

$$\sum_{n<\omega} n^2 c_n \left(1 - \frac{n}{\omega}\right)^2 \to 0 \quad \text{as} \quad \omega \to \infty,$$

and this is equivalent to summability $(C, 2)$.

LEMMA 5. *Suppose* $f(x)$ *to be periodic, of period* 2π, *and integrable. Let its Fourier series be*

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

Let $$\phi(t) = \frac{f(x_0+t) + f(x_0-t) - 2f(x_0)}{t^2}.$$

Suppose that $\phi(t) \to s$ *as* $t \to 0$.

Then the series

$$\sum_{n=1}^{\infty} n^2 (a_n \cos nx_0 + b_n \sin nx_0)$$

is summable $(C, 2+\delta)$ *to* $-s$ *for all* $\delta > 0$. *Further, it is summable* $(\log n, 2)$ *to the same sum.*

This is a particular case of a result due to Zygmund*.

We are now in a position to prove the theorem. We are given that

$$\sum_{n=1}^{\infty} a_n \left(\frac{\sin nh}{nh}\right)^2 \tag{1}$$

converges to $F(h)$ for h in $(-\eta, \eta)$ and that

$$F(h) \to s \quad \text{as} \quad h \to 0. \tag{2}$$

We may evidently suppose that $F(h)$ is bounded in $(-\eta, \eta)$.

By Lemma 2, it follows that

$$\sum_{n=1}^{\infty} a_n n^{-2}$$

is convergent, and hence that

$$\sum_{n=1}^{\infty} \frac{a_n}{n^2} \cos nh \tag{3}$$

converges to, say, $G(h)$ for h in $(-2\eta, 2\eta)$.

Then

$$G(2h) - G(0) = -2h^2 F(h). \tag{4}$$

Now consider the formal product of the series (3) and the Fourier series of an even function $\lambda(h)$ which is equal to 1 in $(-\eta, \eta)$, and to 0 outside $(-2\eta, 2\eta)$, and whose first three derivatives exist. Let this formal product be

$$\sum_{n=1}^{\infty} \frac{b_n}{n^2} \cos nh. \tag{5}$$

The series (5) converges† to a sum which is equal to $G(h)$ in $(-\eta, \eta)$, to $G(h)\lambda(h)$ in $(-2\eta, -\eta)$ and in $(\eta, 2\eta)$, and to 0 outside $(-2\eta, 2\eta)$, so that it is everywhere bounded. Hence‡ the series (5) is a Fourier series.

* Zygmund, **11.**

† Zygmund, **13,** 60 (Théorème III).

‡ Hobson **3,** 675.

Since (5) converges to $G(h)$ for h in $(-\eta, \eta)$ we see, by Lemma 5* and by equations (2) and (4), that

$$\sum_{n=1}^{\infty} b_n$$

is summable $(C, 2+\delta)$ to s. Further, the series is summable $(\log n, 2)$ to s.

The series

$$\sum_{n=1}^{\infty} \frac{b_n - a_n}{n^2} \cos nh$$

is convergent to 0 in $(-\eta, \eta)$; hence† it is uniformly convergent in any interval contained in $(-\eta, \eta)$, and thus, by Lemma 4, $\sum_{n=1}^{\infty} (b_n - a_n)$ is summable $(C, 2)$ to 0. Since a series summable $(C, 2)$ is summable $(\log n, 2)$ or $(C, 2+\delta)$, the theorem follows.

We now give the example, referred to in the introduction, of a series summable $(R, 3)$ but not Abel summable.

Let $(\alpha)_n$ denote $\alpha(\alpha+1)\dots(\alpha+n-1)$, and let A denote $2\pi(\log 3)^{-1}$.

Define $a_n = (n!)^{-1} n^3 \{(Ai-1)_n + \alpha(Ai-2)_n + \beta(Ai-3)_n\}$, where α, β are constants.

Then

$$a_n \sim \{\Gamma(Ai-1)\}^{-1} n^{Ai+1} = O(n) \quad \text{as} \quad n \to \infty.$$

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \{(Ai-1)_n + \alpha(Ai-2)_n + \beta(Ai-3)_n\} \times \left\{\frac{1}{(n-1)!} + \frac{3}{(n-2)!} + \frac{1}{(n-3)!}\right\} x^n$$

$$\sim (Ai-1)\,Ai(Ai+1)(1-x)^{-Ai-2} \quad \text{as} \quad x \to 1,$$

so that $\sum_{n=0}^{\infty} a_n$ is not Abel summable. Also

$$\sum_{n=0}^{\infty} a_n n^{-3} e^{\pm in\hbar} = (1-e^{\pm ih})^{-Ai+1} + \alpha(1-e^{\pm ih})^{-Ai+2} + \beta(1-e^{\pm ih})^{-Ai+3}$$

$$= \mp ih \exp(\mp \tfrac{1}{2}\pi A - Ai \log h) \times [(1 \pm \tfrac{1}{2} ih + \dots)^{-Ai+1} \mp i\alpha h (1 \pm \tfrac{1}{2} ih + \dots)^{-Ai+2} - \beta h^2 (1 \pm \tfrac{1}{2} ih + \dots)^{-Ai+3}].$$

The expression in square brackets may be expanded in ascending powers of h. By suitably choosing α and β we can arrange that the coefficients of

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* Taking $x_0 = 0$.

† Zygmund, **13**, 87 (Théorème I).

h and h^2 are zero. We then have

$$\sum_{n=0}^{\infty} a_n n^{-3} \sin nh = -\tfrac{1}{2}(e^{\frac{1}{2}\pi A}+e^{-\frac{1}{2}\pi A})\, h \exp(-Ai \log h)+O(h^4).$$

Since $$\exp(-Ai \log h) = \exp(-Ai \log 3h),$$

we have

$$\sum_{n=0}^{\infty} a_n \left(\frac{\sin nh}{nh}\right)^3 = \tfrac{1}{4}h^{-3}\left\{3\sum_{n=0}^{\infty} a_n n^{-3} \sin nh - \sum_{n=0}^{\infty} a_n n^{-3} \sin 3nh\right\}$$

$$= O(h) \to 0 \quad \text{as} \quad h \to 0.$$

Thus $\sum_{n=0}^{\infty} a_n$ is summable $(R, 3)$.

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ON THE THEORY OF TRIGONOMETRIC SERIES (VI)†

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1. In §§2–6 of the present paper, we are concerned with two uniqueness theorems. It is known (II, Theorem II) that, when the Poisson sums of a trigonometric series satisfy certain conditions, and when the coefficients a_n, b_n of the series are $o(n)$, then the series is a Fourier series. In Theorem II of the present paper, we replace this condition on the coefficients by the much more general conditions:

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{m=1}^{n} (a_m \cos mx + b_m \sin mx) = 0,$$

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{m=1}^{n} (b_m \cos mx - a_m \sin mx) = 0$$

for all x. Where, in II, Theorem II, we were concerned with the differential coefficient of a function, we have now to consider the approximate differential coefficient of a function. We established, in V, some properties of the approximate differential coefficient of a function which belongs to certain classes of functions which occur in the theory of trigonometric series, and we make some application of these properties in the present paper. In II, Theorem II was reduced to Theorem I, in which, instead of supposing that $a_n = o(n)$, $b_n = o(n)$, we suppose that $\Sigma A_n(x)/n^2$ is the Fourier series of a continuous function (condition F). In the present paper, Theorem II is likewise reduced to Theorem I, in which the condition F is replaced by a much less restrictive one. The condition F may be divided into two; namely, (*a*) $\Sigma A_n(x)/n^2$ is a Fourier series, and (*b*) the

† The first four papers of this series have appeared in Vols. 34 and 35 of these *Proceedings*; the fifth paper has appeared in *Fundamenta Math.*, 21 (1933), 168–210. See also *Fundamenta Math.*, 23 (1934), 235–6. The references in this paper are to *Proc. London Math. Soc.* (2), 34 (1932), 441–456, which will be called I; to *Proc. London Math. Soc.* (2), 34 (1932), 457–491, which will be called II; and to the paper in *Fundamenta Math.*, which will be called V. A knowledge of V is not assumed; a knowledge of I and II is desirable, as constant reference will be made to these papers.

Poisson sum of that series is continuous. Of these conditions, (a) is of secondary importance, and may be replaced, roughly speaking, by any condition which implies that $\Sigma A_n(x)/n^2$ is a Fourier series if its Poisson sum satisfies the conditions of a uniqueness theorem. We replace it by the condition that $\Sigma A_n(x)/n^4$ is to be the Fourier series of a continuous function. (The condition $a_n = o(n^3)$, $b_n = o(n^3)$ is also possible, as is shown in Theorem XIV.) The condition (b) is more important. What is essential is that the Poisson sum shall not have an *ordinary* discontinuity at any point, even on one side (property A). Other kinds of discontinuity are permissible. It is the property A which replaces (b).

In §§ 6–8 we first prove a result (Theorem III) which is constantly used in the sequel. This result could be deduced from known theorems of M. Riesz and Zygmund (*cf.* § 8), but, for reasons explained in § 8, I wish to avoid appealing to these theorems. It is the method of proving Theorem III to which I draw particular attention. This method is short, involves no calculations, and explains the theorem which it establishes. Riemann's theorem, or rather the useful part of Riemann's theorem, which asserts that the asymptotic behaviour at a point of the partial sums of a trigonometric series whose coefficients are $o(1)$ depends only on the sum of the doubly integrated series in a neighbourhood of the point, is shown to be an easy consequence of Theorem III.

In §§ 9–15 we are concerned with theorems which complete II, §§ 6–9. It was shown, for example, in II, Theorem III, that, if $a_n = o(n^2)$, and

$$\lim_{r\to 1} \sum_{n=1}^{\infty} a_n \left(\frac{\sin n\theta}{n\theta}\right) r^n = \phi(\theta) \quad (-\eta < \theta < \eta),$$

where $\phi(\theta) \to 0$ as $\theta \to 0$, then Σa_n is summable to zero by Poisson's method. It was observed in II (p. 472) that the conclusion could be strengthened, and that Σa_n is summable by Cesàro's method for suitable indices. I can now give the "right" index of summation. In the theorem just quoted, $\Sigma a_n = 0$ $(C, 2)$. Similarly, in II, Theorems IV, V, VII, the conclusion that Σa_n is summable P can be replaced by summability $(C, 3)$, $(C, 4)$, $(C, 5)$ respectively.

In II, Theorems V, VII, an additional condition on $\Sigma a_n/n^2$ was required in order that the conclusion may follow, while in II, Theorems III, IV, no such additional condition occurs. We show, in the present paper (Theorems XVII, XX), that this lies in the nature of the problem. For example, even if $a_n = o(n^2)$, the condition

$$\Sigma a_n \left(\frac{\sin n\theta}{n\theta}\right)^3 = 0 \quad (-\eta < \theta < \eta)$$

by itself, so far from implying

$$\lim \Sigma a_n r^n = 0,$$

does not even imply the existence of

$$\lim \Sigma \frac{a_n}{n^2} r^n.$$

The construction of an example to illustrate this is rendered at once possible and easy by a theorem of Hardy and Littlewood†.

Preliminary lemmas.

2. We are concerned in the first place with what is, in certain respects, a generalization of the results of I, §§ 3, 4. The notation is that of I, §§ 1, 2. Given two Fourier-Lebesgue series

$$f(x) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + \beta_n \sin nx), \tag{1}$$

$$\phi(x) \sim \tfrac{1}{2}\gamma_0 + \sum_{n=1}^{\infty} (\gamma_n \cos nx + \delta_n \sin nx), \tag{2}$$

let there be an open interval d in which f and ϕ are equivalent. Then the series

$$\tfrac{1}{2}(a_0-\gamma_0) + \sum_{n=1}^{\infty} \{(a_n-\gamma_n)\cos nx + (\beta_n-\delta_n)\sin nx\},$$

and any series obtained from this by k formal differentiations, with k a positive integer, is uniformly summable by Poisson's method to zero in any assigned closed interval contained in d. Let x denote a point of d. Let

$$p(r, x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + \beta_n \sin nx) r^n \quad (0 < r < 1),$$

and let $\overline{p}(x)$, $\underline{p}(x)$ denote the upper and lower Poisson sums of (1) at x. Let $p(r, x;\ k)$, $\overline{p}(x;\ k)$, $\underline{p}(x;\ k)$ have the corresponding meanings for the series obtained by k formal differentiations of (1). Let $\pi(r, x)$, $\overline{\pi}(x)$, $\underline{\pi}(x)$, etc., have the corresponding meanings for the series (2). Then

$$\overline{p}(x) = \overline{\pi}(x),\ \ \overline{p}(x;\ k) = \overline{\pi}(x;\ k),\ \ \overline{\lim}\,(1-r)\,p(r, x) = \overline{\lim}\,(1-r)\,\pi(r, x),\ \text{etc.}$$

Consequently, in order to prove any property which can be expressed in terms of $\overline{p}(x)$, $\underline{p}(x)$, $\overline{\lim}\,(1-r)\,p(r, x)$, $\overline{p}(x;\ k)$, etc., for one or more values of x in d, it is sufficient to prove the property when the p's are replaced by

† Hardy and Littlewood, *Journal London Math. Soc.*, 1 (1926), 134–138; 6 (1931), 281–286.

π's. The principle which justifies this substitution of (2) for (1) will be called the principle L.

We recall the definition in accordance with which a trigonometric series

$$\tfrac{1}{2}C_0+\sum_{n=1}^{\infty}(C_n\cos nx+D_n\sin nx)$$

is said to possess the property F, when

$$-\sum_{n=1}^{\infty}(C_n\cos nx+D_n\sin nx)/n^2$$

is the Fourier series of a continuous function. We say that the trigonometric series possesses the property F in an interval d, if the doubly integrated series is the Fourier series of a function which is continuous in d. Writing $C_n(x)=C_n\cos nx+D_n\sin nx$, we shall use the symbol

$$\sum_{n-1}^{\infty}C_n(x)\quad(P) \tag{3}$$

only when

$$\sum_{n-1}^{\infty}C_n(x)r^n\quad(0<r<1)$$

tends, as $r\to 1$, either to a finite limit, or to $+\infty$, or to $-\infty$. The symbol (3) will then denote the finite limit, or $+\infty$, or $-\infty$ respectively. When (3) has a meaning for all x in the set of points under consideration, we say that the Poisson sum of $\Sigma C_n(x)$ is *unique* in the set; and we use a similar mode of expression for other functions which are defined by a limiting process. For example, if $f(x)$ is a finite function, defined for all x, we say that its derivative $f'(x)$ is unique, if, for each x,

$$\frac{f(x+h)-f(x)}{h} \tag{4}$$

tends, as $h\to 0$, either to a finite limit, or to $+\infty$, or to $-\infty$. If, for each x, the expression (4) tends, as $h\to 0$ on a set of unit metric density at the origin, either to a finite limit, or to $+\infty$, or to $-\infty$, we say that the approximate derivative $f_a(x)$ of $f(x)$ is unique; and the value of $f_a(x)$ is respectively the finite limit, or $+\infty$, or $-\infty$.

LEMMA 1. *Let $C-\Sigma A_n(x)/n^2$ possess the property F in an interval $d\equiv(a<x<b)$, and let it be summable P in that interval to a unique function $F(x)>0$. Then*

$$\mathfrak{F}(x)=Cx+\Sigma B_n(x)/n^3\quad(P)\quad(a<x<b)$$

is non-diminishing, and $F(x)$ is finite p.p. (i.e. almost everywhere) in d.

In virtue of the property F in d, the function $G(x)$ which satisfies

$$G(x) \sim \tfrac{1}{2}Cx^2 + \Sigma A_n(x)/n^4,$$

$$G(x) = \tfrac{1}{2}Cx^2 + \Sigma A_n(x)/n^4 \quad (P) \quad (a < x < b),$$

is continuous in d. By I, Theorem II,

$$\bar{D}^2 G(x) \geqslant F(x) > 0 \quad (a < x < b).$$

By II, Lemma 3, $G(x)$ is convex in d. By the principle L, we may suppose that $G(x)$ is convex in $(-\infty, \infty)$. Hence

$$Cx + \Sigma B_n(x)/n^3 = \mathfrak{F}(x)$$

is non-diminishing. We have

$$C - \Sigma \frac{A_n(x)}{n^2} r^n = \frac{1}{\pi} \int_0^\pi \frac{(1-r^2) r \sin t}{\delta^2} [F(\xi+t) - F(\xi-t)]\, dt.$$

At a point ξ at which $\mathfrak{F}'(\xi)$ exists, *i.e.* p.p., we have, by an elementary argument,

$$C - \Sigma \frac{A_n(x)}{n^2} \quad (P) = \mathfrak{F}'(\xi). \tag{5}$$

For such a ξ in d, the first member of (5) is $F(\xi)$ by definition. Hence the result.

LEMMA 2. *Let $-\Sigma A_n(x)/n^2$ possess the property F in an interval d, and let*

$$F(x) = C - \Sigma A_n(x)/n^2 \quad (P) \quad (a < x < b).$$

Then, at a point $\xi \subset d$ at which $F(\xi)$ is finite,

(A)
$$\overline{\lim}\, \Delta^2 F(\xi, h)/h \geqslant \pi \underline{\lim}\, (1-r) P(r, \xi),$$

$$\underline{\lim}\, \Delta^2 F(\xi, h)/h \leqslant \pi \overline{\lim}\, (1-r) P(r, \xi).$$

(B)
$$\bar{D}^2 F(\xi) \geqslant \underline{P}(\xi), \quad \underline{D}^2 F(\xi) \leqslant \bar{P}(\xi).$$

Proof of (A). We adapt the proof of I, Theorem I. We may suppose that $\xi = 0$, and we make the same simplifications as there. It is sufficient to prove that

$$\underline{\lim}\, F(h)/h > 0 \quad (h > 0) \tag{6}$$

implies
$$\overline{\lim}\, (1-r) P(r, 0) \geqslant 0. \tag{7}$$

The relation (6) implies that $F(x)$ is non-negative in a neighbourhood of

the origin. By Lemma 1, $\mathfrak{F}(x)$ is non-diminishing in that neighbourhood. By the principle L, we may suppose that $\mathfrak{F}(x)$ is non-diminishing in $(-\infty, \infty)$. Write

$$\mathfrak{F}(t) = Ct + \Sigma \frac{a_n \sin nt}{n},$$

$$\begin{aligned} A(r) &= C + \Sigma a_n r^n \\ &= \frac{1}{\pi} \int_0^\pi \frac{1-r^2}{\delta} \, d\mathfrak{F}(t) \\ &\geqslant 0 \qquad\qquad (0 < r < 1). \end{aligned}$$

The remainder of the proof is the same as for I, Theorem I.

Proof of (B). We adapt the proof of I, Theorem II. On examining the proof, we see that everything remains unaltered except the equation at the head of p. 446, which now becomes

$$\underline{\lim} \frac{d}{dr} \left(\frac{rA(r)}{1-r^2} \right) = \underline{\lim} \frac{1}{\pi} \int_0^\pi \frac{1-r^2}{\delta^2} \, d\mathfrak{F}(t),$$

and we must show that the second member is positive if

$$\underline{\lim} \, F(h)/h^2 > 0. \tag{8}$$

The function $\mathfrak{F}(t)$ is non-diminishing, and in an interval $(0, \eta)$ say, we have, as in the proof of Lemma 1, $\mathfrak{F}'(t) = F(t)$ p.p. Hence

$$\mathfrak{F}(t) = \int_0^t F(u) \, du + \lambda(t),$$

where $\lambda(t)$ is non-diminishing. Hence

$$\begin{aligned} \underline{\lim} \frac{1}{\pi} \int_0^\pi \frac{1-r^2}{\delta^2} \, d\mathfrak{F}(t) &= \underline{\lim} \frac{1}{\pi} \int_0^\eta \frac{1-r^2}{\delta^2} \, d\mathfrak{F}(t) \\ &\geqslant \underline{\lim} \frac{1}{\pi} \int_0^\eta \frac{1-r^2}{\delta^2} \, F(t) \, dt \\ &> 0, \end{aligned}$$

by (8).

Lemma 3. *Let* $-\Sigma A_n(x)/n^2$ *possess the property* F *in an interval* d. *Let*

$$F(x) = C - \Sigma A_n(x)/n^2 \quad (P) \quad (a < x < b)$$

be finite and integrable. Then, at a point $\xi \subset d$ *at which* $D^2 F(\xi)$ *exists, we have* $P(\xi) = D^2 F(\xi)$.

By the property F, the function $G(x)$ which satisfies

$$G(x) \sim \tfrac{1}{2} Cx^2 + \Sigma A_n(x)/n^4,$$

$$G(x) = \tfrac{1}{2} Cx^2 + \Sigma A_n(x)/n^4 \quad (P) \quad (a < x < b)$$

is continuous in d. By I, Theorem II,

$$\bar{D}^2 G(x) \geqslant F(x) \geqslant \underline{D}^2 G(x) \quad (a < x < b).$$

By II, Lemma 5, $G(x)$ is, in d, the integral of an integral. By the principle L, we may suppose that $G(x)$ has this property in $(-\infty, \infty)$; *i.e.* we may suppose that $-\Sigma A_n(x)/n^2$ is a Fourier series, in which case the result of the lemma is known; *cf.* I, Theorem III, Cor. 2.

3. LEMMA 4. *Let $F(x) < \infty$ be defined in $a < x < b$ and be u.s.c. (i.e. upper semi-continuous) in that interval. Let $F(x)$ be finite p.p., and let $\bar{D}^2 F(x) > 0$ at all points of $a < x < b$ with the possible exception of the points of an enumerable set E at which*

$$\overline{\lim}\, \Delta^2 F(x, h)/h \geqslant 0 \geqslant \underline{\lim}\, \Delta^2 F(x, h)/h.$$

Then $F(x)$ is continuous and convex in $a < x < b$.

We first show that $F(x)$ is finite in (a, b). If not, there are two possibilities.

(i) There exist two points α and $\beta > \alpha$ such that $F(x) = -\infty$, $F(\beta) = -\infty$. Since $F(x)$ is u.s.c., $\lim_{x \to \alpha} F(x) = -\infty$. Since, however, $F(x)$ is finite p.p., given $N > 0$, we can find a $\gamma < \alpha$ such that $F(\gamma)$ is finite and $F(\gamma) < -N$. Similarly, we can find a $\delta > \beta$ such that $F(\delta)$ is finite and $F(\delta) < -N$. By II, Lemma 14, we have $F(x) < -N$ in $(\alpha\beta)$, and, since N is arbitrary, $F(x) = -\infty$ in an interval, contrary to hypothesis.

(ii) There is only one point α such that $F(\alpha) = -\infty$. Then $F(x)$ being finite and u.s.c. in $a < x < b$, it is convex in that interval by II, Lemma 14, and hence $F(\alpha+0)$ is finite or $+\infty$. For a similar reason, $F(\alpha-0)$ is finite or $+\infty$. Since $F(\alpha) = -\infty$, this contradicts the fact that $F(x)$ is u.s.c. at α.

Thus $F(x)$ is finite in (a, b). By Lemma 14, the result follows.

Definition 1. A function $f(x)$, which at every point of the interval $a \leqslant x \leqslant b$ has a unique value, will be said to possess the property A in that

interval if it satisfies the following two conditions:

(i) If $a \leqslant x < b$ and $f(x+0)$ has a unique value, then $f(x) = f(x+0)$.

(ii) If $a < x \leqslant b$ and $f(x-0)$ has a unique value, then $f(x) = f(x-0)$.

A function will be said to possess the property A in an open interval if it possesses this property in every closed interval contained in the open interval.

Definition 2. A function $f(x)$, which at every point of the interval $a \leqslant x \leqslant b$ has a unique value, will be said to possess the property of Darboux in that interval if, given two points α and $\beta > \alpha$, $f(x)$ takes, in (α, β), every value which lies between $f(\alpha)$ and $f(\beta)$.

We extend the definition to an open interval as before. A function which possesses the property of Darboux in an interval possesses the property A in that interval.

Definition 3. A function $f(x)$, which at every point of the interval $a \leqslant x \leqslant b$ has a unique value, will be said to possess the property R^* in that interval if, given any perfect set in (a, b), there is a portion of the set on which $f(x)$ is u.s.c.

We extend the definition to an open interval as before. If, in Definition 3, we replace u.s.c. by "continuous", we obtain the definition of the property R.

LEMMA 5. *Let $F(x) < \infty$ be defined in $a \leqslant x \leqslant b$, and possess the property A and the property R^* in that interval. Let $F(x)$ be finite p.p. and let $\bar{D}^2 F(x) > 0$ p.p. Let $\underline{D}^2 F(x) > -\infty$ in that interval with the possible exception of an enumerable set E at which*

$$\overline{\lim}\, \Delta^2 F(x, h)/h \geqslant 0 \geqslant \underline{\lim}\, \Delta^2 F(x, h)/h. \tag{9}$$

Then $F(x)$ is continuous and convex in (a, b).

If S is the set of zero measure at which $\bar{D}^2 F(x) \leqslant 0$, then we can find an increasing absolutely continuous function $g(x)$ such that $g'(x) = +\infty$ for $x \subset S$, while $0 < g(x) < \epsilon$ for an assigned $\epsilon > 0$. Writing

$$G(x) = \int_a^x g(t)\, dt,$$

and $H(x) = F(x) + G(x)$, we have $\bar{D}^2 H(x) > 0$ for all $x \subset CE$, while $H(x)$ possesses the other properties of $F(x)$. If, then, we show that $H(x)$ is continuous and convex, the lemma will follow, since $H(x) \to F(x)$ uniformly as $\epsilon \to 0$. We may therefore suppose that $\bar{D}^2 F(x) > 0$, except at the points of E at which (9) holds.

By the definition of the property R^*, there is a non-dense closed set F_1 such that, if δ denote a contiguous open interval of F_1, then $F(x)$ is u.s.c. in δ. By Lemma 4, $F(x)$ is continuous and convex in δ. Let ξ denote an isolated point of F_1. Then $F(\xi+0)$, $F(\xi-0)$ are each finite or $+\infty$; for, since $F(\xi)<\infty$, it follows that, unless $F(\xi+0)$ and $F(\xi-0)$ are finite and equal both to each other and to $F(\xi)$, $F(x)$ does not possess the property A. It follows that $F(x)$ is continuous and convex in the open intervals contiguous to the derived set of F_1. By repeating the argument, we infer that $F(x)$ is continuous and convex in the open intervals contiguous to P_1, the perfect kernel of F_1. Let $(\alpha\beta)$ denote such a contiguous interval. Then $F(\beta-0)$ is finite or $+\infty$. Since $F(\beta)<\infty$, the property A requires that $F(\beta-0)$ is finite, and that $F(\beta)=F(\beta-0)$. Similarly for α. Thus $F(x)$ is continuous and convex in the closed contiguous intervals of P_1.

By the definition of the property R^*, there is a closed set $F_2 \subset P_1$, and non-dense in P_1, such that, if δ denotes a contiguous interval of F_2, and d a closed interval in δ, then $F(x)$ is u.s.c. on $P_1 d$. Now $F(x)$, being convex in the closed contiguous intervals of $P_1 d$, attains its maximum in such an interval at an end point, *i.e.* a point of $P_1 d$. It follows that $F(x)$ is u.s.c. in d, and hence, by Lemma 4, convex and continuous in d. Hence it possesses this property in the open interval δ. By repeating the preceding argument, we infer that $F(x)$ is continuous and convex in each of the closed contiguous intervals of P_2, the perfect kernel of F_2. By transfinite induction, we manifestly establish the desired result.

Uniqueness theorems.

4. LEMMA 6. *Let* $-\Sigma A_n(x)/n^2$ *possess the property* F. *Let* $\underline{P}(x)$ *be finite in* $a \leqslant x \leqslant b$, *except possibly at the points of an enumerable set* E *at which*

$$\lim (1-r)\, P(r, x) = 0. \tag{10}$$

Then, (A) $$F(x) = -\Sigma A_n(x)/n^2 \quad (P) \quad (a \leqslant x \leqslant b)$$

is unique, satisfies $F(x)<\infty$, *and possesses the property* R^* *in* (a, b).

(B) *If further,* $F(x)$ *possesses the property* A *in* (a, b), *and* $\underline{P}(x) \geqslant \phi(x)$, *where* $\phi(x)$ *is integrable* D (*i.e. in the sense of Denjoy-Perron*) *in* (a, b), *then* $F(x)$ *is continuous in* (a, b), *and* $D^2 F(x)$ *exists p.p. in that interval.*

Proof of (A). For $x \subset E$,

$$P(r, x) = o\left(\frac{1}{1-r}\right), \quad \Sigma \frac{A_n(x)}{n} r^n = o\left(\log \frac{1}{1-r}\right),$$

and so $F(x)$ is finite. The rest of (A) is established by II, Lemma 12.

Proof of (B). We may suppose that $\phi(x)$ is finite except at the points of E, at which $\phi(x) = -\infty$. Let $m(x)$ be a minor function of $\phi(x)$ in (a, b), and let

$$\mu(x) = \int_a^x m(t)\,dt.$$

Write $G(x) = F(x) - \mu(x)$. By the definition of the property R^*, there is a non-dense closed set F_1 such that, if δ is a contiguous interval of F_1, and d is a closed interval interior to δ, then $F(x)$ is u.s.c. in d; and, since $F(x) < \infty$, it follows that $F(x)$ is bounded above in d. By Lemma 1, it is finite p.p. in d. By Lemma 2 (B),

$$\bar{D}^2 F(x) \geqslant \underline{P}(x) \geqslant \phi(x) \tag{11}$$

at any point x of d at which $F(x)$ is finite. And $\bar{D}^2 F(x) \geqslant \phi(x)$ must hold at a point at which $F(x)$ is not finite, *i.e.* $F(x) = -\infty$, since then $\bar{D}^2 F(x) = +\infty$. Thus the last inequality holds for all x in d. For x not belonging to E, we have $\bar{D}^2 \mu(x) \leqslant \phi(x)$. Hence

$$\bar{D}^2 G(x) \geqslant 0 \quad (x \subset d.\, CE).$$

For $x \subset Ed$, $F(x)$ is finite, and, by Lemma 2 (A),

$$\overline{\lim}\, \Delta^2 F(x, h)/h \geqslant 0 \geqslant \underline{\lim}\, \Delta^2 F(x, h)/h.$$

Hence

$$\overline{\lim}\, \Delta^2 G(x, h)/h \geqslant 0 \geqslant \underline{\lim}\, \Delta^2 G(x, h)/h.$$

By Lemma 4, $G(x)$ is continuous and convex in d, and hence in δ. By the argument of Lemma 5, we can infer that $G(x)$ is continuous and convex in (a, b). Hence $F(x)$ is continuous in (a, b). Further, $D^2 G(x)$ exists p.p. Since $D^2 \mu(x)$ exists p.p., so does $D^2 F(x)$.

Theorem I. *Let* $-\Sigma A_n(x)/n^2$ *possess the property* F. *Let* $P(x)$ *be finite except at the enumerable set* E *at which* (10) *holds. Let* $\bar{P}(x) \geqslant \psi(x)$, *where* $\psi(x)$ *is integrable* D. *Let*

$$F(x) = -\Sigma A_n(x)/n^2 \quad (P)$$

(*which is unique by Lemma* 6) *possess the property* A. *Then* $\Sigma A_n(x)$ *is a Fourier series.*

We may suppose that $\psi(x)$ is finite except at the points of E. By II, Lemma 8, there is a non-dense closed set F_1 with the following property. If δ denotes a contiguous interval of F_1, and d denotes a closed interval interior to δ, there is a $K = K(d)$ such that, for $x \subset d$, $\underline{P}(x) \geqslant K$. By Lemma 6,

$D^2F(x)$ exists p.p. in d, and therefore p.p. in δ. By the same lemma, $F(x)$ is continuous in d. By Lemma 3, it follows that $\underline{P}(x) = \bar{P}(x) \geqslant \psi(x)$ p.p. in d, and hence p.p. in δ. Thus p.p. in CF_1, $\underline{P}(x) \geqslant \psi(x)$, while in CF_1, $\underline{P}(x)$ is finite except possibly at the enumerable set E at which (10) holds. Let Π_1 be the perfect kernel of F_1. Since $F_1 - \Pi_1$ is enumerable, it follows that, in $C\Pi_1$, $\underline{P}(x)$ has the properties just mentioned.

By II, Lemma 8, there is a closed set $F_2 \subset \Pi_1$, and non-dense in Π_1, with the following property. If δ denotes a contiguous interval of F_2, and d is a closed interval interior to δ, there is a $K = K(d)$ such that $\underline{P}(x) \geqslant K$ for $x \subset \Pi_1 d$. We thus have, in the interval d,

$$\underline{P}(x) \geqslant K \qquad (x \subset \Pi_1 d),$$

$$\underline{P}(x) \geqslant \psi(x) \quad (\text{p.p. for } x \subset C.\Pi_1 d),$$

while $\underline{P}(x)$ is finite except at the enumerable set at which (10) holds. By Lemma 6, $F(x)$ is continuous in d, and $D^2F(x)$ exists p.p. in d. The argument used above may then be repeated, and we infer eventually that $F(x)$ is continuous, and, further, that $\underline{P}(x) \geqslant \psi(x)$ p.p. Writing

$$G(x) = \Sigma \frac{A_n(x)}{n^4} \quad (P),$$

where $G(x)$ is continuous, and the series is the Fourier series of $G(x)$, we have by I, Theorem II,

$$\bar{D}^2 G(x) \geqslant F(x) \geqslant \underline{D}^2 G(x),$$

and so by II, Lemma 5, $G(x)$ is the integral of the integral of $F(x)$. Hence $-\Sigma A_n(x)/n^2$ is the Fourier series of the continuous function $F(x)$, so that the theorem reduces to II, Theorem I.

5. We write $$a_{-n} = a_n, \quad b_{-n} = -b_n \quad (n = 1, 2, \ldots)$$

and $$c_n = \tfrac{1}{2}(a_n - ib_n) \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

THEOREM II. *Let*

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{1}^{n} c_m e^{imx} = 0$$

for all x. Let $\bar{P}(x) \geqslant \psi(x)$, where $\psi(x)$ is integrable D. Let $\underline{P}(x)$ be finite except at an enumerable set E at which (10) *holds. Then $\Sigma A_n(x)$ is a Fourier series.*

Lemma 7. *Under the conditions of the theorem,*

$$\mathfrak{F}(x) = \Sigma\, B_n(x)/n^3 \tag{12}$$

converges for all x.

Let $$\sigma_n(x) = \sum_1^n B_m(x),$$

so that, for an assigned x, $\sigma_n(x) = o(n^2)$. Then

$$\begin{aligned}\sum_p^q B_n(x)/n^3 &= \sum_p^q \frac{\sigma_n(x)-\sigma_{n-1}(x)}{n^3}\\ &= -\frac{\sigma_{p-1}(x)}{p^3} + \sum_{n=p}^{q-1} \sigma_n(x)\left(\frac{1}{n^3}-\frac{1}{(n+1)^3}\right) + \frac{\sigma_q(x)}{q^3}\\ &= o\left(\frac{1}{p}\right) + o\left(\sum_p^{q-1}\frac{1}{n^2}\right) + o\left(\frac{1}{q}\right),\end{aligned}$$

whence the result.

In order to deduce Theorem II from Theorem I, we have to show that $F(x)$, which is unique as in Theorem I, possesses the property A. The method by which we do this is as follows. We show that $\mathfrak{F}(x)$ is approximately continuous, that it has the property R^*, and that $F(x)$ is the approximate derivative of $\mathfrak{F}(x)$. From these facts we can infer that $F(x)$ possesses the property of Darboux, and, *a fortiori*, the property A, in virtue of the following result (V, Theorem IV), which we state as a lemma.

Lemma 8. *If* $\mathfrak{F}(x)$ *is defined in the open interval* I, *and* (i) $\mathfrak{F}(x)$ *is approximately continuous*, (ii) $\mathfrak{F}(x)$ *has the property* R^*, (iii) $\mathfrak{F}_a(x) = F(x)$ *is unique, then, given two points of* I, α *and* $\beta > \alpha$, *there is a* ξ, *satisfying* $\alpha < \xi < \beta$, *such that* $\mathfrak{F}(\beta)-\mathfrak{F}(\alpha) = (\beta-\alpha)\,F(\xi)$.

We now observe that in order to prove that $F(x)$ possesses the property of Darboux, we may suppose, without loss of generality, that $F(\alpha) > 0$, $F(\beta) < 0$; and we must show that there is a ξ such that $F(\xi) = 0$. Since $F(\alpha)$ is the approximate derivative of $\mathfrak{F}(x)$ at α, we can find an arbitrarily small $h > 0$ such that $\mathfrak{F}(\alpha+h)-\mathfrak{F}(\alpha) > 0$. Further, we can find $k > 0$ such that $\alpha+h < \beta-k$, and such that $\mathfrak{F}(\beta)-\mathfrak{F}(\beta-k) < 0$. Hence, since the approximately continuous function $\mathfrak{F}(x)$ possesses the property of Darboux [*cf.* Denjoy, *Bull. Soc. Math. de France*, 43 (1915), 161–248 (179)], we can find two points x_1, x_2, satisfying $\alpha \leqslant x_1 < x_2 \leqslant \beta$, such that $\mathfrak{F}(x_2)-\mathfrak{F}(x_1) = 0$. The result then follows from Lemma 8.

It remains now to prove the two propositions: (A) $\mathfrak{F}(x)$ is approximately continuous and possesses the property R^*: (B) $F(x)$ is the approximate derivative of $\mathfrak{F}(x)$.

Proof of (A):

LEMMA 9. (V, Lemma 19.) *If* $\alpha_n = o(1)$, $\beta_n = o(1)$, *then*

$$\Phi(x) \sim -\Sigma(\beta_n \cos nx - \alpha_n \sin nx)/n$$

is approximately continuous at a point at which it is convergent.

Since the coefficients in (12) are $o(1/n)$, it follows by Lemma 9 that $\mathfrak{F}(x)$ is approximately continuous. To prove that $\mathfrak{F}(x)$ has the property R, and *a fortiori* the property R^*, we observe that there is a finite positive function $M(x)$ such that, for each x,

$$\left| \sum_1^n B_m(x) \right| < n^2 M(x) \quad (n = 1, 2, \ldots).$$

Then the proof of Lemma 7 gives

$$\left| \sum_{p+1}^{\infty} \frac{B_n(x)}{n^3} \right| < M(x) \frac{4}{p}.$$

Write $$\mathfrak{F}_p(x) = \sum_1^p B_m(x)/m^3.$$

Then $\mathfrak{F}_p(x)$ is continuous, $\mathfrak{F}_p(x) \to \mathfrak{F}(x)$ as $p \to \infty$, and

$$|\mathfrak{F}_p(x) - \mathfrak{F}_q(x)| < M(x)\left(\frac{4}{p} + \frac{4}{q}\right).$$

Let E_m denote the closed set of points in $(0, 2\pi)$ at which

$$|\mathfrak{F}_p(x) - \mathfrak{F}_q(x)| \leqslant \frac{8m}{p} \quad (q = p+1, p+2, \ldots),$$

where m is a positive integer. Since $M(x)$ is finite, $\Sigma E_m = (0, 2\pi)$, and there is uniform convergence in each E_m. Given any perfect set $P \subset (0, 2\pi)$, there is an m such that E_m contains a portion ϖ of P. On this portion there is uniform convergence. Hence $\mathfrak{F}(x)$ is continuous on ϖ, and thus possesses the property R.

Proof of (B).

LEMMA 10. *The series* $-\Sigma A_n(x)/n^2$ *is either convergent or properly divergent to* $-\infty$; *i.e.*

$$\lim_{m \to \infty} \sum_1^m A_n(x)/n^2$$

is finite or $+\infty$.

We write $$r = e^{-\xi} \quad (0 < r < 1)$$

so that $\xi \to 0$ as $r \to 1$. We define the integer $N = N(\xi)$ by $N\xi \leqslant 1 < (N+1)\xi$. Then

$$\overline{\lim_{\xi \to 0}} \left| \sum_1^N \frac{A_n(x)}{n^2} - \sum_1^\infty \frac{A_n(x)}{n^2} e^{-n\xi} \right|$$

$$\leqslant \overline{\lim} \left| \sum_{M+1}^N \frac{A_n(x)}{n^2} (1 - e^{-n\xi}) \right| + \overline{\lim} \sum_{N+1}^\infty \left| \frac{A_n(x)}{n^2} e^{-n\xi} \right| = J_1 + J_2,$$

where M is an arbitrarily assigned integer. Choose $M = M(\epsilon, x)$ so that

$$\left| \frac{1}{n^2} \sum_1^n A_m(x) \right| < \epsilon \quad (n \geqslant M),$$

and write $$s_n = s_n(x) = \sum_1^n A_m(x).$$

Then

$$J_1 = \overline{\lim} \left| \sum_{M+1}^N (s_n - s_{n-1}) \frac{1 - e^{-n\xi}}{n^2} \right|$$

$$\leqslant \overline{\lim} \left| s_M \frac{1 - e^{-M\xi}}{(M+1)^2} \right| + \overline{\lim} \left| \sum_{M+1}^{N-1} s_n \left\{ \frac{1 - e^{-n\xi}}{n^2} - \frac{1 - e^{-(n+1)\xi}}{(n+1)^2} \right\} \right|$$

$$+ \overline{\lim} \left| s_N \frac{1 - e^{-N\xi}}{N^2} \right|$$

$$\leqslant \epsilon \, \overline{\lim} \sum_{M+1}^{N-1} n^2 \left| \frac{e^{-(n+1)\xi} - e^{-n\xi}}{n^2} + (1 - e^{-(n+1)\xi}) \left(\frac{1}{n^2} - \frac{1}{(n+1)^2} \right) \right|$$

$$\leqslant \epsilon \, \overline{\lim} \sum_{M+1}^{N-1} n^2 \left(\frac{\xi}{n^2} + \frac{2(n+1)\xi}{n^3} \right) \leqslant 5\epsilon.$$

And

$$J_2 \leqslant \overline{\lim} \left| \sum_{N+1}^\infty (s_n - s_{n-1}) \frac{e^{-n\xi}}{n^2} \right|$$

$$\leqslant \overline{\lim} \left| \frac{s_N e^{-N\xi}}{N^2} \right| + \overline{\lim} \sum_{N+1}^\infty \left| s_n \left(\frac{e^{-n\xi}}{n^2} - \frac{e^{-(n+1)\xi}}{(n+1)^2} \right) \right|$$

$$\leqslant \epsilon \, \overline{\lim} \sum_{N+1}^\infty n^2 \left(\frac{e^{-n\xi}}{n^2} - \frac{e^{-(n+1)\xi}}{(n+1)^2} \right)$$

$$\leqslant \epsilon \, \overline{\lim} \sum_{N+1}^\infty n^2 \left(\frac{e^{-n\xi} - e^{-(n+1)\xi}}{n^2} + \frac{2e^{-(n+1)\xi}}{n^3} \right)$$

$$\leqslant \epsilon \, \overline{\lim} \sum_{N+1}^\infty 3\xi e^{-n\xi} \leqslant 3\epsilon.$$

Thus $$\overline{\lim_{r\to 1}}\left|\sum_1^N \frac{A_n(x)}{n^2}-\sum_1^\infty \frac{A_n(x)}{n^2}r^n\right|=0 \quad \left(N=\left[\frac{1}{\log 1/r}\right]\right),$$

and since, as $r\to 1$, the sum of the infinite series tends to a finite limit or to $+\infty$, the lemma follows.

We can now write

$$F(x)=-\sum_1^\infty \frac{A_n(x)}{n^2},$$

where $F(x)<\infty$; *i.e.* we can omit the sign (P) in the original definition of $F(x)$.

LEMMA 11. *The function* $\mathfrak{F}(x)$ *has the approximate derivative* $\mathfrak{F}_a(x)=F(x)$.

This is V, Lemma 25, with a different notation.

This completes the proof of Theorem II.

Theorems on the Cesàro-summation of differentiated series.

6. The symbolism

$$f(x)\sim \tfrac{1}{2}a_0+\sum_1^\infty (a_n\cos nx+b_n\sin nx)$$

means that the series on the right of $\sim$ is the Fourier series of the function on the left. It is sometimes convenient to express $f(x)$ as the sum of a simple periodic function and of a Fourier series without a constant term. We shall then write

$$f(x)\sim \tfrac{1}{2}A_0(x)+\sum_1^\infty A_n(x).$$

Here $A_n(x)$ $(n=1, 2, \ldots)$ has the same meaning as before; $\frac{1}{2}A_0(x)$ is a periodic function of x, and the preceding symbol $\sim$ means that $\sum_1^\infty A_n(x)$ is the Fourier series of $f(x)-\frac{1}{2}A_0(x)$. We shall be concerned only with the case in which $A_0(x)$ is equal to an algebraic polynomial in aninterval (a, b) of length less than 2π.

Let $$\gamma_p(t)=\int_0^1 (1-u)^{p-1}\cos tu\,du \quad (p>0). \tag{13}$$

We find on successive integration by parts that, when the integer k is odd,

$$\gamma_{1+k}(t)=\sum_{r=1}^{\frac{1}{2}(k+1)} A_r t^{-2r}+B_k\cos t/t^{1+k}; \tag{14}$$

and that, when k is even,

$$\gamma_{1+k}(t) = \sum_{r=1}^{\frac{1}{2}k} A_r' t^{-2r} + B_k \sin t/t^{1+k}; \tag{15}$$

the A's and B's being independent of t. When $k = 0$, the sum in (15) is to be omitted.

Theorem III. *Let*

$$f(x) \sim \tfrac{1}{2}A_0(x) + \sum_1^\infty A_n(x), \tag{16}$$

where

(i) $$a_n = o\left(\frac{1}{n}\right), \quad b_n = o\left(\frac{1}{n}\right);$$

(ii) $$\int_0^{2\pi} A_0(x)\, e^{-inx}\, dx = o\left(\frac{1}{n}\right);$$

(iii) $A_0(x)$ *is equal to an algebraic polynomial in* $a < x < b$, $(b-a < 2\pi)$.

Let the series (16) *converge to zero in* $a < x < b$. *Then, if* k *is zero or a positive integer, the series obtained from* (16) *by* $k+1$ *formal differentiations is uniformly summable* (C, k) *to zero in every closed interval interior to* (a, b).

It is sufficient to prove uniform summability in the centre third of (a, b). We may suppose that $(a, b) \equiv (-3a, 3a)$. Let x_0 denote a variable confined to the interval $(-a, a)$. Then

$$f(x_0+t) = 0 \quad (-2a \leqslant t \leqslant 2a). \tag{17}$$

Let ω denote a positive integer. The ω-th Rieszian mean of index $k > 0$ of (16) at $x = x_0$ is†

$$\frac{\omega}{\pi} \int_{-\infty}^{\infty} f(t)\, \gamma_{1+k}\, \omega(t-x_0)\, dt.$$

When $k > 0$, it is sufficient to prove that

$$\frac{d^{k+1}}{dx_0^{\,k+1}}\, \omega \int_{-\infty}^{\infty} f(t)\, \gamma_{1+k}\, \omega(t-x_0)\, dt$$

tends to zero uniformly in x_0; *i.e.* that

$$\omega^{k+2} \int_{-\infty}^{\infty} f(x_0+t)\, \gamma_{1+k}^{(1+k)}(\omega t)\, dt \tag{18}$$

tends to zero uniformly in x_0. There are two cases.

† We suppose that, in (16), $\frac{1}{2}A_0(x)$ is replaced by its Fourier series, which is then added to $\sum_1^\infty A_n(x)$.

(a) If k is odd, we differentiate (14) $k+1$ times. We obtain numerical multiples of

$$\cos t/t^{k+1}, \quad \sin t/t^{k+2}, \tag{19}$$

and terms having in their denominators powers of t higher than t^{k+2}. These latter terms give rise to contributions in (18), which, in virtue of (17), tend uniformly to zero in x_0 as $\omega \to \infty$. The contribution of the second term in (19) is

$$\int_{-\infty}^{\infty} f(x_0+t) \frac{\sin \omega t}{t^{k+2}}\, dt = -\int_{-\pi}^{\pi} f(x_0+t)\, \frac{1}{(k+1)!} \left[\frac{d^k}{dt^k} \frac{1}{(2 \sin \tfrac{1}{2}t)^2}\right] \sin \omega t\, dt. \tag{20}$$

In virtue of (17), we can apply the lemma of Riemann-Lebesgue, which tells us that (20) tends uniformly to zero in x_0 as $\omega \to \infty$.

Finally, the contribution of the first term in (19) is

$$\omega \int_{-\infty}^{\infty} f(x_0+t) \frac{\cos \omega t}{t^{k+1}}\, dt = \frac{\omega}{k!} \int_{-\pi}^{\pi} f(x_0+t)\, D^{k-1} \frac{1}{(2 \sin \tfrac{1}{2}t)^2} \cos \omega t\, dt. \tag{21}$$

Since $k-1$ is even, the function

$$D^{k-1} \frac{1}{(2 \sin \tfrac{1}{2}t)^2} \tag{22}$$

is even, and has the same value at $t = \pi$ as at $t = -\pi$. By (17), we may replace the function (22) in the second member of (21) by an even function $\phi(t)$ such that

$$\phi(t) = D^{k-1} \frac{1}{(2 \sin \tfrac{1}{2}t)^2} \quad (-\pi \leqslant t \leqslant -2\alpha, \ \ 2\alpha \leqslant t \leqslant \pi),$$

and such that $\phi(t)$ has two continuous derivatives in $-\pi \leqslant t \leqslant \pi$. Since $\phi(-\pi) = \phi(\pi)$, the Fourier constants of $\phi(t)$ are $o(1/n^2)$, and if $s_n(t)$ denotes the n-th partial sum of the Fourier series of $\phi(t)$, then

$$|\phi(t) - s_n(t)| = o\left(\frac{1}{n}\right). \tag{23}$$

Let $n = [\frac{1}{2}\omega]$. Then we have to consider

$$\omega \int_{-\pi}^{\pi} f(x_0+t)\, \phi(t) \cos \omega t\, dt$$

$$= \omega \int_{-\pi}^{\pi} f(x_0+t) \{\phi(t) - s_n(t)\} \cos \omega t\, dt + \omega \int_{-\pi}^{\pi} f(x_0+t)\, s_n(t) \cos \omega t\, dt.$$

The first term of the second member is $o(1)$ by (23). The second term is evaluated as follows. We have

$$\phi(t) = \sum_0^{\infty} c_n \cos nt \quad \left(\sum_0^{\infty} |c_n| < C\right), \tag{24}$$

so that

$$\omega \int_{-\pi}^{\pi} f(x_0+t)\, s_n(t) \cos \omega t\, dt = \sum_0^n c_\nu \omega \int_{-\pi}^{\pi} f(x_0+t) \cos \nu t \cos \omega t\, dt. \quad (25)$$

Since $\nu < \frac{1}{2}\omega$,

$$\omega \int_{-\pi}^{\pi} f(x_0+t) \cos \nu t \cos \omega t\, dt = \frac{\omega}{2} \int_{-\pi}^{\pi} f(x_0+t)[\cos(\omega+\nu)t + \cos(\omega-\nu)t]\, dt$$
$$= o(1)$$

by (i) and (ii), uniformly in ν and x_0 as $\omega \to \infty$, so that, by (24), the expression (25) tends to zero with the required uniformity.

(*b*) If $k > 0$ is even, we differentiate (15) $k+1$ times. By the same discussion as before, the only term to consider is (21). Since $k-1$ is odd, (22) is an odd function which is zero at π and $-\pi$. We may now take $\phi(t)$ to be odd.

When $k = 0$, we use Dirichlet's integral, and we have to consider

$$-\frac{1}{\pi} \int_{-\pi}^{\pi} f(x_0+t) \frac{d}{dt} \left(\frac{\sin(\omega+\frac{1}{2})t}{2 \sin \frac{1}{2}t} \right) dt$$
$$= -\frac{1}{\pi} \int_{-\pi}^{\pi} f(x_0+t) \left[-\frac{\omega}{2} \sin \omega t - \frac{\sin \omega t}{(2 \sin \frac{1}{2}t)^2} + \omega(\tfrac{1}{2} \cot \tfrac{1}{2}t) \cos \omega t \right] dt.$$

Of these three terms, the first is $o(1)$ by (i), the second is $o(1)$ as in the case of (20), and the third is $o(1)$ as in the case of (21).

COROLLARY. *In the preceding theorem, condition* (ii) *may be omitted.*

Suppose that $A_0(x)$ satisfies (iii) but not (ii). We can find a periodic function $B_0(x)$ which is equal to $A_0(x)$ in $a < x < b$, and which satisfies

$$\int_0^{2\pi} B_0(x)\, e^{-inx}\, dx = o\left(\frac{1}{n}\right).$$

Then $\frac{1}{2}B_0(x) + \Sigma A_n(x)$ satisfies the conditions of Theorem III, whence the result.

7. An easy consequence is

THEOREM IV. *Let*

$$f(x) \sim \tfrac{1}{2}A_0(x) + \sum_1^\infty A_n(x) \quad (26)$$

satisfy the conditions (i) *that* $a_n = o(n^{-\rho})$, $b_n = o(n^{-\rho})$; (ii) *that* $A_0(x)$ *is equal*

to an algebraic polynomial in $a<x<b$, $(b-a<2\pi)$, *where* ρ *is a positive integer. Let* k *be zero or a positive integer. Then the series obtained from* (26) *by* $k+\rho$ *formal differentiations is uniformly summable* (C, k) *to zero in any interval completely interior to* (a, b).

For $\rho=1$, this is the corollary to Theorem III. For $\rho>1$, we have, on writing $A(r, x)$ for the Poisson function of $\frac{1}{2}A_0(x)$,

$$A(r, x)+\sum_1^\infty A_n(x)\,r^n=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\frac{1-r^2}{\delta'}\,dt$$

$$[\delta'=1-2r\cos(t-x)+r^2];$$

and if we differentiate $\rho-1$ times with respect to x, and write

$$A_n^{(\rho-1)}(x)=\frac{d^{\rho-1}}{dx^{\rho-1}}\,A_n(x),$$

we get

$$A^{(\rho-1)}(r, x)+\sum_1^\infty A_n^{(\rho-1)}(x)\,r^n=\frac{(-1)^{\rho-1}}{2\pi}\int_{-\pi}^{\pi} f(t+x)\,D^{(\rho-1)}\left(\frac{1-r^2}{\delta}\right)dt,$$

where $\delta=1-2r\cos t+r^2$.

Since $f(t)=0$ in (a, b), we see that

$$\tfrac{1}{2}A_0^{(\rho-1)}(x)+\sum_1^\infty A_n^{(\rho-1)}(x) \tag{27}$$

is summable to zero by Poisson's method, for $a<x<b$; for we can clearly replace the series $A^{(\rho-1)}(r, x)$ by its limit as $r\to 1$, namely $\frac{1}{2}A_0^{(\rho-1)}(x)$. Since the coefficients of (27) are $o(1/n)$, it converges to zero in that interval. The result now follows on applying the corollary, Theorem III, to the series (27).

As an example of Theorem IV, let $\rho=2$. Let

$$\sum_1^\infty A_n(x) \quad [a_n=o(1),\ b_n=o(1)] \tag{28}$$

be a trigonometric series. Write

$$F(x)=-\Sigma\,\frac{A_n(x)}{n^2}.$$

Let

$$\Phi(x)=-\Sigma\,\frac{A_n'(x)}{n^2} \quad [a_n'=o(1),\ b_n'=o(1)]$$

be any trigonometric series such that $F(x)=\Phi(x)$ in $a<x<b$, $(b-a<2\pi)$. By Theorem IV,

$$\Sigma\{A_n(x)-A_n'(x)\}$$

converges to zero uniformly in any interval interior to (a, b). This is the theorem of Riemann which is expressed by saying that the behaviour of the n-th partial sum of (28) at a point (in an interval), as $n \to \infty$, depends only on the values of $F(x)$ in the neighbourhood of the point (interval). As is well known, this result is usually deduced from an asymptotic integral representation of the n-th partial sum of (28), in which only the values of $F(x)$ in a neighbourhood of the point (interval) in question occur. If it be desired to obtain such an integral representation, we can proceed as follows. We may suppose that the interval (a, b), mentioned above, lies within $(-\pi, \pi)$. The function $\Phi(x)$ is any function which satisfies the two conditions, (a) $\Phi(x) = F(x)$ in (a, b); (b) the Fourier constants of $\Phi(x)$ are $o(1/n^2)$. A simple class of such functions is the class defined by

$$\Phi(x) = F(x)\,\phi(x),$$

where

(α) $\phi(x) = 1\ (a < x < b)$; $\phi(x) = 0\ (-\pi \leqslant x \leqslant \alpha;\ \beta \leqslant x \leqslant \pi;$

$$-\pi < \alpha < a < b < \beta < \pi),$$

(β) the Fourier constants of ϕ are $o(1/n^3)$.

To prove this, we must verify that

$$\int_{-\pi}^{\pi} F\phi e^{-int}\,dt = o\left(\frac{1}{n^2}\right).$$

This is done by an argument used at the end of (a), Theorem III. If $s_n(t)$ is the n-th partial sum of the Fourier series of $\phi(t)$, then

$$|\phi(t) - s_n(t)| = o\left(\frac{1}{n^2}\right),$$

and, taking $n = [\frac{1}{2}\omega]$, we have

$$\int_{-\pi}^{\pi} F\phi e^{-i\omega t}\,dt = \int_{-\pi}^{\pi} F(\phi - s_n)\,e^{-i\omega t}\,dt + \int_{-\pi}^{\pi} F s_n\,e^{-i\omega t}\,dt.$$

The result follows at once. Manifestly,

$$\sum_1^n A_n'(x) = \frac{d^2}{dx^2}\,\frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)\phi(t)\,\frac{\sin(n+\frac{1}{2})(t-x)}{\sin\frac{1}{2}(t-x)}\,dt$$

$$= \frac{1}{2\pi}\int_{\alpha}^{\beta} F(t+x)\,\phi(t+x)\,\frac{d^2}{dt^2}\,\frac{\sin(n+\frac{1}{2})\,t}{\sin\frac{1}{2}t}\,dt.$$

We thus have

THEOREM V (Neder†). *If $\phi(x)$ is defined as above, then*

$$\sum_1^n A_m(x) - \frac{1}{2\pi}\int_\alpha^\beta F(t+x)\,\phi(t+x)\,D^2\left(\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\right)dt$$

converges to zero as $n\to\infty$, uniformly for x in any closed interval interior to (a, b).

It seems necessary to add that, so far as I am aware, the only direct use to which this asymptotic integral representation has been put is the deduction of the theorem of Riemann previously mentioned. The usual method of obtaining this representation cannot, however, be characterised as altogether felicitous.

8. It was proved by M. Riesz‡ that, if

$$\sum_0^\infty c_n z^n \quad [c_n = o(n^\rho), \quad \rho > 0]$$

is convergent within the unit circle, and tends uniformly to zero as z tends radially to a point of the arc $a < \theta < b$, then

$$\sum_0^\infty c_n e^{in\theta}$$

is uniformly summable (C, ρ) to zero in any interval of θ interior to (a, b). The proof of M. Riesz involves Cauchy's theorem and the introduction of a "multiplicator", *i.e.* a function analogous to $\phi(t)$ in the integral of Theorem V.

The theorem of M. Riesz was generalized by Zygmund§, who proved that, if

$$\tfrac{1}{2}a_0 + \sum_1^\infty A_n(x) \quad [a_n = o(n^\rho),\ b_n = o(n^\rho);\ \rho \geqslant 0]$$

is uniformly summable by Poisson's method to zero in $a < x < b$, then it is uniformly summable (C, ρ) in any interval of θ interior to (a, b). The proof of Zygmund depends on an elaborated theory of formal multiplication, and a full proof by that method would be long and complicated. The particular case of Zygmund's theorem when ρ is an integer can be deduced very simply from the corollary to Theorem III.

† Neder, *Math. Annalen*, 84 (1921), 117–136.

‡ M. Riesz, *Journal für Math.*, 140 (1911), 89–99.

§ Zygmund, *Math. Zeitschrift*, 24 (1925), 47–104 (92, 93).

Theorem VI (Zygmund). *If*

$$\tfrac{1}{2}a_0+\sum_1^\infty A_n(x) \quad [a_n=o(n^\rho),\ b_n=o(n^\rho)], \tag{29}$$

where ρ is 0 or a positive integer, is uniformly summable to zero by Poisson's method in $a<x<b$, then it is uniformly summable (C, ρ) in any interval completely interior to (a, b).

In virtue of the uniform summability, we can integrate (29) formally $\rho+1$ times, and infer that

$$P(x)+\frac{a_0x^{\rho+1}}{2(\rho+1)!}+\sum_1^\infty \frac{A_n(x)}{n^{\rho+1}} \quad \begin{cases} A_n(x)=(-1)^{\frac{1}{2}(\rho-1)}A_n(x) & (\rho \text{ odd}) \\ \qquad =(-1)^{\frac{1}{2}(\rho+2)}B_n(x) & (\rho \text{ even}), \end{cases}$$

where $P(x)$ is a polynomial of degree ρ, is summable by Poisson's method to zero in $a<x<b$, and therefore, since its coefficients are $o(1/n)$, converges to zero in that interval. By the corollary to Theorem III, with $k=\rho$, the result follows.

Theorems on the relation between various methods of summation.

9. We require two lemmas.

Lemma 12. *Let the Poisson sums of*

$$\Sigma A_n(x) \quad [a_n=o(n),\ b_n=o(n)] \tag{30}$$

be bounded in an interval (a, b). Then they are equal p.p.; and if

$$\Sigma A_n'(x)$$

is the Fourier series of a function which in (a, b) is equivalent to the Poisson sum of (30), then the series obtained by differentiating

$$\Sigma\{A_n(x)-A_n'(x)\}$$

formally ρ times ($\rho=0, 1, \ldots$) is uniformly summable $(C, \rho+1)$ to zero in any closed interval interior to (a, b).

Let $\bar{P}(x)$, $\underline{P}(x)$ denote the Poisson sums of (30). Then, by II, Lemma 15, $\bar{P}(x)=\underline{P}(x)=f(x)$ say, p.p. in (a, b), and

$$F(x)\sim -\Sigma\frac{A_n(x)}{n^2} \tag{31}$$

has the form

$$F(x)=\int_a^x dy\int_a^y f(t)\,dt+px+q \quad (a<x<b) \tag{32}$$

in (a, b). Further, the series (31) converges to $F(x)$ in (a, b). Moreover,

$$\Phi(x) = -\Sigma \frac{A_n'(x)}{n^2}$$

has the form

$$\Phi(x) = \int_a^x dy \int_a^y f(t)\,dt + Px + Q \quad (a < x < b)$$

in (a, b). Hence there is a linear function $Ax+B$ such that

$$Ax + B - \Sigma \frac{A_n(x) - A_n'(x)}{n^2}$$

converges to zero in (a, b). By the corollary to Theorem III, with $k = 1, 2, \ldots$, the lemma follows.

Let $f(x)$ be defined in a neighbourhood of the point x_0. It may happen that

$$\tfrac{1}{2}[f(x_0+h)+f(x_0-h)] = \sum_{r=0}^{p-1} \frac{A_{2r}}{(2r)!} h^{2r} + [A_{2p}+o(1)] \frac{h^{2p}}{(2p)!}$$

as $h \to 0$, or that

$$\tfrac{1}{2}[f(x_0+h)-f(x_0-h)] = \sum_{r=0}^{p-1} \frac{A_{2r+1}}{(2r+1)!} h^{2r+1} + [A_{2p+1}+o(1)] \frac{h^{2p+1}}{(2p+1)!}$$

as $h \to 0$. Write $A_s = f_s(x_0)$.

LEMMA 13 (Gronwall†). *Let*

$$f(x) \sim \tfrac{1}{2}a_0 + \Sigma A_n(x). \tag{33}$$

If $f_p(x_0)$ exists, then the series obtained by differentiating (33) *formally p times is summable $(C, p+1)$ at x_0 to $f_p(x_0)$.*

The ω-th Rieszian mean of (33) of index $p+1$ at $x = x_0$ is

$$\frac{\omega}{\pi} \int_{-\infty}^{\infty} f(t)\,\gamma_{2+p}\,\omega(t-x_0)\,dt.$$

We must therefore show that

$$\frac{d^p}{dx_0^p} \frac{\omega}{\pi} \int_{-\infty}^{\infty} f(t)\,\gamma_{2+p}\,\omega(t-x_0)\,dt$$

tends to $f_p(x_0)$ as $\omega \to \infty$, or that

$$\frac{\omega^{p+1}}{\pi} \int_0^{\infty} [f(x_0+t)+(-1)^p f(x_0-t)]\,\gamma_{2+p}^{(p)}(\omega t)\,dt$$

† Gronwall, *Journal für Math.*, 147 (1917), 16–35.

tends to $f_p(x_0)$ as $\omega \to \infty$. We may take $x_0 = 0$. If p is even we may suppose that $f(t)$ is even, if p is odd, that $f(t)$ is odd. We must therefore show that

$$\frac{2}{\pi}\,\omega^{p+1}\int_0^\infty f(t)\,\gamma_{2+p}^{(p)}(\omega t)\,dt \tag{34}$$

tends to $f_p(x_0)$ as $\omega \to \infty$. Suppose that $p = 2\nu$. There is an even function $\phi(t)$ such that, in a neighbourhood of the origin,

$$\phi(t) = \sum_{r=0}^{\nu} \frac{A_{2r}}{(2r)!}\,t^{2r},$$

and such that the Fourier constants of $\phi(t)$ are $o(1/n^{p+2})$. By subtracting the Fourier series of ϕ from (33), we see that we may take $f(t) = o(t^p)$ as $t \to 0$, and we must show that (34) tends to zero. We see from (13) and (15) that $\gamma_{2+p}^{(p)}(t)$ is bounded for $t > 0$, and $O(1/t^{p+2})$ as $t \to \infty$. Hence, if $\eta > 0$,

$$\omega^{p+1}\int_0^\eta f(t)\,\gamma_{2+p}^{(p)}(\omega t)\,dt = o\int_0^\infty t^p\,|\,\gamma_{2+p}^{(p)}(t)\,|\,dt = o(1),$$

$$\omega^{p+1}\int_\eta^\infty f(t)\,\gamma_{2+p}^{(p)}(\omega t)\,dt = O\,\frac{1}{\omega}\int_\eta^{\eta+2\pi} |f(t)|\,\left\{\sum_{s=0}^{\infty}(t+2\pi s)^{-p-2}\right\}dt = o(1).$$

Similarly when p is odd.

10. Theorem VII. *Let* $a_n = o(n^2)$. *Let*

$$\lim_{r\to 1}\sum_1^\infty a_n\left(\frac{\sin n\theta}{n\theta}\right)r^n = \phi(\theta),$$

and let $\phi(\theta) \to 0$ *as* $\theta \to 0$. *Then*

$$\sum_1^\infty a_n = 0 \quad (C, 2).$$

The Poisson sum of $\qquad \sum_1^\infty \frac{a_n}{n}\sin n\theta,$

which is equal to $\theta\phi(\theta)$, is bounded in $(-\eta, \eta)$ say. If

$$\sum_1^\infty \frac{a_n}{n}\sin n\theta \quad [a_n = o(n)]$$

be the Fourier series of an odd function which is equal to $\theta\phi(\theta)$ in $(-\eta, \eta)$, then, by Lemma 12,

$$\Sigma(a_n - a_n) = 0 \quad (C, 2),$$

and by Lemma 13, with $p=1$,

$$\Sigma a_n = 0 \quad (C, 2),$$

since $\phi_1(0)=0$. Adding the last two equations, we obtain the desired result.

Let p denote a positive integer. We shall say that the series $\sum_1^\infty a_n$ is summable to s by the method (R, p) if

$$\lim_{r\to 1} \sum_1^\infty a_n \left(\frac{\sin n\theta}{n\theta}\right)^p r^n = \phi(\theta), \quad \lim_{\theta\to 0} \phi(\theta) = s.$$

We have just seen that, when $a_n = o(n^2)$, summability $(R, 1)$ implies summability $(C, 2)$. We further have

THEOREM VIII. *Under the conditions of Theorem VII,*

$$\lim_{r\to 1} \sum_1^\infty a_n \left(\frac{\sin n\theta}{n\theta}\right)^p r^n = \phi_p(\theta)$$

exists† *(for θ in a neighbourhood of the origin), and $\phi_p(\theta)\to 0$ as $\theta\to 0$.*

Thus, when $a_n = o(n^2)$, summability $(R, 1)$ implies summability (R, p) for any integer $p>1$.

We proved in II, Theorem III, that, under the conditions of Theorem VII, with the notation

$$F(\theta) \sim -\Sigma \frac{a_n}{n^3} \sin n\theta, \tag{35}$$

we have, in a neighbourhood $(-\eta, \eta)$ of the origin,

$$F(\theta) = A\theta + \int_0^\theta dy \int_0^y t\phi(t)\, dt. \tag{36}$$

Since $F(\theta)$ is in $(-\eta, \eta)$ the integral of an a.c. (*i.e.* absolutely continuous) function, we can find a series

$$-\Sigma \frac{a_n}{n^3} \sin n\theta \quad [a_n = o(n)] \tag{37}$$

which is the Fourier series of the integral of an a.c. function and which converges to $F(\theta)$ in $(-\eta, \eta)$. Then

$$-\Sigma \frac{a_n - a_n}{n^3} \sin n\theta \tag{38}$$

converges to zero in $(-\eta, \eta)$.

† The proof shows that $\Sigma a_n \left(\frac{\sin n\theta}{n\theta}\right)^p$ converges to $\phi_p(\theta)$.

For $p = 2$, we observe that, since the coefficients in (38) are $o(1/n)$, we can apply Theorem III with $k = 0$, and infer that

$$-\Sigma \frac{a_n - \alpha_n}{n^2} \cos n\theta$$

converges to zero in $(-\eta, \eta)$. Hence

$$\Sigma(a_n - \alpha_n)\left(\frac{\sin n\theta}{n\theta}\right)^2 = \frac{1}{2\theta^2} \Sigma \frac{a_n - \alpha_n}{n^2} (1 - \cos 2n\theta)$$

$$= 0$$

for $-\eta < 2\theta < \eta$. Thus

$$\Sigma(a_n - \alpha_n) = 0 \quad (R, 2). \tag{39}$$

The series (37) being the Fourier series of the integral of an a.c. function, the series

$$-\Sigma \frac{\alpha_n}{n^2} \cos n\theta$$

is the Fourier series of an a.c. function which in $(-\eta, \eta)$ is equal to

$$A + \int_0^\theta t\phi(t)\, dt.$$

Hence

$$\Sigma \alpha_n \left(\frac{\sin n\theta}{n\theta}\right)^2 = \frac{1}{2\theta^2} \Sigma \frac{\alpha_n}{n^2} (1 - \cos 2n\theta)$$

$$= \frac{1}{2\theta^2} \int_0^{2\theta} t\phi(t)\, dt,$$

which tends to 0 with θ. Thus

$$\Sigma \alpha_n = 0 \quad (R, 2). \tag{40}$$

By adding (39) and (40), we obtain the theorem for $p = 2$.

For $p \geqslant 3$, we can integrate (35) term by term $p - 3$ times. If we write

$$\mathfrak{F}(\theta) = \int_0^\theta \ldots \int_0^t t\phi(t)\, dt + P(\theta) \quad (-\eta < \theta < \eta),$$

where there are $p - 1$ signs of integration, and $P(\theta)$ is a polynomial of degree $p - 2$, then this function will be equal to the series obtained by integrating the second member of (35) $p - 3$ times. We then infer that

$$\frac{\Delta^p \mathfrak{F}(0, \theta)}{(2\theta)^p} = \Sigma a_n \left(\frac{\sin n\theta}{n\theta}\right)^p,$$

where we use the notation

$$\Delta f(x, h) = f(x+h) - f(x-h);$$

$$\Delta^p f(x, h) = \Delta . \Delta^{p-1} f(x, h) = \Delta^{p-1} f(x+h, h) - \Delta^{p-1} f(x-h, h);$$

so that, in particular,

$$\Delta^2 f(x, h) = f(x+2h) + f(x-2h) - 2f(x).$$

We have

$$\begin{aligned}
\Delta^p \mathfrak{F}(0, \theta) &= \Delta^{p-1} \mathfrak{F}(\theta, \theta) - \Delta^{p-1} \mathfrak{F}(-\theta, \theta) \\
&= 2\theta \Delta^{p-1} \mathfrak{F}'(\xi, \theta) \qquad (-\theta < \xi < \theta) \\
&= (2\theta)^2 \Delta^{p-2} \mathfrak{F}''(\eta, \theta) \qquad (-2\theta < \eta < 2\theta) \\
&= (2\theta)^{p-2} \Delta^2 \mathfrak{F}^{(p-2)}(\nu, \theta) \quad [-(p-2)\theta < \nu < (p-2)\theta].
\end{aligned}$$

Since $$\mathfrak{F}^{(p-2)}(\theta) = A + \int_0^\theta t\phi(t)\, dt,$$

we have

$$\frac{\Delta^p \mathfrak{F}(0, \theta)}{(2\theta)^p} = \frac{\Delta^2 \mathfrak{F}^{(p-2)}(\nu, \theta)}{(2\theta)^2} = \frac{1}{(2\theta)^2} \int_{\nu-2\theta}^{\nu+2\theta} t\phi(t)\, dt,$$

which tends to zero with θ.

11. LEMMA 14. *Let* $\Sigma A_n(x) \quad [a_n = o(n),\ b_n = o(n)]$

be a trigonometric series, and let $\underline{P}(x) > -K$ *for* $a < x < b$. *Then* $\underline{P}(x)$ *is integrable L in any closed interval interior to* (a, b), *and if*

$$F(x) \sim -\Sigma A_n(x)/n^2,$$

then, for $a < \alpha < \beta < b$,

$$F(x) = \int_\alpha^x dy \int_\alpha^y \underline{P}(t)\, dt + \int_\alpha^x M(y)\, dy + C' \quad (\alpha \leqslant x \leqslant \beta),$$

where $M(x)$ *is a non-decreasing function.*

We observe that $$-\Sigma \frac{A_n(x)}{n^2} \quad (P) \tag{41}$$

has a unique value $F(x)$, for $a < x < b$, which satisfies $F(x) < \infty$ and is finite p.p. Further, $F(x)$ possesses the property R^* by II, Lemma 12, and the symbol (P) may be omitted in (41) since the coefficients are $o(1/n)$.

Write $$\mathfrak{F}(x) = \Sigma \frac{B_n(x)}{n^3}.$$

Then, since the coefficients in this series are $o(1/n^2)$, we have $\Delta^2 \mathfrak{F}(x, h)/h \to 0$. On the other hand,

$$\frac{\Delta \mathfrak{F}(x, h)}{2h} = -\Sigma \frac{A_n(x)}{n^2} \left(\frac{\sin nh}{nh}\right);$$

and this tends, as $h \to 0$, to $F(x)$, whether $\Sigma A_n(x)/n^2$ is convergent or properly divergent. Thus $F(x)$ is the unique differential coefficient of $\mathfrak{F}(x)$, and consequently possesses the property of Darboux, and, *a fortiori*, the property A. Further, by I, Theorem II, $\bar{D}^2 F(x) \geqslant \underline{P}(x) > -K$ at a point x of (a, b) at which $F(x)$ is finite, and $\bar{D}^2 F(x) > -K$ at a point at which $F(x) = -\infty$. By the proof of Lemma 5†, $F(x)$ is continuous in (a, b), and $F(x) + \frac{1}{2}Kx^2$ is convex. It follows by II, Lemma 4, that $D^2 F(x)$ exists p.p. in (a, b), and is integrable L in (α, β); and so, by II, Lemma 6, that $\underline{P}(x)$ is also integrable L in (α, β).

To prove the last part of the lemma, we observe that, since $\underline{P}(x) > K$, it is possible to find a de la Vallée Poussin minor function $m(x)$ such that

$$\overline{\lim} \frac{m(x+h) - m(x)}{h} < P(x) \quad (\alpha \leqslant x \leqslant \beta),$$

and thus $$\bar{D}^2 \left[F(x) - \int_a^x m(t)\, dt \right] > 0 \quad (\alpha \leqslant x \leqslant \beta).$$

Hence $$F(x) - \int_a^x m(t)\, dt$$

is convex in (α, β), and, if we make $m(x)$ tend uniformly to

$$\int_a^x \underline{P}(t)\, dt,$$

it follows that $$F(x) - \int_a^x dy \int_a^y \underline{P}(t)\, dt$$

is convex in (α, β). A function which is continuous and convex in a closed interval is, by II, Lemma 4, the integral of a non-diminishing function. This proves the lemma.

† The proof of Lemma 5 shows that the conclusion of the lemma is true if all the conditions referring to $\bar{D}^2 F(x)$, $\underline{D}^2 F(x)$, $\overline{\lim}\, \Delta^2 F(x, h)/h$, $\underline{\lim}\, \Delta^2 F(x, h)/h$ are replaced by the single condition "$\bar{D}^2 F(x) > 0$ everywhere".

THEOREM IX. *Let* $a_n = o(n^2)$. *Let*

$$\lim_{r\to 1} \sum_1^\infty a_n \left(\frac{\sin n\theta}{n\theta}\right) r^n = \phi(\theta),$$

and let $\phi(\theta) \to +\infty$ *as* $\theta \to 0$. *Then*

$$\lim_{r\to 1} \Sigma a_n r^n = +\infty.$$

Let $(-2\eta, 2\eta)$ be a neighbourhood of the origin in which $\phi(\theta) > 0$. Then

$$\lim_{r\to 1} \sum_1^\infty \frac{a_n}{n} \sin n\theta\, r^n = \theta\phi(\theta) \quad (-2\eta < \theta < 2\eta),$$

and so, writing $$F(\theta) \sim -\Sigma \frac{a_n}{n^3} \sin n\theta,$$

we have, by Lemma 14†,

$$F(\theta) = \int_0^\theta dy \int_0^y t\phi(t)\, dt + \int_0^\theta M(y)\, dy \quad (-\eta \leqslant \theta \leqslant \eta),$$

where $M(y)$ is non-diminishing in $(0, \eta)$, and is even.

We can find a series

$$-\Sigma \frac{\alpha_n}{n^3} \sin n\theta \quad [\alpha_n = o(n)]$$

which is the integral of an a.c. function, and which in $(-\eta, \eta)$ is equal to

$$\int_0^\theta dy \int_0^y t\phi(t)\, dt.$$

We can find a series

$$-\Sigma \frac{\beta_n}{n^3} \sin n\theta \quad [\beta_n = O(n)]$$

which is the integral of a b.v. function, (*i.e.* a function of bounded variation), and which in $(-\eta, \eta)$ is equal to

$$\int_0^\theta M(y)\, dy.$$

Then $$-\Sigma \frac{a_n - \alpha_n - \beta_n}{n^3} \sin n\theta$$

converges to zero in $(-\eta, \eta)$. Its coefficients are $O(1/n)$, so that it is a

† We use the fact that $F(\theta)$ is odd, and is the sum of a Fourier series.

Fourier series. If we write down Poisson's formula for this function, we see that the series obtained by any number of formal differentiations is summable (P) to zero in $(-\eta, \eta)$. In particular,

$$\lim_{r\to 1} \Sigma (a_n - a_n - \beta_n) r^n = 0. \tag{42}$$

The series $$\Sigma \frac{a_n}{n} \sin n\theta$$

is the Fourier series of a function $f(\theta)$ which, in the neighbourhood of the origin, is equal to $\theta\phi(\theta)$. We have

$$\Sigma \frac{a_n}{n} \sin n\theta\, r^n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \frac{1-r^2}{1-2r\cos(t-\theta)+r^2}\, dt,$$

$$\Sigma a_n r^n = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \frac{d}{dt}\left(\frac{1-r^2}{\delta}\right) dt \quad (\delta = 1-2r\cos t + r^2),$$

and
$$\lim \Sigma a_n r^n = \lim \frac{1}{2\pi} \int_{-\eta}^{\eta} t\phi(t) \frac{(1-r^2)\, 2r \sin t}{\delta^2}\, dt$$

$$= +\infty. \tag{43}$$

Finally, $$-\Sigma \frac{\beta_n}{n^2} \cos n\theta$$

is the Fourier series of a function $g(\theta)$ of bounded variation. Then

$$-\Sigma \frac{\beta_n}{n^2} \cos n\theta\, r^n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) \frac{1-r^2}{1-2r\cos(t-\theta)+r^2}\, dt,$$

$$\Sigma \frac{\beta_n}{n} \sin n\theta\, r^n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) \frac{(1-r^2)\, 2r \sin(t-\theta)}{\{1-2r\cos(t-\theta)+r^2\}^2}\, dt,$$

$$\Sigma \beta_n \cos n\theta\, r^n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) \frac{d}{d\theta}\left[\frac{(1-r^2)\, 2r \sin(t-\theta)}{\{1-2r\cos(t-\theta)+r^2\}^2}\right] dt,$$

and we may replace $d/d\theta$ by $-d/dt$. Putting $\theta = 0$, we get

$$\Sigma \beta_n r^n = -\frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) \frac{d}{dt}\left[\frac{(1-r^2)\, 2r \sin t}{\delta^2}\right] dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1-r^2)\, 2r \sin t}{\delta^2}\, dg,$$

and so
$$\underline{\lim}\, \Sigma \beta_n r^n = \underline{\lim}\, \frac{1}{2\pi} \int_{-\eta}^{\eta} \frac{(1-r^2)\, 2r \sin t}{\delta^2}\, dg.$$

Now $g(\theta) = M(\theta)$ in $(-\eta, \eta)$, and $M(\theta)$ is even and non-decreasing in $(0, \eta)$. Hence

$$\underline{\lim} \Sigma \beta_n r^n = \underline{\lim} \frac{1}{\pi} \int_0^\eta \frac{(1-r^2)\, 2r \sin t}{\delta^2}\, dM$$

$$\geqslant 0. \tag{44}$$

On adding (42), (43), and (44), we obtain the desired result.

The proof of Theorem IX enables us to infer

THEOREM X. *If* $a_n = o(n^2)$, *and*

$$\underline{\lim_{r\to 1}} \Sigma a_n \left(\frac{\sin n\theta}{n\theta}\right) r^n = \phi(\theta),$$

then

$$\underline{\lim_{r\to 1}} \Sigma a_n r^n \geqslant \underline{\lim_{\theta\to 0}} \phi(\theta).$$

If, however,

$$\lim_{r\to 1} \Sigma a_n \left(\frac{\sin n\theta}{n\theta}\right) r^n = \phi(\theta),$$

then

$$\overline{\lim_{\theta\to 0}} \phi(\theta) \geqslant \overline{\lim_{r\to 1}} \Sigma a_n r^n \geqslant \underline{\lim_{r\to 1}} \Sigma a_n r^n \geqslant \underline{\lim_{\theta\to 0}} \phi(\theta).$$

We need prove only the first part of the enunciation. If $\underline{\lim} \phi(\theta) = -\infty$, the theorem is true. If $\underline{\lim} \phi(\theta) > -\infty$, we may suppose that $\underline{\lim} \phi(\theta) > 0$, and it is then sufficient to show that $\underline{\lim} \Sigma a_n r^n \geqslant 0$. This follows from the proof of Theorem IX; the only difference is that (43) must be replaced by

$$\underline{\lim} \Sigma a_n r^n \geqslant \underline{\lim} \frac{1}{2\pi} \int_{-\eta}^{\eta} t\phi(t) \frac{(1-r^2)\, 2r \sin t}{\delta^2}\, dt$$

$$\geqslant 0.$$

12. THEOREM XI. *If* $a_n = o(n^3)$, *and*

$$\lim_{r\to 1} \Sigma a_n \left(\frac{\sin n\theta}{n\theta}\right)^2 r^n = \phi(\theta),$$

where $\phi(\theta) \to 0$ *as* $\theta \to 0$, *then*

$$\Sigma a_n = 0 \quad (C, 3).$$

We proved in II, Theorem IV, that, with the notation

$$F(\theta) \sim -\Sigma \frac{a_n}{n^4} \cos 2n\theta, \tag{45}$$

we have, in a neighbourhood $(-\eta, \eta)$ of the origin,

$$F(\theta) = C + \int_0^\theta dy \int_0^y \{A - 2t^2\phi(t)\}\, dt. \tag{46}$$

Now $F(\frac{1}{2}\theta)$ is in $(-2\eta, 2\eta)$ the integral of an a.c. function, and is even. We can therefore find a series

$$B - \Sigma \frac{a_n}{n^4} \cos n\theta \quad [a_n = o(n^2)],$$

which is the integral of an a.c. function, and which converges to $F(\frac{1}{2}\theta)$ in $(-2\eta, 2\eta)$. Hence

$$B - \Sigma \frac{a_n}{n^4} \cos 2n\theta$$

is the integral of an a.c. function, and converges to $F(\theta)$ in $(-\eta, \eta)$. Then

$$-B - \Sigma \frac{a_n - \alpha_n}{n^4} \cos 2n\theta \tag{47}$$

converges to zero in $(-\eta, \eta)$. Theorem III, with $k = 3$, gives

$$\Sigma (a_n - \alpha_n) = 0 \quad (C, 3).$$

Further,

$$4\Sigma \frac{a_n}{n^2} \cos 2n\theta$$

is the Fourier series of a function $f(\theta)$, which, in the neighbourhood of the origin, is equal to $A - 2\theta^2\phi(\theta)$, and for which, with the notation of Lemma 13, $f_2(0) = 0$. Hence, by Lemma 13, with $p = 2$,

$$\Sigma a_n = 0 \quad (C, 3).$$

By addition, we obtain the result.

Theorem XII. *If p denotes a positive integer, then, under the conditions of Theorem XI,*

$$\lim_{r \to 1} \Sigma a_n \left(\frac{\sin n\theta}{n\theta}\right)^p r^n = \phi_p(\theta) \quad (p > 2)$$

exists† for θ in a neighbourhood of the origin, and $\phi_p(\theta) \to 0$ as $\theta \to 0$.

Thus, when $a_n = o(n^3)$, summability $(R, 2)$ implies summability (R, p) for $p > 2$.

† The proof shows that $\Sigma a_n \left(\frac{\sin n\theta}{n\theta}\right)^p$ converges to $\phi_p(\theta)$.

For $p \geqslant 4$, the result follows on integrating (45) $p-4$ times, by considerations analogous to those used in the case $p \geqslant 3$ of Theorem VIII. For $p = 3$, we observe that, since (47) converges to zero in $(-\eta, \eta)$, and since the coefficients of that series are $o(1/n)$, it follows by Theorem III, with $k = 0$, that

$$\Sigma \frac{a_{n^3}-a_n}{n^3} \sin 2n\theta$$

converges to zero in $(-\eta, \eta)$. Since $\sin 3\nu\theta - 3\sin\nu\theta = -4\sin^3\nu\theta$, it follows that

$$\Sigma (a_{n^3}-a_n)\left(\frac{\sin n\theta}{n\theta}\right)^3 = 0$$

for θ in a neighbourhood of the origin; and, *a fortiori*, that

$$\Sigma (a_{n^3}-a_n) = 0 \quad (R, 3). \tag{48}$$

Further,

$$\Sigma \frac{a_{n^3}}{n^3} \sin 2n\theta$$

is the Fourier series of an a.c. function which is equal to

$$\psi(\theta) = \int_0^\theta \{A - 2t^2\phi(t)\}\,dt = A\theta + o(\theta^3)$$

in a neighbourhood of the origin, so that

$$\Sigma a_{n^3}\left(\frac{\sin 2n\theta}{2n\theta}\right)^3 = -\frac{\psi(3\theta)-3\psi(\theta)}{32\theta^3}$$

$$= o(1).$$

Thus

$$\Sigma a_{n^3} = 0 \quad (R, 3). \tag{49}$$

On adding (48) and (49), we obtain the desired result.

THEOREM XIII. *If* $a_{n^3} = o(n^3)$, *and*

$$\lim_{r\to 1} \Sigma a_{n^3}\left(\frac{\sin n\theta}{n\theta}\right)^2 r^n = \phi(\theta) \tag{50}$$

for θ *in a neighbourhood of the origin, then*

(i) $$\overline{\lim}\,\phi(\theta) \geqslant \underline{\lim}\,\Sigma a_{n^3} r^n; \quad \overline{\lim}\,\Sigma a_{n^3} r^n \geqslant \underline{\lim}\,\phi(\theta);$$

(ii) $$\overline{\lim}\,\theta\phi(\theta) \geqslant \pi\,\underline{\lim}\,(1-r)\Sigma a_{n^3} r^n; \quad \pi\,\overline{\lim}\,(1-r)\Sigma a_{n^3} r^n \geqslant \underline{\lim}\,\theta\phi(\theta)$$

$$(\theta > 0).$$

In proving (i), we need consider only the second inequality, and we may suppose that

$$\underline{\lim}\,\phi(\theta) > 0. \tag{51}$$

We then have to show that

$$\overline{\lim}\,\Sigma a_n r^n \geqslant 0. \tag{52}$$

Let $\phi(\theta) > 0$ in $(-\eta, \eta)$, where $\theta \neq 0$. Then

$$\overline{\lim}\,\Sigma \frac{a_n}{n^2} r^n - \overline{\lim}\,\Sigma \frac{a_n}{n^2} \cos 2n\theta\, r^n \geqslant \lim \Sigma \frac{a_n}{n^2}(1-\cos 2n\theta) r^n$$

$$\geqslant 0 \qquad (0<|\theta|<\eta). \tag{53}$$

If now $$\overline{\lim}\,\Sigma \frac{a_n}{n^2} r^n = +\infty,$$

then (52) is necessarily true, since

$$\Sigma \frac{a_n}{n^2} r^n = \int_0^r \frac{d\sigma}{\sigma} \int_0^\sigma (\Sigma a_n t^{n-1})\, dt.$$

We may therefore suppose that

$$\overline{\lim}\,\Sigma \frac{a_n}{n^2} r^n < K,$$

where K is finite, in which case, by (53),

$$\underline{p}(2\theta) = \underline{\lim} -\Sigma \frac{a_n}{n^2} \cos 2n\theta\, r^n > -K \quad (-\eta < \theta < \eta).$$

We can now apply Lemma 14. This tells us that

$$G(\theta) \sim \Sigma \frac{a_n}{n^4} \cos n\theta$$

has in $(-2\eta, 2\eta)$ the form

$$G(\theta) = \int_0^\theta dy \int_0^y \underline{p}(t)\, dt + \int_0^\theta M(y)\, dy,$$

where $M(\theta)$ is non-diminishing in $(0, 2\eta)$. This expression shows that $D^2 G(\theta)$ exists p.p. in $(-2\eta, 2\eta)$. Consequently, the series

$$-\Sigma \frac{a_n}{n^2} \cos 2n\theta \tag{54}$$

is summable by Poisson's method [to $D^2 G(2\theta)$], p.p. in $(-\eta, \eta)$. If θ_0 is a point at which it is summable, then (50), for $\theta = \theta_0$, tells us that

$$\lim_{r\to 1} \Sigma \frac{a_n}{n^2} r^n = A \tag{55}$$

exists; and hence, again from (50), we infer that (54) is summable by Poisson's method everywhere in $(-\eta, \eta)$. We accordingly have

$$F(2\theta) = \lim_{r\to 1} -\Sigma \frac{a_n}{n^2} \cos 2n\theta\, r^n = \tfrac{1}{2}\theta^2 \phi(\theta) - A \quad (0 < |\theta| < \eta)$$

and, by (55),

$$F(0) = -A. \tag{56}$$

Since $G(\theta)$ is continuous in a neighbourhood of the origin, the series (54) satisfies the conditions of Lemma 2 in $(-\eta, \eta)$. The second inequality in Lemma 2 (B) gives

$$\begin{aligned} \overline{\lim}\, \Sigma a_n r^n &\geqslant \underline{\lim}\, 2\left[\frac{F(\theta) - F(0)}{\theta^2}\right] \\ &\geqslant \underline{\lim}\, \phi(\theta) \\ &> 0. \end{aligned}$$

To prove (ii), we again consider the second inequality. We suppose that

$$\underline{\lim}\, \theta\phi(\theta) > 0 \quad (A > 0).$$

This implies $\phi(\theta) > 0$ in $0 < |\theta| < \eta$. The above proof remains unaltered, and we apply the second inequality of Lemma 2 (A), which gives

$$\begin{aligned} \pi\, \overline{\lim}\, (1-r) \Sigma a_n r^n &\geqslant \underline{\lim}\, 2\left[\frac{F(\theta) - F(0)}{\theta}\right] \\ &\geqslant \underline{\lim}\, \theta\phi(\theta) \\ &> 0. \end{aligned}$$

THEOREM XIV. *In Theorem I, the condition for $\Sigma A_n(x)/n^4$ to be the Fourier series of a continuous function may be replaced by the condition*

$$a_n = o(n^3), \quad b_n = o(n^3). \tag{57}$$

The proof, which we shall only indicate, consists, in the first place, in replacing in Lemmas 1, 2, and 3 the condition for $\Sigma A_n(x)/n^4$ (P) to be continuous in an interval by the condition (57). That this is valid follows, for Lemma 1, by Lemma 14; for Lemma 2, by Theorem XIII; for Lemma 3,

by II, Lemma 15. It then appears that Lemma 6 remains true under the new conditions. The deduction of Theorem XIV then proceeds as in Theorem I; we infer that

$$F(x) = -\Sigma A_n(x)/n^2 \quad (P)$$

is continuous. The coefficients of this series being $o(n)$, it follows from II, Theorem II, that $-\Sigma A_n(x)/n^2$ is the Fourier series of the continuous function $F(x)$. The required result then follows from II, Theorem I.

13. THEOREM XV. *If* $a_n = o(n^4)$, *and*

$$\text{(i)} \quad \lim_{r\to 1} \Sigma a_n \left(\frac{\sin n\theta}{n\theta}\right)^3 r^n = \phi(\theta) \quad [\phi(\theta)\to 0],$$

$$\text{(ii)} \quad \lim_{r\to 1} \Sigma \frac{a_n}{n^2}\left(\frac{\sin n\theta}{n\theta}\right) r^n = \lambda(\theta) \quad [\lambda(\theta)\to 0],$$

then

$$\Sigma a_n = 0 \quad (C, 4).$$

We proved in II, Theorem V, that, in a neighbourhood of the origin,

$$F(\theta) \sim -\Sigma \frac{a_n}{n^5} \sin n\theta$$

has the form

$$F(\theta) = A\theta + \int_0^\theta dy \int_0^y t^3 \mu(t)\, dt \quad [\mu(\theta)\to 0 \text{ as } \theta\to 0].$$

We can find a series

$$-\Sigma \frac{b_n}{n^5} \sin n\theta \quad [b_n = o(n^3)]$$

which is the integral of an a.c. function, and which is equal to $F(\theta)$ in a neighbourhood of the origin. Then

$$-\Sigma \frac{a_n - b_n}{n^5} \sin n\theta$$

converges to zero in $(-\eta, \eta)$ say. By Theorem III, with $k = 4$,

$$\Sigma(a_n - b_n) = 0 \quad (C, 4). \tag{58}$$

Further,

$$\Sigma \frac{b_n}{n^3} \sin n\theta$$

is the Fourier series of a function $f(\theta)$, which, in the neighbourhood of the

origin, is of the form $\theta^3\mu(\theta)$, and for which, with the notation of Lemma 13, $f_3(0)=0$. By that lemma, with $p=3$,

$$\Sigma b_n = 0 \quad (C, 4). \tag{59}$$

From (58) and (59), the result follows.

THEOREM XVI. *If $a_n = o(n^4)$ and, in a neighbourhood of the origin,*

$$\text{(i)} \quad \lim_{r\to 1} \Sigma a_n \left(\frac{\sin n\theta}{n\theta}\right)^3 r^n = \phi(\theta) \quad [\phi(\theta)\to l \ (\textit{finite})],$$

$$\text{(ii)} \quad \lim_{r\to 1} \Sigma \frac{a_n}{n^3} \sin n\theta\, r^n(1-r^{2n}) = 0,$$

then a necessary and sufficient condition that

$$\lim \Sigma a_n r^n = l$$

is $$\text{(iii)} \qquad \lim (1-r)^2 \Sigma a_n r^n = 0.$$

The condition (ii) is satisfied in particular, (*a*) if $a_n = o(n^2)$, or (*b*) if $\Sigma a_n n^{-3}\sin n\theta$ is summable by Poisson's method in a neighbourhood of the origin.

We may suppose that $l=0$ in (i). The condition (iii) is necessary, and we have to prove that it is sufficient. We have

$$\lim_{r\to 1} \Sigma \frac{a_n}{n^3} (3\sin n\theta - \sin 3n\theta) r^n = 4\theta^3\phi(\theta).$$

Write

$$b_n = \frac{3a_n}{n^3}$$

when n is not a multiple of 3, and

$$b_n = \frac{3a_{3\nu}}{(3\nu)^3} - \frac{a_\nu}{\nu^3}$$

when $n=3\nu$. Then

$$\Sigma b_n \sin n\theta\, r^n - \Sigma \frac{a_n}{n^3}(3\sin n\theta - \sin 3n\theta) r^n = \Sigma \frac{a_\nu}{\nu^3} \sin 3\nu\theta r^\nu (1-r^{2\nu})$$

$$\to 0$$

as $r\to 1$ by (ii), for $-\eta<\theta<\eta$ say. Thus

$$\lim \Sigma b_n \sin n\theta\, r^n = 4\theta^3\phi(\theta) \quad (-\eta<\theta<\eta).$$

Manifestly, $b_n = o(n)$. Hence, by II, Lemma 15,

$$F(\theta) \sim -\Sigma \frac{b_n}{n^2} \sin n\theta \tag{60}$$

has the form $$F(\theta) = A\theta + \int_0^\theta dy \int_0^y t^3 \phi(t)\, dt$$

in a neighbourhood of the origin. By writing down Poisson's formula for the series (60) and differentiating five times, we see that

$$\lim \Sigma n^3 b_n r^n = 0.$$

Let $$P(r) = \Sigma a_\nu r^\nu.$$

Then $$\Sigma n^3 b_n r^n = 3P(r) - 27P(r^3).$$

Given $\epsilon > 0$, there is an $r(\epsilon)$ such that, for $r(\epsilon) \leqslant r < 1$,

$$|27P(r^3) - 3P(r)| < \epsilon.$$

Writing $\rho = [r(\epsilon)]^3$, we have

$$|P(\rho) - 3^{-2} P(\rho^{3^{-1}})| < \epsilon 3^{-3}; \tag{61}$$

and, since $\rho < \rho^{3^{-n}} < 1$ for $n = 1, 2, \ldots$, we also have

$$|P(\rho^{3^{-n}}) - 3^{-2} P(\rho^{3^{-n-1}})| < \epsilon 3^{-3},$$

so that $$|3^{-2n} P(\rho^{3^{-n}}) - 3^{-2n-2} P(\rho^{3^{-n-1}})| < \epsilon 3^{-2n-3}.$$

By addition, we find that

$$|P(\rho) - 3^{-2n-2} P(\rho^{3^{-n-1}})| < \epsilon. \tag{62}$$

Adding the inequalities (61) for n, $n+1$, ..., $n+p$, we get

$$|3^{-2n} P(\rho^{3^{-n}}) - 3^{-2n-2p} P(\rho^{3^{-n-1}})| < \epsilon 3^{-2n}.$$

Hence $$\lim_{n \to \infty} 3^{-2n} P(\rho^{3^{-n}}) = K(\rho)$$

exists, and (62) gives

$$|P(\rho) - K(\rho)| \leqslant \epsilon. \tag{63}$$

Now $$1 - \rho^{3^{-n}} = (1-\rho) \Big/ \sum_{r=0}^{3^n - 1} \rho^{r 3^{-n}}$$

and $$\lim_{n \to \infty} 3^{-n} \sum_{r=0}^{3^n - 1} \rho^{r 3^{-n}} = \int_0^1 \rho^x\, dx = \frac{(1-\rho)}{\log 1/\rho},$$

so that $$\lim (1-\rho^{3^{-n}})^2 3^{2n} = \left(\log \frac{1}{\rho}\right)^2,$$

and $$\lim (1-\rho^{3^{-n}})^2 P(\rho^{3^{-n}}) = K(\rho)\left(\log \frac{1}{\rho}\right)^2.$$

The first member is zero by (iii). Thus $K(\rho) = 0$, and (63) gives $|P(\rho)| \leqslant \epsilon$. Hence $\lim P(\rho) = 0$.

It should be noticed that we have, independently of (iii),

$$\left| P(\rho) - \frac{\lim (1-\rho^{3^{-n}})^2 P(\rho^{3^{-n}})}{(\log 1/\rho)^2} \right| \leqslant \epsilon.$$

This holds when ρ is replaced by a σ satisfying $\rho \leqslant \sigma < 1$. The limit in the above inequality, when ρ is replaced by σ, takes all its values in $\rho \leqslant \sigma \leqslant \rho^{3^{-1}}$. In that interval, $P(\rho)$ is bounded. Hence we have, independently of (iii),

$$|(1-r)^2 \Sigma a_n r^n| < K.$$

Neither the condition (ii) nor the condition (iii) is superfluous in the above theorem. This is shown by

THEOREM XVII. *There is a Fourier series of a bounded function*

$$f(\theta) \sim \Sigma \frac{a_n}{n^2} \cos n\theta + C \tag{64}$$

such that (i) $$\Sigma a_n \left(\frac{\sin n\theta}{n\theta}\right)^3 = 0$$

for θ in a neighbourhood of the origin, and such that $\Sigma a_n/n^2$ is not summable by Poisson's method.

Let $F(\theta)$ be the odd function defined in $(0, \pi)$ by

$$F(0) = 0, \quad F(\theta) = 3^{-n} \text{ for } 2\pi 3^{-n-1} \leqslant \theta \leqslant \pi 3^{-n},$$

while $F(\theta)$ is linear in each of the remaining intervals and continuous in $(0, \pi)$. Then

$$F(3\theta) = 3F(\theta) \quad (0 < \theta < \tfrac{1}{3}\pi). \tag{65}$$

Manifestly, $F(\theta)$ is a.c. Let $f(\theta)$ be defined p.p. by $f(\theta) = F'(\theta)$. Then $f(\theta)$ is bounded. Let (64) be its Fourier series. In virtue of (65), (i) is satisfied. We have

$$\frac{1}{2t} \int_{-t}^{t} f(\theta)\, d\theta = \frac{F(t)}{t},$$

which clearly does not tend to a limit as $t \to 0$. By a theorem of Hardy and Littlewood (*loc. cit.*), $\Sigma a_n/n^2$ is not summable P.

14. Theorem XVIII. *If* $a_n = o(n^5)$, *and*

$$\text{(i)} \quad \lim_{r\to 1} \Sigma a_n \left(\frac{\sin n\theta}{n\theta}\right)^4 r^n = \phi(\theta) \quad [\phi(\theta) \to 0],$$

$$\text{(ii)} \quad \lim_{r\to 1} \Sigma \frac{a_n}{n^2} \left(\frac{\sin n\theta}{n\theta}\right)^2 r^n = \lambda(\theta) \quad [\lambda(\theta) \to 0],$$

then

$$\Sigma a_n = 0 \quad (C, 5).$$

It was shown in II, Theorem VII, that

$$F(\theta) \sim -\Sigma \frac{a_n}{n^6} \cos n\theta$$

has the form

$$F(\theta) = \int_0^\theta dy \int_0^y \{A + t^4 \mu(t)\}\, dt + C \quad [\mu(\theta) \to 0 \text{ as } \theta \to 0]$$

in a neighbourhood $-\eta < \theta < \eta$ of the origin. We can find a series

$$B - \Sigma \frac{b_n}{n^6} \cos n\theta \quad [b_n = o(n^4)]$$

which is the integral of an a.c. function and is equal to $F(\theta)$ in $(-\eta, \eta)$. Then

$$-B - \Sigma \frac{a_n - b_n}{n^6} \cos n\theta$$

converges to zero in $(-\eta, \eta)$; and, since the coefficients in this series are $o(1/n)$, we have, by Theorem III, with $k = 5$,

$$\Sigma (a_n - b_n) = 0 \quad (C, 5). \tag{66}$$

Further,

$$\Sigma \frac{b_n}{n^4} \cos n\theta$$

is the Fourier series of a function $f(\theta)$, which, in the neighbourhood of the origin, is equal to $A + \theta^4 \mu(\theta)$, so that, with the notation of Lemma 13, $f_4(0) = 0$. By that lemma, with $p = 4$,

$$\Sigma b_n = 0 \quad (C, 5). \tag{67}$$

The result follows from (66) and (67).

THEOREM XIX. *If* $a_n = o(n^5)$, *and if, in a neighbourhood of the origin,*

$$\text{(i)} \quad \lim_{r\to 1} \Sigma a_n \left(\frac{\sin n\theta}{n\theta}\right)^4 r^n = \phi(\theta) \quad [\phi(\theta)\to l \text{ (finite)}],$$

$$\text{(ii)} \quad \lim_{r\to 1} \Sigma \frac{a_n}{n^4} \sin^2 n\theta\, r^n(1-r^n) = 0,$$

then a necessary and sufficient condition that

$$\lim \Sigma a_n r^n = l$$

is that (iii) $\lim (1-r)^2 \Sigma a_n r^n = 0.$

The condition (ii) is satisfied in particular, (*a*) if $a_n = o(n^3)$, or (*b*) if $\Sigma a_n n^{-4} \cos n\theta$ is summable by Poisson's method in a neighbourhood of the origin.

We may suppose that $l = 0$ in (i). The condition (iii) is necessary, and we have to prove that it is sufficient. We have

$$\lim_{r\to 1} \Sigma \frac{a_n}{n^4} [4(1-\cos n\theta) - (1-\cos 2n\theta)] r^n = 8(\tfrac{1}{2}\theta)^4 \phi(\tfrac{1}{2}\theta)$$
$$= \theta^4 \psi(\theta),$$

where $\psi(\theta)\to 0$ as $\theta\to 0$. Let

$$b_n = 4\frac{a_n}{n^4}$$

when n is odd, and $\quad b_n = 4\frac{a_{2\nu}}{(2\nu)^4} - \frac{a_\nu}{\nu^4}$

when $n = 2\nu$. Then

$$\Sigma b_n(1-\cos n\theta) r^n - \Sigma \frac{a_n}{n^4} [4(1-\cos n\theta) - (1-\cos 2n\theta)] r^n$$

$$= \Sigma \frac{a_n}{n^4} (1-\cos 2n\theta) r^n (1-r^n)$$

$$= o(1)$$

as $r\to 1$, by (ii). Hence,

$$\lim_{r\to 1} \Sigma b_n(1-\cos n\theta) r^n = \theta^4 \psi(\theta).$$

Here, $b_n = o(n)$. It was shown in the proof of II, Theorem IV, that the

preceding relation implies that

$$F(\theta) \sim -\Sigma \frac{b_n}{n^2} \cos n\theta \tag{68}$$

has the form $$F(\theta) = \int_0^\theta dy \int_0^y \{A - t^4 \psi(t)\} dt + C$$

in a neighbourhood of the origin. Writing down Poisson's formula for the series (68), and differentiating six times, we get

$$\lim \Sigma n^4 b_n r^n = 0.$$

Now $$\Sigma n^4 b_n r^n = 4\Sigma a_n r^n - 16\Sigma a_n r^{2n}.$$

Writing $$P(r) = \Sigma a_n r^n,$$

we have, for $r(\epsilon) \leqslant r < 1$,

$$|4P(r^2) - P(r)| < \epsilon.$$

This is treated by the argument of Theorem XVI, the number 2 taking the place of 3 in that argument, and it gives $\lim P(r) = 0$.

Neither the condition (ii) nor the condition (iii) is superfluous in the above theorem. This is shown by

THEOREM XX. *There is a Fourier series of a bounded function*

$$f(\theta) \sim \Sigma \frac{a_n}{n^2} \cos n\theta + C \tag{69}$$

such that (i) $$\Sigma a_n \left(\frac{\sin n\theta}{n\theta}\right)^4 = 0$$

for θ in a neighbourhood of the origin, and such that $\Sigma a_n/n^2$ is not summable by Poisson's method.

Let $F(\theta)$ be the odd function defined in $(0, \pi)$ by

$$F(0) = 0, \quad F(\theta) = 2^{-n} \text{ for } \tfrac{3}{2}\pi 2^{-n-1} \leqslant \theta \leqslant \pi 2^{-n},$$

while $F(\theta)$ is linear in each of the remaining intervals, and continuous in $(0, \pi)$. Then

$$F(2\theta) = 2F(\theta) \quad (0 < \theta < \tfrac{1}{2}\pi), \tag{70}$$

and $F(\theta)$ is a.c. Let $f(\theta)$ be defined p.p. by $f(\theta) = F'(\theta)$. Then $f(\theta)$ is bounded. Let (69) be its Fourier series. Then, since

$$\frac{1}{2t} \int_{-t}^{t} f(\theta)\, d\theta = \frac{F(t)}{t}$$

does not tend to a limit as $t \to 0$, it follows by the theorem of Hardy and Littlewood that $\Sigma a_n/n^2$ is not summable by Poisson's method. Write

$$G(\theta) = \int_0^\theta F(t)\,dt.$$

Then
$$G(\theta) = \Sigma \frac{a_n}{n^4}(1-\cos n\theta),$$

$$\Sigma a_n \left(\frac{\sin n\theta}{n\theta}\right)^4 = \frac{4G(2\theta)-G(4\theta)}{8\theta^4}. \tag{71}$$

But
$$\begin{aligned} G(4\theta) &= \int_0^{4\theta} F(t)\,dt \\ &= \tfrac{1}{2}\int_0^{4\theta} F(2t)\,dt \\ &= \tfrac{1}{4}\int_0^{2\theta} F(t)\,dt \\ &= \tfrac{1}{4}G(2\theta). \end{aligned}$$

Hence the second member of (71) is zero, and (i) is satisfied.

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WYTHOFF'S CONSTRUCTION FOR UNIFORM POLYTOPES

(*Preliminary note*)

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1. A polyhedron is said to be uniform if its faces are regular and if it has a group of symmetries which is transitive on the vertices. Thus the uniform polyhedra are the regular and Archimedean solids, prisms and antiprisms*. A polytope is said to be uniform if its bounding figures are uniform and it has a group of symmetries which is transitive on the vertices. Various families of uniform polytopes have been discovered by Schläfli, Gosset, Mrs. Boole Stott, Schoute, and Elte†.

Wythoff‡ gave a new construction for Mrs. Stott's polytopes. The object of this paper is to apply that construction exhaustively, obtaining all the Schläfli-Gosset-Stott-Schoute-Elte polytopes and many others. It may be true that all *finite* polytopes are Wythoffian; but this would certainly be difficult to prove, since non-Wythoffian degenerate polytopes ("nets") are known§.

2. If a discrete set of primes is symmetrical by reflexion in every one of them, it effects a partition of space into congruent *fundamental regions* . If the reflexions leave a direction invariant, they transform into itself any prime normal to this direction, and so they can be regarded as operating in a space of fewer dimensions. We shall always suppose this reduction to have been carried out. If the reflexions leave a point invariant, they

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* Kepler, **1**, 116.

† Schläfli, **1**, 386, 389, 390. Gosset, **1**, 45–48. Stott, **1**. Schoute, **1**, 35–67; **2**, 73–104; **3**. Elte, **1**, 104, 117, 118, 123.

‡ Wythoff, **1**.

§ Coxeter, **2**, 184 (§ 20 . 3).

‖ *Cf.* Klein, **1**, 20.

transform into itself any sphere* drawn around this point, and the angular fundamental region is conveniently replaced by its intersection with the sphere. It can be proved† that this spherical region is a simplex (all of whose dihedral angles are submultiples of π).

If the reflexions leave no point invariant, the fundamental region may be finite or infinite. If finite, it is a (Euclidean) simplex or simplicial prism‡; if infinite, it can be regarded as joining a finite simplex or simplicial prism to a simplex at infinity§.

3. Spherical and Euclidean simplexes whose dihedral angles are submultiples of π have been completely enumerated||, and are conveniently represented by diagrams consisting of linked dots¶. Each dot denotes a bounding prime of the simplex; a pair of dots not directly linked denotes a pair of perpendicular primes. A link marked k denotes a dihedral angle π/k $(k > 2)$; but it is convenient to omit the mark when (as so often happens) $k = 3$. Such a diagram is what topologists call a *graph***.

In the spherical case, the graph is a *forest* (with very simple trees). When the forest contains more than one tree††, the vertices of the simplex fall into sets which lie in absolutely perpendicular spaces, each set forming a simplex represented by one of the trees.

In the Euclidean case, the graph is either a tree or an isolated circuit (of unmarked links)‡‡. The new graph obtained by juxtaposing several such trees or circuits can be regarded as representing a prism whose constituents are the corresponding simplexes (left undetermined as to size).

The most general case that we shall consider is when the connected pieces of the graph fall into two categories: trees representing spherical simplexes, and trees or circuits representing Euclidean simplexes. The pieces of the first category together represent a spherical simplex which we regard as lying on a sphere of infinite radius. The region represented by the whole graph is obtained by joining this simplex at infinity to the prism represented by the pieces of the second category.

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* We use the word *sphere* for the analogue in any number of dimensions.

† Coxeter, **5**, 596 (Theorem 4).

‡ Called *prismotope* in Schoute, **3**.

§ Coxeter, **5**, 599 (Theorem 7).

|| Coxeter, **2**, 144.

¶ Coxeter, **3**, 133.

** Whitney, **1**. (We use the word *dot* for *vertex*, to avoid confusion with vertices of polytopes.)

†† Coxeter, **2**, 137 (§ 15 . 3).

‡‡ Coxeter, **2**, 138 (§ 15 . 4).

4. With every discrete group generated by reflexions we can associate the graph that represents its fundamental region. The dots of the graph correspond to a set of generators of the group, and the abstract definition can be read off in the manner described by Coxeter, **2**, 164. If the graph is disconnected, the group is the direct product of the groups represented by the several trees or circuits. For the orders of these groups, see Coxeter, **5**, 618, 619, or **6**.

5. Take first the case when the fundamental region is a simplex. We use capital letters to denote its vertices, and corresponding small letters for the opposite bounding primes. Consider any vertex A, and its transforms under all the operations of the group, *i.e.* the A-vertices of all the simplexes that arise by repeated reflexion in the primes*. These points are the vertices of a uniform polytope which can be symbolized by drawing a little ring around the a-dot of the graph. The graph with both the a-dot and the b-dot ringed symbolizes the polytope whose typical vertex is the point of intersection of the edge AB of the simplex with the bisector of the dihedral angle ab. When the a-, b-, and c-dots are ringed, the typical vertex is the point of concurrence of the three lines in which the plane ABC is met by the bisectors of the angles bc, ca, ab†, and so on. Finally, the graph with all dots ringed symbolizes the polytope whose vertices are the in-centres of all the simplexes‡.

If the graph is disconnected, the polytope is the prism whose constituents are symbolized by those connected pieces which contain ringed dots. There may be unringed pieces, but they will have no effect. It is, therefore, natural to make an unringed graph symbolize a mere point. In particular, the null graph symbolizes a point.

Consider the sub-graph derived by removing certain dots along with all the links that are incident in those dots. The corresponding polytope is an element of the polytope symbolized by the whole graph, and every element can be obtained in this manner§. Unringed pieces of the sub-graph have no effect on the shape of the element; but, in order to calculate the *number* of elements of a particular type, we must make the unringed pieces as extensive as possible (and discard all sub-graphs that could be derived from others by reducing unringed pieces). The number is then Γ/γ, where Γ and γ are the orders of the groups represented by the whole

* *Cf.* Klein, **1**, 23.

† For a diagram, see Robinson, **1**, 71.

‡ *Cf.* Möbius, **1**, 661, 677, 691.

§ *Cf.* Schoute, **1**, 16; **2**, 15, 80.

graph and sub-graph regardless of rings. In particular, the *vertices* are given by removing all the ringed dots (and their links)*; *e.g.*, the polytope

has $\Gamma/1 = 696729600$ vertices.

In order that π_r and π_s $(r > s)$ may be *incident* elements (of r and s dimensions), all ringed pieces of the sub-graph symbolizing π_s must occur in the sub-graph symbolizing π_r. The number of such π_r's incident in each π_s is then γ/γ', where γ is the order of the group represented by the sub-graph symbolizing π_s (regardless of rings), while γ' is the order of the group represented by the common part of the two sub-graphs. Thus the number is 1 whenever the sub-graph symbolizing π_r contains the whole of that symbolizing π_s; *e.g.*, in the particular polytope mentioned above, any vertex belongs to one edge of each of eight distinct types, symbolized by the separate ringed dots.

6. When the fundamental region is a prism, we must stipulate that every connected piece of the graph has at least one ringed dot. The rule of § 5 then gives a definite point in each constituent of the prism, and the typical vertex of the derived polytope is constructed from such points in an obvious manner. It is always possible to adjust the relative size of the constituents so as to make the polytope equilateral, and so uniform. In order to obtain the elements of the polytope (which is a degenerate prism†), we now have to remove at least one dot from *every* connected piece.

This construction can be extended to the case when the graph involves pieces of both categories, but then it yields merely semi-degenerate‡ prisms.

7. In order to enumerate the polytopes that can be derived by such a construction, it will be sufficient to consider *connected* graphs, in which case the fundamental regions must occur among the simplexes considered in Coxeter, **3**. All other polytopes of this kind can then be derived as prisms.

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* *Cf.* Robinson, **1**, 73.

† Coxeter, **1**, 353; **2**, 187.

‡ Coxeter, **1**, 354. The semi-degenerate prism $[\delta_{n-1}, \alpha_1]$ is described by Gosset, **1**, 45.

7.1. If the simplex* is $(k_1, k_2, \ldots, k_{m-1})$, so that the graph is a simple chain of $m-1$ links, we shall number the dots $0, 1, \ldots, m-1$. When the 0-dot is ringed, the polytope is $\{k_1, k_2, \ldots, k_{m-1}\}$†. When the n-dot is ringed, it is $t_n\{k_1, k_2, \ldots, k_{m-1}\}$‡. Hence we naturally let

$$t_{f,g,\ldots,l}\{k_1, k_2, \ldots, k_{m-1}\} \quad (0 \leqslant f < \ldots < l < m)$$

denote the polytope symbolized when the f-, g-, ..., l-dots are all ringed§. Robinson‖ has shown that the operator $t_{f,\ldots,l}$ is equivalent to Mrs. Stott's $ce_f \ldots e_l$, with the convention that ce_0 is identity. In fact, e_i adds a ring to the i-dot, while c (or c_0) removes the ring from the 0-dot.

7.2. If the simplex is $\begin{pmatrix} 3^n \\ 3^p \\ 3^q \end{pmatrix}$, so that the graph consists of three simple chains, of n, p, q links, all emanating from one special dot¶, we shall call the special dot 0 and number the remaining dots 1, 2, ... along each chain. The polytope derived by ringing certain dots will be denoted by a symbol of the form

$$(n_{r,\ldots,t}\, p_{u,\ldots,w}\, q_{x,\ldots,z}) \quad \text{or} \quad (n_{r,\ldots,t}\, p_{u,\ldots,w}\, q_{x,\ldots,z})_0$$

$$(n \geqslant r > \ldots > t > 0,\ \ p \geqslant u > \ldots > w > 0,\ \ q \geqslant x > \ldots > z > 0),$$

where the suffixes indicate which dots are ringed**.

In particular,

$$(n_n\, p\, q) = n_{pq}\dagger\dagger,$$

$$(n_{n-l}\, p\, q) = t_l\, n_{pq} \quad (l < n),$$

$$(n\, p\, q)_0 = 0_{npq},$$

$$(n\, p_1\, q_1) = t_{n+1}\, n_{pq},$$

$$(n\, p_{p'+1}\, q_{p'+1}) = t_{p'\,p'}\, n_{pq} \quad (p' < p,\ p' < q)\ddagger\ddagger.$$

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* Coxeter, **3**, 133.

† Todd, **1**.

‡ Coxeter, **1**, 354. $t_1\{3, 3, 5\}$ is the *octicosahedric* of Gosset, **1**, 46.

§ This agrees with the notation $t_{0,1}$, $t_{1,2}$ of Coxeter, **4**. See also Du Val, **1**, 48 (foot-note).

‖ Robinson, **1**, 72.

¶ Coxeter, **3**, 133.

** Thus the polytope used as an example in § 5 is $(4_{4,3,2,1}\, 2_{2,1}\, 1_1)_0$.

†† The polytopes n_{21} ($n = 0, 1, 2, 3, 4, 5$) are described by Gosset, **1**, 45, 47, 48. 1_{22}, 2_{31}, 1_{32}, 2_{41} are Elte's V_{72}, V_{126}, V_{576}, V_{2160}, respectively. Du Val, **1**, 73, mistakenly describes n_{pq} as $(n_1\, p\, q)$.

‡‡ Coxeter, **1**, 412 ("the case $p' = q'$"). The symbols t_{01}, t_{02} of Du Val, **1**, 37–39, should have contained commas: $t_{0,1}\, 5_{21} = (5_{3,1}\, 2\, 1)$, $t_{0,2}\, 5_{21} = (5_{3,3}\, 2\, 1)$.

The rest of the polytopes $(2_1\,2\,1)$, ..., $(5_{5,4,3,2,1}\,2_{2,1}\,1_1)_0$,

$(3_1\,3\,1)$, ..., $(3_{3,2,1}\,3_{3,2,1}\,1_1)_0$,

$(2_1\,2\,2)$, ..., $(2_{2,1}\,2_{2,1}\,2_{2,1})_0$

are now published for the first time*.

It is sometimes convenient to allow the final zero suffix to be brought inside the brackets and appended to one of n, p, q (*i.e.* to associate the 0-dot with a particular one of the three chains); *e.g.*†

$$(n_0\,p\,q) = (n\,p_0\,q) = (n\,p\,q_0) = (n\,p\,q)_0.$$

With this convention,

$$(n_{n-f,\ldots,n-l}\,1\,1) = t_{f,\ldots,l}\beta_{n+3} \qquad (l \leqslant n)$$

and

$$(n_{n-f,\ldots,n-l}\,1_1\,1_1) = t_{f,\ldots,l,n+1}\beta_{n+3}.$$

Since‡

$$h\gamma_{n+3} = 1_{n1} = (n\;1_1\;1)$$

and

$$t_{f,\ldots,l}\gamma_{n+3} = t_{n+2-l,\ldots,n+2-f}\beta_{n+3} = (n_{l-2,\ldots,f-2}\,1\,1) \quad (f \geqslant 2),$$

it is natural to define

$$h_{f,\ldots,l}\gamma_{n+3} = (n_{l-2,\ldots,f-2}\,1_1\,1) \quad (1 < f < \ldots < l < n+3).$$

Then, in Schoute's notation§,

$$h_{f,\ldots,l}\gamma_m = e_f \ldots e_l HM_m.$$

We are now numbering the dots in the graph for $\begin{pmatrix} 3^{m-3} \\ 3 \\ 3 \end{pmatrix}$ as follows:

7.3. The graph for $\begin{pmatrix} 3^{m-4},\ 4 \\ 3 \\ 3 \end{pmatrix}$ differs from this in having the final link marked 4. By ringing the dots, we obtain derivatives of δ_m analogous to

* Except those mentioned in the preceding foot-note.

† *Cf.* Coxeter, **1**, 372 (7.36), 411 (12.31).

‡ Coxeter, **1**, 372 (7.43).

§ Schoute, **2**, 87. Robinson, **1**, 73–74 (§ 3).

‖ Coxeter, **3**, 135 (i).

the above derivatives of β_m and γ_m. In particular*, $h\delta_m$ is

and by ringing the f-, ..., l- dots we obtain Schoute's "*hmpd.* nets"†

$$h_{f,\ldots,l}\delta_m = e_{f-1}\ldots e_{l-1}NH_{m-1} \quad (1<f<\ldots<l<m).$$

7.4. The simplex‡ $\begin{pmatrix} 3, & & 3 \\ & 3^{m-5}, & \\ 3, & & 3 \end{pmatrix}$ leads first to alternative symbols for some of the above polytopes, in accordance with the rule that

can be replaced by

respectively. But

(with m dots altogether) appears to be a new discovery (for $m>5$). We shall call it§ $q\delta_m$, and ring the f-, ..., l- dots to derive

$$q_{f,\ldots,l}\delta_m \quad (1<f<\ldots<l<m-2).$$

(Most of these polytopes will have two alternative symbols, on account of the symmetry of the unringed graph; *e.g.*, $q_3\delta_6 = q_2\delta_6$.)

In terms of Mrs. Stott's *partial operations*||, we find that

$$h_{f,\ldots,l} = \tfrac{1}{2}e_0 ce_f \ldots e_l \quad \text{and} \quad q_{f,\ldots,l} = \tfrac{1}{2}e_{m-1}h_{f,\ldots,l}.$$

The vertices of $q\delta_m$ (edge $\sqrt{2}$) can be selected from the totality of points having $m-1$ integral coordinates by the following rule: if r_n of the coordinates are congruent to n (mod 4), so that $r_0+r_1+r_2+r_3 = m-1$, then

$$r_1 \equiv r_2 \equiv r_3 \pmod 2.$$

---

* Coxeter, **1**, 365. $h\delta_4$ is the *simple tetroctahedric check* of Gosset, **1**, 46.

† Schoute, **2**, 98.

‡ Coxeter, **3**, 135 (ii).

§ *h* for *half*; *q* for *quarter*.

|| Stott, **1**, 15.

It is therefore proper to let $q\delta_4$ denote

(Andreini's* Fig. 15).

7.5. The simplex† $\begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix}$ gives nothing new. In fact, with an obvious extension of the $(n\,p\,q)$ notation,

$$\begin{array}{llll|llll}
(1_1\,1\,1\,1) & = h\delta_5 & = \{3,\,3,\,4,\,3\} & & (1\,1\,1\,1)_0 & = t_2\delta_5 & = \{3,\,4,\,3,\,3\} \\
(1_1\,1\,1\,1) & = q\delta_5 & = t_1\delta_5 & & (1_1\,1\,1\,1)_0 & = h_2\delta_5 & = t_{0,1}\{3,\,3,\,4,\,3\} \\
(1_1\,1_1\,1\,1) & = h_3\delta_5 & = t_2\{3,\,3,\,4,\,3\} & & (1_1\,1_1\,1\,1)_0 & = q_2\delta_5 & = t_{1,2}\delta_5 \\
(1_1\,1_1\,1_1\,1) & = h_{2,3}\delta_5 & = t_{1,2}\{3,\,3,\,4,\,3\} & & (1_1\,1_1\,1_1\,1)_0 & = h_{2,3}\delta_5 & = t_{1,2}\{3,\,3,\,4,\,3\} \\
(1_1\,1_1\,1_1\,1_1) & = t_{1,3}\delta_5 & = t_1\{3,\,4,\,3,\,3\} & & (1_1\,1_1\,1_1\,1_1)_0 & = t_{1,2,3}\delta_5 & = t_{0,1}\{3,\,4,\,3,\,3\}.
\end{array}$$

7.6. Finally, a circuit‡ of m dots and m links with one dot ringed symbolizes§ $a_{m-1}h$; and by ringing further dots we derive Schoute's *simplex nets*. The symbolism is perfect, since the numbers of links between consecutive ringed dots form Schoute's *partition cycle*¶. In particular, a polygon with every vertex ringed is the symbol for the space-filling investigated by Hinton**.

8. As an example, let us enumerate the elements of the six-dimensional polytope $(2_2\,2\,1_1)$ or

(for which $\Gamma = 51840$).

(In the figures below, for greater clarity, the omitted links have been allowed to appear as broken lines.)

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* Andreini, **1**, 107. Fig. 13 will be mentioned later. Fig. 16 is not uniform. Figs. 12, 14, 17, *et seq.*, are respectively $h\delta_4$, $t_{1,2}\delta_4$, $t_{0,1}\delta_4$, $t_1\delta_4$, $h_3\delta_4$, $t_{0,2}\delta_4$, $t_{0,1,2}\delta_4$, $t_{0,1,3}\delta_4$, $h_{2,3}\delta_4$, $h_2\delta_4$, $t_{0,1,2,3}\delta_4$.

† Coxeter, **3**, 135 (iii).

‡ *Ibid.*, (iv).

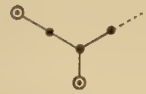
§ Coxeter, **1**, 366.

‖ Schoute, **1**, 35–67.

¶ Schoute, **1**, 49.

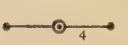
** Hinton, **1**. (He considers only explicitly the cases $m = 3, 4, 5$. For the general case, see Schoute, **1**, 59.)

ELEMENTS OF $(2_2\,2\,1_1)$.

	$h\gamma_5$	$\Gamma/1920 = 27$		α_0	$\Gamma/120 = 432$
	$[\alpha_4, \alpha_1]$	$\Gamma/120.2 = 216$		α_4	$\Gamma/120 = 432$
	$[\alpha_2, \alpha_1]$	$\Gamma/6.2.6 = 720$		α_1	$\Gamma/2.2.6 = 2160$
	$t_{0,3}\alpha_4$	$\Gamma/120.2 = 216$		α_3	$\Gamma/24.2 = 1080$
	$h_4\gamma_5$	$\Gamma/1920 = 27$		β_4	$\Gamma/192 = 270$
	α_5	$\Gamma/720 = 72$		β_2	$\Gamma/2.2.6 = 2160$
	α_4	$\Gamma/120 = 432$		$[\alpha_2, \alpha_1]$	$\Gamma/6.2.2 = 2160$
	α_3	$\Gamma/24.2 = 1080$		$[\alpha_3, \alpha_1]$	$\Gamma/24.2 = 1080$
	α_2	$\Gamma/6.6 = 1440$		α_2	$\Gamma/6.2 = 4320$
	α_1	$\Gamma/2.24 = 1080$		α_3	$\Gamma/24 = 2160$

As a check, we can apply the extension of Euler's theorem, obtaining

$$432-(1080+2160)+(1440+2160+4320)$$
$$-(720+1080+1080+2160+2160)+(216+432+432+270+1080)$$
$$-(27+216+27+72)=0.$$

9. When only one dot is ringed, the vertex figure* of the polytope can be obtained by the following rule. If all the links incident in the ringed dot are unmarked, remove it and them, and ring the dots to which they lead. (*E.g.*, the vertex figure of $\alpha_4 h$ or is or $t_{0,3}\alpha_4$, formerly called $e\alpha_4$.) On the other hand, if any of those links are marked (in which case the graph is necessarily a tree), the rule as just stated gives a forest, symbolizing a prism with one constituent for each removed link; whenever the link is marked k, the corresponding constituent must be given edge $2\cos\pi/k$; *e.g.*, the vertex figure of the cuboctahedron  is the rectangle or $[\alpha_1, \beta_1]$.

The calculations of Coxeter, **1**, 381-383, were based on the assumption that the polytopes n_{21} exist. We can now justify this assumption without the use of coordinates.

10. The vertex figure is again simple when *every* dot is ringed. If the polytope is finite, its vertex figure is a simplex having the same vertices as the spherical polar of the fundamental region. If the polytope is degenerate, its vertex figure is the reciprocal of the fundamental region with respect to a concentric† sphere of suitable radius.

11. Since the product of an even number of reflexions is a *positive* operation, it is always possible to regard the fundamental regions as being alternately black and white. We have seen that the graph with all dots ringed symbolizes a polytope with one vertex in every region. Another polytope can be constructed by taking a vertex in every region of one colour‡. But it is seldom possible to choose the position of the typical vertex in its region so as to make the polytope equilateral, and so uniform. (When this is possible, the vertex will not, in general, be the in-centre of the region.)

Apart from three semi-degenerates§, the actual cases are as follows.

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* Coxeter, **1**, 336.

† *I.e.*, concentric with the *inscribed* sphere that the region must have in this case.

‡ Klein, **1**, 23. Möbius, **1**, 656, 669, 688.

§ The *tetroctahedric semi-check* of Gosset, **1**, 47, corresponds to the graph that consists of a triangle and an isolated dot.

Fundamental region	Graph	Polytope
(2^{m-1})*	• • • ... (m dots)	$h\gamma_m$
γ_{m-1}	∞ ∞ ... ($m-1$ pieces)	$h\delta_m$
(k)	k	$\{k\}$
$(2, k)$	k	$\{k\}$-antiprism
(3, 3)		$\{3, 5\}$
(3, 4)	4	The snub cube $s\begin{Bmatrix}3\\4\end{Bmatrix}$
(3, 5)	5	The snub dodecahedron $s\begin{Bmatrix}3\\5\end{Bmatrix}$
(3, 6)	6	$s\begin{Bmatrix}3\\6\end{Bmatrix}$ (Kepler's† L)
(4, 4)	4 4	$s\begin{Bmatrix}4\\4\end{Bmatrix}$ (Kepler's N)
α_2		$\{3, 6\}$
$\begin{pmatrix}3\\3\\3\end{pmatrix}$		$s\{3, 4, 3\}$ (Gosset's *tetricosahedric*)‡
$[\alpha_2, \alpha_1\sqrt{\frac{2}{3}}]$	∞	Gosset's *complex tetroctahedric check*‡ (Andreini's§ Fig. 13)
$\begin{pmatrix}3\\3\\3\\3\end{pmatrix}$		A new degenerate polytope $s\{3, 4, 3, 3\}$

* Meaning (2, 2, ... , 2).
† Kepler, 1, 117 (XIX).
‡ Gosset, 1, 46.
§ Andreini, 1, 106.

We might symbolize such a *snub** polytope by ringing all the dots in the graph, and then removing the dots (leaving just the rings and links). For the dots represent reflections which are not symmetries of the polytope.

12. $s\{3, 4, 3\}$ is bounded by 24 icosahedra and 96+24 tetrahedra. Its 96 vertices can be selected from the 120 vertices of $\{3, 3, 5\}$ by removing 24 that belong to an inscribed $\{3, 4, 3\}$. The icosahedra are the actual vertex figures of $\{3, 3, 5\}$ at the 24 removed vertices, and the tetrahedra all occur among the 600 bounding tetrahedra of $\{3, 3, 5\}$.

$s\{3, 4, 3, 3\}$ has, at each vertex, five α_4's, one β_4, and four $s\{3, 4, 3\}$'s.

Mrs. Stott has pointed out that $s\{3, 4, 3\}$ can be constructed by taking one vertex in every edge of $\{3, 4, 3\}$, not at the mid-point (which would give $t_1\{3, 4, 3\}$), but at one of the two points of "golden section" (ratio $\tau : 1$†). $s\{3, 4, 3, 3\}$ is similarly derivable from $\{3, 4, 3, 3\}$. This process can be applied to any regular polytope for which k_2 is even, the remaining cases being

$$s\beta_3 = \{3, 5\} \text{ (ratio } \tau : 1), \quad s\delta_3 = s\begin{Bmatrix}4\\4\end{Bmatrix} \text{ (ratio } \sqrt{3} : 1),$$

$$s\{3, 6\} = \{3, 6\} \text{ (ratio } 2 : 1).$$

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* The word "snub" is a free translation of Kepler's *simus*.

† Here $\tau = \frac{1}{2}(\sqrt{5}+1)$. Convenient coordinates for the vertices of $s\{3, 4, 3\}$ (edge 2) are the even permutations of $\pm(\tau^2, \tau, 1, 0)$.

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ON THE REPRESENTATIONS OF A NUMBER AS THE SUM OF FOUR OR MORE N-NUMBERS

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Linfoot and Evelyn have investigated in a series of papers* the number $\nu_r(n)$ of representations of n as the sum of r N-numbers, an N-number being a positive integer not divisible by the N-th power of any prime. Putting

$$\nu_r(n) = \frac{n^{r-1}}{(r-1)!\,\zeta^r(N)}\, S(n)+\Delta, \tag{1}$$

where

$$S(n) = \prod_{p^N+n}\left\{1+\frac{(-1)^{r+1}}{(p^N-1)^r}\right\}\prod_{p^N|n}\left\{1+\frac{(-1)^r}{(p^N-1)^{r-1}}\right\}, \tag{2}$$

they show by elementary methods that

$$\Delta = O(n^{r-2+2/(N+1)+\epsilon}), \tag{3}$$

for every $\epsilon > 0$. By using Gelbcke's refinement of Winogradoff's method of dealing with Waring's problem, they find that

$$\Delta = O(n^{r-2+1/N+(N-1)/(Nr)+\epsilon}), \tag{4}$$

provided that $r > \sqrt{N}+1$. This is an improvement on (3) if $r \geqslant N+2$.

In this paper we use the method by which Hardy and Littlewood dealt with the representations of a number as a sum of primes, to prove that

$$\Delta = O(n^{r-2+1/N}\log^a n) \quad (r \geqslant 4), \tag{5}$$

where

$$a = \begin{cases} 3 & (N=2,\ r=4), \\ 2 & \text{otherwise.} \end{cases} \tag{6}$$

This is an improvement on both (3) and (4).

* For a list of these papers, see *Quart. J. of Math.*, 3 (1932), 152.

Note on the notation of sums. To avoid any ambiguity as to which are the variables of summation, the following rule has been observed throughout this paper:

To every Σ there is only one variable of summation, and it is always the first letter under the Σ.

1. *The function $\zeta(s, w)$ and its average order.*

Let $s = \sigma + it$, and let $A_1, \ldots$ denote sufficiently large positive numbers depending at most on N and r.

Let $0 < w \leqslant 1$, and let $\zeta(s, w)$ be defined for $\sigma > 1$ by

$$\zeta(s, w) = \sum_{m=1}^{\infty} (m+w)^{-s}, \tag{7}$$

and elsewhere by analytic continuation.

Then*, if x is half an odd positive integer,

$$\zeta(s, w) - \sum_{m<x} (m+w)^{-s} + \frac{(x+w)^{1-s}}{1-s}$$

$$= -\frac{1}{2i}\int_{x-i\infty}^{x} (\cot \pi z - i)(z+w)^{-s}\,dz - \frac{1}{2i}\int_{x}^{x+i\infty} (\cot \pi z + i)(z+w)^{-s}\,dz. \tag{8}$$

If $y > 0$,
$$|\cot \pi (x - iy) - i| < 2e^{-2\pi y}, \tag{9}$$

and
$$|(x - iy + w)^{-s}| \leqslant |x - iy + w|^{-\sigma} e^{y|t|/(x+w)}, \tag{10}$$

so that, if $\sigma > 0$ and $2\pi(x+w) > |t|$,

$$\left|\frac{1}{2i}\int_{x-i\infty}^{x} (\cot \pi z - i)(z+w)^{-s}\,dz\right| < (x+w)^{-\sigma}\int_0^{\infty} e^{-y\{2\pi - |t|/(x+w)\}}\,dy$$

$$= \frac{(x+w)^{-\sigma}}{2\pi - |t|/(x+w)}. \tag{11}$$

Similarly,

$$\left|\frac{1}{2i}\int_{x}^{x+i\infty} (\cot \pi z + i)(z+w)^{-s}\,dz\right| < \frac{(x+w)^{-\sigma}}{2\pi - |t|/(x+w)}, \tag{12}$$

and hence, by (8), (11), and (12),

$$\left|\zeta(s, w) - \sum_{m<x} (m+w)^{-s} + \frac{(x+w)^{1-s}}{1-s}\right| < \frac{2(x+w)^{-\sigma}}{2\pi - |t|/(x+w)}. \tag{13}$$

* *Cf.* Titchmarsh, *The zeta function of Riemann* (Cambridge, 1930), 2.12.

Now let $t>0$ and $\frac{1}{2}<\sigma<\max(1-1/N, \frac{2}{3})$, and take $x=[t]+\frac{1}{2}$. Then

$$\frac{2(x+w)^{-\sigma}}{2\pi-|t|/(x+w)}<\frac{2(\frac{1}{2})^{-\sigma}}{2\pi-2}<1, \tag{14}$$

and

$$\left|\frac{(x+w)^{1-s}}{1-s}\right|<\frac{t+\frac{3}{2}}{\sqrt{\{(1-\sigma)^2+t^2\}}}<A_1, \tag{15}$$

so that, by (13),

$$\left|\zeta(s, w)-\sum_{m\leqslant t}(m+w)^{-s}\right|<1+A_1,$$

which shows that, if $T>0$,

$$\int_0^T|\zeta(s, w)|\,dt<(1+A_1)\,T+\int_0^T\left|\sum_{m\leqslant t}(m+w)^{-s}\right|dt. \tag{16}$$

Now

$$\int_0^T\left|\sum_{m\leqslant t}(m+w)^{-s}\right|^2dt=\int_0^T\sum_{m\leqslant t}\sum_{n\leqslant t}(m+w)^{-\sigma-it}(n+w)^{-\sigma+it}dt$$

$$=I_1+I_2, \tag{17}$$

where

$$I_1=\int_0^T\sum_{m\leqslant t}(m+w)^{-2\sigma}\,dt<\zeta(2\sigma)\,T, \tag{18}$$

and

$$I_2=2\int_0^T\sum_{n\leqslant t}\sum_{m<n}\{(m+w)(n+w)\}^{-\sigma}\cos\left(t\log\frac{n+w}{m+w}\right)dt$$

$$=2\sum_{n\leqslant T}\sum_{m<n}(m+w)^{-\sigma}(n+w)^{-\sigma}\int_n^T\cos\left(t\log\frac{n+w}{m+w}\right)dt$$

$$\leqslant 4\sum_{n\leqslant T}(n+w)^{-\sigma}\sum_{m=1}^{n-1}(m+w)^{-\sigma}\left(\log\frac{n+w}{m+w}\right)^{-1}. \tag{19}$$

In this summation,

$$\log\frac{n+w}{m+w}>\frac{n-m}{n+w}, \tag{20}$$

and

$$\sum_{m=1}^{n-1}(m+w)^{-\sigma}(n-m)^{-1}\leqslant n^{1-\sigma}\sum_{m=1}^{n-1}m^{-1}(n-m)^{-1}$$

$$=2n^{-\sigma}\sum_{m=1}^{n-1}m^{-1}<2n^{-\sigma}(1+\log n). \tag{21}$$

so that, if $T \geqslant 1$,

$$I_2 \leqslant 8 \sum_{n \leqslant T} (n+w)^{1-\sigma} n^{-\sigma} (1+\log n)$$

$$\leqslant 16 (1+\log T) \sum_{n \leqslant T} n^{1-2\sigma}$$

$$< \frac{8}{1-\sigma} T^{2-2\sigma} (1+\log T). \tag{22}$$

We now put $\sigma - \frac{1}{2} = \epsilon$, so that $0 < \epsilon < \max(\frac{1}{2} - 1/N, \frac{1}{6})$; then, if $T \geqslant 1$,

$$I_2 < A_2 T (1+T^{-2\epsilon} \log T) < A_2 T \{1+(2\epsilon)^{-1}\} < A_2 \epsilon^{-1} T. \tag{23}$$

But if $0 < T < 1$, $I_2 = 0$, and so, if $T > 0$,

$$I_2 < A_2 \epsilon^{-1} T. \tag{24}$$

And, since $\zeta(2\sigma) = \zeta(1+2\epsilon) < 1+(2\epsilon)^{-1} < \epsilon^{-1}$, we have, by (17), (18), and (24),

$$\int_0^T \left| \sum_{m \leqslant t} (m+w)^{-s} \right|^2 dt < (1+A_2)\epsilon^{-1} T. \tag{25}$$

Schwarz's inequality thus gives

$$\int_0^T \left| \sum_{m \leqslant t} (m+w)^{-s} \right| dt < \sqrt{(1+A_2)}\, \epsilon^{-\frac{1}{2}} T, \tag{26}$$

so that, by (16) and (26),

$$\int_0^T |\zeta(s, w)|\, dt < A_3 \epsilon^{-\frac{1}{2}} T. \tag{27}$$

2. *The application of the Farey dissection.*

Let
$$F(\tau) = \sum_M e^{2\pi i M\tau} \quad (I\tau > 0), \tag{28}$$

where M runs through all N-numbers. Then $\nu_r(n)$ is the coefficient of $e^{2\pi i n\tau}$ in the expansion of $F^r(\tau)$ in powers of $e^{2\pi i\tau}$, so that

$$\nu_r(n) = \int_{i/n}^{i/n+1} F^r(\tau)\, e^{-2\pi i n\tau}\, d\tau. \tag{29}$$

Dividing the interval (0, 1) by means of the Farey series of order $[\sqrt{n}]$, we thus have

$$\nu_r(n) = \sum_{q \leqslant \sqrt{n}} \sum_m{}' \xi_q^{-mn} \int_{i/n-\vartheta_1}^{i/n+\vartheta_2} F^r(m/q+u)\, e^{-2\pi i n u}\, du, \tag{30}$$

where the accent indicates that m runs through the positive integers not greater than q and prime to q,

$$\xi_q = e^{2\pi i/q}, \tag{31}$$

and ϑ_1, ϑ_2 are functions of n, m, and q such that

$$(2q\sqrt{n})^{-1} \leqslant \vartheta_l < (q\sqrt{n})^{-1} \quad (l = 1, 2). \tag{32}$$

3. *The connection between* $F(m/q+u)$ and $\zeta(s, w)$.

By (28) and (31),

$$F(m/q+u) = \sum_M \xi_q^{mM} e^{2\pi i Mu}. \tag{33}$$

Now, by Mellin's integral formula,

$$e^{2\pi i Mu} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Gamma(s)(-2\pi i Mu)^{-s}\,ds,$$

so that
$$F(m/q+u) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Gamma(s)\,D(s)(-2\pi i u)^{-s}\,ds, \tag{34}$$

where
$$D(s) = \sum_M \xi_q^{mM} M^{-s}. \tag{35}$$

If $\mu(h)$ denotes Möbius's function, we have $\sum_{h^N|x} \mu(h) = 1$ or 0 according as x is or is not an N-number. It follows that

$$\begin{aligned} D(s) &= \sum_{h=1}^{\infty}\sum_{l=1}^{\infty} \mu(h)\,\xi_q^{mhNl}\,(h^N l)^{-s} \\ &= \sum_{b=1}^{q} \sum_{l\equiv b \,(\mathrm{mod}\, q)} l^{-s} \sum_{h=1}^{\infty} \mu(h)\,\xi_q^{mh^N b}\,h^{-Ns}. \end{aligned} \tag{36}$$

But, by (7),

$$\sum_{l\equiv b \,(\mathrm{mod}\, q)} l^{-s} = \sum_{k=0}^{\infty} (b+kq)^{-s} = b^{-s} + q^{-s}\,\zeta(s, b/q), \tag{37}$$

so that, by (36),

$$D(s) = \sum_{b=1}^{q} \{b^{-s} + q^{-s}\,\zeta(s, b/q)\}\,G_b(s), \tag{38}$$

where
$$G_b(s) = \sum_{h=1}^{\infty} \mu(h)\,\xi_q^{mh^N b}\,h^{-Ns} \quad (\sigma > 1/N). \tag{39}$$

This shows that $D(s)$ is regular for $\sigma > 1/N$ except at the point $s = 1$, and that the integrand in (34) tends to zero uniformly in the strip

$1/N+1/\log n \leqslant \sigma \leqslant 2$, when $t \to \pm\infty$. Hence

$$F(m/q+u) = Z+U, \tag{40}$$

where Z is the residue of $\Gamma(s)\,D(s)\,(-2\pi iu)^{-s}$ at $s=1$, and

$$U = \frac{1}{2\pi i}\int_{1/N+1/\log n-i\infty}^{1/N+1/\log n+i\infty} \Gamma(s)\,D(s)\,(-2\pi iu)^{-s}\,ds \tag{41}$$

$$\{\log n > (1-1/N)^{-1}\}.$$

4. *The evaluation of Z.*

The function $\zeta(s, b/q)$ has a simple pole of residue 1 at $s=1$, so that, by (38), $D(s)$ has a simple pole of residue $q^{-1}\sum_{b=1}^{q} G_b(1)$ at $s=1$, and hence

$$\begin{aligned} Z &= (-2\pi iqu)^{-1}\sum_{b=1}^{q} G_b(1) \\ &= (-2\pi iqu)^{-1}\sum_{b=1}^{q}\sum_{h=1}^{\infty}\mu(h)\,\xi_q^{mh^N b}\,h^{-N} \\ &= (-2\pi iqu)^{-1}\sum_{h=1}^{\infty}\mu(h)\,h^{-N}\sum_{b=1}^{q}\xi_q^{mh^N b} \\ &= (-2\pi iu)^{-1}\sum_{h^N\equiv 0 \pmod q}\mu(h)\,h^{-N}. \end{aligned} \tag{42}$$

This last sum is zero if q is not an $(N+1)$-number. If q is an $(N+1)$-number, let q_1 be its greatest quadratfrei divisor. Then

$$\begin{aligned} \sum_{h^N\equiv 0 \pmod q}\mu(h)\,h^{-N} &= \sum_{h\equiv 0 \pmod{q_1}}\mu(h)\,h^{-N} = \sum_{l=1}^{\infty}\mu(q_1 l)\,(q_1 l)^{-N} \\ &= \mu(q_1)\,q_1^{-N}\sum_{(l,\,q_1)=1}\mu(l)\,l^{-N} \\ &= \mu(q_1)\,q_1^{-N}\,\frac{1}{\zeta(N)\prod_{p\mid q}(1-p^{-N})}. \end{aligned}$$

We thus have, for every q,

$$Z = \frac{1}{\zeta(N)}\,\mu(q_1)(-2\pi iu)^{-1}\,E_N(q)\prod_{p\mid q}(p^N-1)^{-1}, \tag{43}$$

where

$$q_1 = \prod_{p\mid q} p, \tag{44}$$

and $E_N(q)=1$ or 0 according as q is or is not an $(N+1)$-number.

5. *The inequality for U.*

Henceforward it will be supposed that

$$n > A_4, \tag{45}$$

and

$$r \geqslant 4. \tag{46}$$

We shall now show that

$$|U| < A_5 n \log^{\beta} n \,(q|u|)^{1-1/N}, \tag{47}$$

where

$$\beta = \begin{cases} \frac{3}{2} & (N=2), \\ 1 & (N>2), \end{cases} \tag{48}$$

but the method of proof of this inequality differs in the cases $N=2$, $N>2$. First suppose that $N=2$.

By Stirling's formula, we have

$$\begin{aligned} \Gamma(\tfrac{1}{2}+u+it) &= O\{|(\tfrac{1}{2}+u+it)^{u+it} e^{-\frac{1}{2}-u-it}|\} \\ &= O\{(1+|t|)^u e^{-t\,\mathrm{am}(\frac{1}{2}+u+it)}\}, \end{aligned}$$

uniformly in the strip $0 < u < \frac{1}{2}$, and

$$\begin{aligned} t\,\mathrm{am}(\tfrac{1}{2}+u+it) &= |t| \arctan \frac{2|t|}{1+2u} \\ &= |t| \left(\tfrac{1}{2}\pi - \arctan \frac{1+2u}{2|t|}\right) > \tfrac{1}{2}\pi|t| - \tfrac{1}{2} - u > \tfrac{1}{2}\pi|t| - 1. \end{aligned}$$

Hence

$$|\Gamma(\tfrac{1}{2}+1/\log n+it)| < A_6 e^{-\frac{1}{2}\pi|t|}(1+|t|)^{1/\log n}. \tag{49}$$

Let

$$u = i/n + \vartheta. \tag{50}$$

Then

$$|\mathrm{am}(-iu)| = \tfrac{1}{2}\pi - \arcsin\{1/(n|u|)\} < \tfrac{1}{2}\pi - (n|u|)^{-1},$$

so that, if $\sigma > 0$,

$$|(-2\pi iu)^{-s}| < |u|^{-\sigma} e^{|t|\,|\mathrm{am}(-iu)|} < |u|^{-\sigma} e^{\frac{1}{2}\pi|t| - |t|/(n|u|)}. \tag{51}$$

Also, by (39) and (45),

$$|G_b(\tfrac{1}{2}+1/\log n+it)| < \zeta(1+2/\log n) < \log n,$$

and hence, by (38),

$$\begin{aligned} |D(\tfrac{1}{2}+1/\log n+it)| &< \log n \sum_{b=1}^{q} b^{-\frac{1}{2}} + \log n\, q^{-\frac{1}{2}} \sum_{b=1}^{q} |\zeta(\tfrac{1}{2}+1/\log n+it,\, b/q)| \\ &< 2q^{\frac{1}{2}} \log n + q^{-\frac{1}{2}} \log n \sum_{b=1}^{q} |\zeta(\tfrac{1}{2}+1/\log n+it,\, b/q)|, \end{aligned}$$

so that, by (27),

$$\int_0^T |D(\tfrac{1}{2}+1/\log n+it)|\,dt < A_7 q^{\frac{1}{2}} \log^{\frac{3}{2}} n \,.\, T. \tag{52}$$

By (49) and (51),

$$\left| \int_{\frac{1}{2}+1/\log n}^{\frac{1}{2}+1/\log n+i\infty} \Gamma(s) D(s) (-2\pi i u)^{-s} ds \right|$$

$$\leqslant A_6 |u|^{-\frac{1}{2}-1/\log n} \int_0^\infty (1+t)^{1/\log n} e^{-t/(n|u|)} |D(\tfrac{1}{2}+1/\log n+it)|\,dt$$

$$= A_6 |u|^{-\frac{1}{2}-1/\log n} \int_0^\infty f(t)\, g'(t)\, dt$$

$$= -A_6 |u|^{-\frac{1}{2}-1/\log n} \int_0^\infty f'(t)\, g(t)\, dt, \tag{53}$$

where
$$f(t) = (1+t)^{1/\log n} e^{-t/(n|u|)}, \tag{54}$$

and
$$g(t) = \int_0^t |D(\tfrac{1}{2}+1/\log n+ix)|\,dx. \tag{55}$$

Now, by (55) and (52),

$$0 \leqslant g(t) < A_7 q^{\frac{1}{2}} \log^{\frac{3}{2}} n \,.\, t,$$

and, by (54), $$-f'(t) < (n|u|)^{-1} (1+t)^{1/\log n} e^{-t/(n|u|)},$$

so that, by (53),

$$\left| \int_{\frac{1}{2}+1/\log n}^{\frac{1}{2}+1/\log n+i\infty} \Gamma(s) D(s) (-2\pi i u)^{-s} ds \right|$$

$$< A_6 A_7 |u|^{-\frac{3}{2}-1/\log n} q^{\frac{1}{2}} n^{-1} \log^{\frac{3}{2}} n \int_0^\infty (1+t)^{1+1/\log n} e^{-t/(n|u|)} dt. \tag{56}$$

Since, by (50), $n|u| \geqslant 1$,

$$\int_0^\infty (1+t)^{1+1/\log n} e^{-t/(n|u|)} dt$$

$$\leqslant e \int_0^\infty (1+t)^{1+1/\log n} e^{-(1+t)/(n|u|)} dt$$

$$= e \int_1^\infty t^{1+1/\log n} e^{-t/(n|u|)} dt < e(n|u|)^{2+1/\log n}\, \Gamma(2+1/\log n). \tag{57}$$

Hence, by (56), (57), and (45),

$$\left| \int_{\frac{1}{2}+1/\log n+i\infty}^{\frac{1}{2}+1/\log n} \Gamma(s)\, D(s)(-2\pi i u)^{-s}\, ds \right| < A_5(q|u|)^{\frac{1}{2}} n \log^{\frac{3}{2}} n. \tag{58}$$

A similar inequality clearly holds for the other part of the integral defining U, so that (58) and (48) prove (47) in the case $N = 2$.

Now consider the case $N > 2$.

Let
$$\zeta(s;\ b,\ q) = b^{-s} + q^{-s}\zeta(s,\ b/q). \tag{59}$$

This function satisfies a functional equation which can be obtained in the same way as that for Riemann's ζ-function. It is

$$\zeta(s;\ b,\ q) = -i(2\pi)^{s-1} q^{-s}\, \Gamma(1-s) \sum_{l=1}^{q} \zeta(1-s;\ l,\ q)(e^{\frac{1}{2}i\pi s}\, \xi_q^{lb} - e^{-\frac{1}{2}i\pi s}\, \xi_q^{-lb}). \tag{60}$$

We thus have, by (38), (59), and (60),

$$\Gamma(s)\, D(s) = -i(2\pi)^{s-1} q^{-s}\, \pi \operatorname{cosec} \pi s \sum_{l=1}^{q} \zeta(1-s;\ l,\ q)$$

$$\times \left\{ e^{\frac{1}{2}i\pi s} \sum_{b=1}^{q} \xi_q^{lb}\, G_b(s) - e^{-\frac{1}{2}i\pi s} \sum_{b=1}^{q} \xi_q^{-lb}\, G_b(s) \right\}. \tag{61}$$

Now, by (39),

$$\sum_{b=1}^{q} \xi_q^{lb}\, G_b(s) = \sum_{h=1}^{\infty} \mu(h)\, h^{-Ns} \sum_{b=1}^{q} \xi_q^{b(l+mh^N)} = q \sum_{h^N m \equiv -l \pmod q} \mu(h)\, h^{-Ns}, \tag{62}$$

and, similarly,

$$\sum_{b=1}^{q} \xi_q^{-lb}\, G_b(s) = q \sum_{h^N m \equiv l \pmod q} \mu(h)\, h^{-Ns}. \tag{63}$$

If l_1, l_2 denote the least positive residues mod q of $-h^N m$, $h^N m$ respectively, we thus have

$$\sum_{l=1}^{q} \zeta(1-s;\ l,\ q) \sum_{b=1}^{q} \xi_q^{lb}\, G_b(s) = q \sum_{h=1}^{\infty} \mu(h)\, h^{-Ns}\, \zeta(1-s;\ l_1,\ q). \tag{64}$$

Hence

$$\operatorname{cosec} \pi s \sum_{l=1}^{q} \zeta(1-s;\ l,\ q)\, e^{\frac{1}{2}i\pi s} \sum_{b=1}^{q} \xi_q^{lb}\, G_b(s)$$

$$= O\left\{ e^{-\frac{1}{2}\pi|t|} q \sum_{h=1}^{\infty} h^{-N\sigma} |\zeta(1-s;\ l_1,\ q)| \right\},$$

and, similarly,

$$\operatorname{cosec}\pi s \sum_{l=1}^{q} \zeta(1-s;\ l,\ q)\, e^{-\frac{1}{2}i\pi s} \sum_{b=1}^{q} \xi_q^{-lb}\, G_b(s)$$

$$= O\left\{ e^{-\frac{1}{2}\pi|t|} q \sum_{h=1}^{\infty} h^{-N\sigma} |\zeta(1-s;\ l_2,\ q)| \right\},$$

so that, by (61), if $\sigma = 1/N + 1/\log n$,

$$|\Gamma(s)\, D(s)| < A_8\, q^{1-\sigma} e^{-\frac{1}{2}\pi|t|} \sum_{h=1}^{\infty} h^{-N\sigma} \sum_{\mu=1}^{2} |\zeta(1-s;\ l_\mu,\ q)|. \tag{65}$$

Since (51) is still true, it follows that

$$\left| \int_{1/N+1/\log n}^{1/N+1/\log n+i\infty} \Gamma(s)\, D(s)\, (-2\pi i u)^{-s}\, ds \right|$$

$$\leqslant A_8\, q^{1-1/N} |u|^{-1/N-1/\log n} \sum_{h=1}^{\infty} h^{-1-N/\log n} \int_0^{\infty} e^{-t/(n|u|)}$$

$$\times \{g_1'(t) + g_2'(t)\}\, dt, \tag{66}$$

where

$$g_\mu(t) = \int_0^t |\zeta(1-1/N-1/\log n - ix;\ l_\mu,\ q)|\, dx \quad (\mu = 1,\ 2). \tag{67}$$

By (67), (59), (45), and (27),

$$0 \leqslant g_\mu(t) < A_9\, t,$$

and hence

$$\int_0^{\infty} e^{-t/(n|u|)}\, g_\mu'(t)\, dt = (n|u|)^{-1} \int_0^{\infty} e^{-t/(n|u|)}\, g_\mu(t)\, dt$$

$$< A_9 (n|u|)^{-1} \int_0^{\infty} t e^{-t/(n|u|)}\, dt = A_9\, n|u|, \tag{68}$$

so that, since $n|u| \geqslant 1$, we have, by (66), (68), and (45),

$$\left| \int_{1/N+1/\log n}^{1/N+1/\log n+i\infty} \Gamma(s)\, D(s) (-2\pi i u)^{-s}\, ds \right| < A_{10} (q|u|)^{1-1/N} n \sum_{h=1}^{\infty} h^{-1-N/\log n}$$

$$< A_{10} (q|u|)^{1-1/N} n \left(1 + \frac{\log n}{N}\right)$$

$$< A_{10} (q|u|)^{1-1/N} n \log n. \tag{69}$$

As before, the other portion of the integral defining U satisfies the same inequality, so that (69) and (48) establish (47) in the case $N > 2$. The inequality (47) is thus true for every N.

6. *The completion of the proof.*

We have, by (40),

$$|F^r(m/q+u)-Z^r| \leqslant r|U|\{|F(m/q+u)|^{r-1}+|Z|^{r-1}\}. \tag{70}$$

To make use of this relation, we need inequalities for $F(m/q+u)$ and Z. It follows from (44) that

$$\prod_{p|q}(p^N-1)^{-1} < \zeta(N)\prod_{p|q}p^{-N} = \zeta(N)q_1^{-N},$$

and that $q_1^N \geqslant q$ if q is an $(N+1)$-number, so that (43) shows that

$$|Z| < (q|u|)^{-1}. \tag{71}$$

Hence, by (46), (47), (48), and (6), since $|u| \geqslant n^{-1}$,

$$\begin{aligned}|U||Z|^{r-1} &< A_5(q|u|)^{2-r-1/N}n\log^\beta n\\ &\leqslant A_5 q^{-2-1/N}|u|^{-2}n^{r-3+1/N}\log^\alpha n.\end{aligned} \tag{72}$$

It follows from (28) that, if ϑ is real,

$$|F(i/n+\vartheta)| < \sum_{m=1}^{\infty} e^{-2\pi m/n} = (e^{2\pi/n}-1)^{-1} < n. \tag{73}$$

Also, by (40), (47), and (71),

$$\begin{aligned}|U||F(m/q+u)| &\leqslant |U|^2+|U||Z|\\ &< A_5{}^2(q|u|)^{2-2/N}n^2\log^{2\beta}n+A_5(q|u|)^{-1/N}n\log^\beta n,\end{aligned}$$

and in the integration in (30), $u = i/n+\vartheta$, where

$$-(q\sqrt{n})^{-1} < \vartheta < (q\sqrt{n})^{-1},$$

and $q \leqslant \sqrt{n}$, so that

$$n^{-1} \leqslant q|u| < 2n^{-\frac{1}{2}}. \tag{74}$$

Hence $$|U||F(m/q+u)| < A_{11}n^{1+1/N}\log^{2\beta}n,$$

so that, by (73) and (46),

$$|U||F(m/q+u)|^{r-1} \leqslant A_{11} n^{r-3+1/N} \log^{2\beta} n\, |F(m/q+u)|^2. \tag{75}$$

Again, by (40), (47), (71), and (74),

$$\begin{aligned} |U||F(m/q+u)|^2 &\leqslant 4(|U|^3+|U||Z|^2) \\ &< A_{12} n^3 \log^{3\beta} n (q|u|)^{3-3/N} + A_{12} n \log^{\beta} n (q|u|)^{-1-1/N} \\ &< 8A_{12} n^{\frac{3}{2}+3/(2N)} \log^{3\beta} n + A_{12} n^{2+1/N} \log^{\beta} n \\ &< A_{13} n^{2+1/N} \log^{\beta} n, \end{aligned}$$

so that, by (73), we have, if $r \geqslant 5$,

$$|U||F(m/q+u)|^{r-1} \leqslant A_{13} n^{r-3+1/N} \log^{\beta} n\, |F(m/q+u)|^2. \tag{76}$$

By (75), (76), (48), and (6),

$$|U||F(m/q+u)|^{r-1} \leqslant A_{14} n^{r-3+1/N} \log^{\alpha} n\, |F(m/q+u)|^2. \tag{77}$$

Now, by definition of ϑ_1 and ϑ_2, and by (28),

$$\begin{aligned} \sum_{q \leqslant \sqrt{n}} \sum_{m}{}' \int_{i/n-\vartheta_1}^{i/n+\vartheta_2} |F(m/q+u)|^2\, du &= \int_0^1 |F(i/n+\vartheta)|^2\, d\vartheta \\ &= \int_0^1 \sum_{M} \sum_{M'} e^{-2\pi(M+M')/n} e^{2\pi i(M-M')\vartheta}\, d\vartheta \\ &= \sum_{M} e^{-4\pi M/n} < \sum_{m=1}^{\infty} e^{-4\pi m/n} < n. \end{aligned} \tag{78}$$

Again,

$$\sum_{q \leqslant \sqrt{n}} q^{-2-1/N} \sum_{m}{}' \int_{i/n-\vartheta_1}^{i/n+\vartheta_2} |u|^{-2}\, du < \sum_{q \leqslant \sqrt{n}} q^{-1-1/N} \int_{-\infty}^{\infty} \frac{d\vartheta}{n^{-2}+\vartheta^2} < A_{15} n. \tag{79}$$

It now appears, from (30), (31), (70), (72), (77), (78), and (79), that

$$\left| \nu_r(n) - \sum_{q \leqslant \sqrt{n}} \sum_{m}{}' \xi_q^{-mn} \int_{i/n-\vartheta_1}^{i/n+\vartheta_2} Z^r e^{-2\pi i n u}\, du \right| < A_{16} n^{r-2+1/N} \log^{\alpha} n. \tag{80}$$

Let

$$b_q = \int_{i/n-\infty}^{i/n+\infty} Z^r e^{-2\pi i n u}\, du. \tag{81}$$

Then, by (71) and (32),

$$\left| b_q - \int_{i/n-\vartheta_1}^{i/n+\vartheta_2} Z^r e^{-2\pi i n u}\, du \right| < 2e^{2\pi} \int_{(2q\sqrt{n})^{-1}}^{\infty} q^{-r} \vartheta^{-r}\, d\vartheta < 2^r e^{2\pi} q^{-1} n^{\frac{1}{2}(r-1)},$$

so that, by (80) and (46),

$$\left| \nu_r(n) - \sum_{q \leqslant \sqrt{n}} \sum_{m}{}' \xi_q^{-mn} b_q \right| < A_{17} n^{r-2+1/N} \log^{\alpha} n. \tag{82}$$

It easily follows from (81) and (43) that b_q is equal to $-2\pi i$ multiplied by the residue of $Z^r e^{-2\pi i n u}$ at $u = 0$, so that

$$b_q = -2\pi i \zeta^{-r}(N) \mu^r(q_1)(-2\pi i)^{-r} E_N(q) \prod_{p|q} (p^N - 1)^{-r} \frac{(-2\pi i n)^{r-1}}{(r-1)!}$$

$$= \frac{n^{r-1}}{\zeta^r(N)(r-1)!} \mu^r(q_1) E_N(q) \prod_{p|q} (p^N - 1)^{-r}. \tag{83}$$

With Ramanujan*, we put

$$c_q(n) = \sum_{m}{}' \xi_q^{-mn}, \tag{84}$$

so that, by (82) and (83),

$$\left| \nu_r(n) - \sum_{q \leqslant \sqrt{n}} b_q c_q(n) \right| < A_{17} n^{r-2+1/N} \log^{\alpha} n. \tag{85}$$

Now it is clear, from (83) and the calculations which led to (71), that $|b_q| < q^{-r} n^{r-1}$, so that, by (84),

$$\left| \sum_{q > \sqrt{n}} b_q c_q(n) \right| < n^{r-1} \sum_{q > \sqrt{n}} q^{1-r} < A_{18} n^{\frac{1}{2}r},$$

and hence, by (85) and (46),

$$\left| \nu_r(n) - \sum_{q=1}^{\infty} b_q c_q(n) \right| < A_{19} n^{r-2+1/N} \log^{\alpha} n. \tag{86}$$

It is known that $c_q(n)$ is a multiplicative function of q, *i.e.* if $(l, m) = 1$, then $c_l(n) c_m(n) = c_{lm}(n)$.

The same is true of $E_N(q) \prod_{p|q} (p^N - 1)^{-r}$. Now, if p is a prime, and $l \geqslant 1$, we have

$$E_N(p^l) = \begin{cases} 1 & (l \leqslant N), \\ 0 & (l > N); \end{cases}$$

and, by (84),

$$c_{p^l}(n) = \begin{cases} p^l - p^{l-1} & (p^l | n), \\ -p^{l-1} & (p^l \nmid n,\ p^{l-1} | n), \\ 0 & (p^{l-1} \nmid n). \end{cases}$$

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* *Trans. Camb. Phil. Soc.*, 22 (1918), 259.

Hence, by (83),

$$\sum_{q=1}^{\infty} b_q c_q(n) = \frac{n^{r-1}}{(r-1)!\,\zeta^r(N)} \prod_p B(p), \tag{87}$$

where

$$B(p) = 1+(-1)^r(p^N-1)^{-r} \sum_{l=1}^{N} c_{p^l}(n) = \begin{cases} 1+(-1)^r(p^N-1)^{-r+1} & (p^N \mid n), \\ 1-(-1)^r(p^N-1)^{-r} & (p^N \nmid n), \end{cases}$$

so that, by (2),

$$\prod_p B(p) = S(n). \tag{88}$$

By (1), (88), and (87),

$$\Delta = \nu_r(n) - \sum_{q=1}^{\infty} b_q c_q(n),$$

so that, by (86),

$$|\Delta| < A_{19} n^{r-2+1/N} \log^{\alpha} n,$$

which implies (5).

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THE NORMAL FORM OF COMPOUND AND INDUCED MATRICES

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1. *Notations and properties.*

The matrices which form the subject of the present communication have been considered* from the point of view of determinants by Cauchy, Schläfli, Sylvester, Zehfuss, and various later writers, and from the point of view of matrices, their latent roots being in question, by Rados, Franklin, Metzler, Stephanos, Burnside, and others. Our present object is to go beyond this and to investigate the elementary divisors of the characteristic determinants of the matrices, which specify completely their invariance under transformations of the type HAH^{-1}, in brief, to obtain the classical canonical form of the matrices.

The matrices which we shall consider are related either to a single generating matrix $A = [a_{ij}]$, in some prescribed field and of given canonical form, or else to a set of such matrices $A_j, j = 1, 2, \ldots, m$, not necessarily all of the same order. Interest in any matrix will centre throughout upon its classical canonical form $C = HAH^{-1}, |H| \neq 0$, and it will be convenient to represent this by the various notations, *e.g.*

$$C = \begin{bmatrix} \lambda_1 & 1 & & & & & \\ & \lambda_1 & 1 & & & & \\ & & \lambda_1 & & & & \\ & & & \lambda_2 & 1 & & \\ & & & & \lambda_2 & & \\ & & & & & \lambda_3 & \end{bmatrix}$$

$$= \operatorname{diag}[C_3(\lambda_1),\ C_2(\lambda_2),\ C_1(\lambda_3)]$$

$$= [(1\ 2\ 3)_{\lambda_1}(4\ 5)_{\lambda_2}(6)_{\lambda_3}].$$

* A list of references is given at the end of this paper.

Special use will be made of the last notation, of umbral brackets exhibiting the row and column indices of elements of C as pertaining to certain *categories* corresponding to the isolated classical sub-matrices $C_p(\lambda_i)$ from which C is constructed. These umbral brackets will be used to express the canonical character not merely of C but also of A, or of any matrix KAK^{-1} similar to A and C.

The matrices which will be considered in relation to A, or to the set A_j, have the following definitions and relevant properties.

(i) *The m-th compound of A, denoted by* $A^{(m)}$.

If A is a rectangular matrix of order $p \times q$, the elements of $A^{(m)}$ are the $\binom{p}{m} \times \binom{q}{m}$ minor determinants of order m contained in A. They are placed in the rows and columns of $A^{(m)}$ according to the rows and columns of A from which they derive, by any rule of index-priority which is the same for rows as for columns, so that, for example, if A were a square symmetric or Hermitian matrix, then $A^{(m)}$ would likewise be symmetric or Hermitian. For the time being we leave aside the question of specifying the precise rule, since it does not affect the canonical form. It may be noted that, if $p = m$, then $A^{(m)}$ is a row vector, while, if $q = m$, then $A^{(m)}$ is a column vector.

Compound matrices, whether square or rectangular, obey a fundamental multiplicative law, namely

$$(AB)^{(m)} = A^{(m)} B^{(m)},$$

which can be extended by iteration to the continued product of any number of conformable rectangular matrices. This law, which we shall establish for rectangular matrices of orders $p \times q$ and $q \times r$, is almost synonymous with the Binet-Cauchy theorem on the multiplication of rectangular arrays. By means of that theorem it can be simply proved as follows:

From a matrix X of order $r \times m$, composed of m arbitrary column vectors

$$x_{(i)} = \{x_{1i}, x_{2i}, \ldots, x_{ri}\} \qquad (i = 1, 2, \ldots, m),$$

let the m-th compound be constructed, namely

$$X^{(m)} = \{|x_{11} x_{22} \ldots x_{mm}|, \ \ldots, \ |x_{r-m+1, r-m+1} \ldots x_{rr}|\}.$$

This is a column vector, the components or elements being the $\binom{r}{m}$

determinants contained in the m columns $x_{(i)}$. Now the ordinary Binet-Cauchy theorem yields the identity

$$(BX)^{(m)} = B^{(m)} X^{(m)},$$

as can be verified at once. Hence we have

$$(ABX)^{(m)} = (AB)^{(m)} X^{(m)},$$

but also $$(ABX)^{(m)} = A^{(m)} (BX)^{(m)} = A^{(m)} B^{(m)} X^{(m)}.$$

The vectors $x_{(i)}$ being arbitrary, we derive at once the result

$$(AB)^{(m)} = A^{(m)} B^{(m)}.$$

In particular, for a square matrix A and its canonical form C we have

$$\begin{aligned} C^{(m)} &= H^{(m)} A^{(m)} (H^{-1})^{(m)} \\ &= H^{(m)} A^{(m)} [H^{(m)}]^{-1} \qquad (|H^{(m)}| \neq 0), \end{aligned}$$

$H^{(m)}$ being non-singular because of Sylvester's theorem that $|H^{(m)}|$ is a power of $|H|$. It follows that $C^{(m)}$ is similar to $A^{(m)}$, though in general, as we shall see, it is not as it stands in the finally reduced classical shape.

(ii) *The m-th induced, or Schläflian, matrix of* A, $A^{[m]}$.

Here we agree to regard the elements in the i-th row of A as the coefficients of q arbitrary variables x_j in the linear form

$$y_i = a_{i1} x_1 + a_{i2} x_2 + \ldots + a_{iq} x_q.$$

Constructing from the y_i all possible powers and products of degree m, we have altogether $\binom{p+m-1}{m}$ forms in $\binom{q+m-1}{m}$ variables, namely powers and products of the x_j. The matrix of this product system is defined to be $A^{[m]}$.

Schläflian matrices, which occur in invariant theory in connection with induced substitutions, also obey a multiplicative law. For in relation to an arbitrary vector

$$x = \{x_1, x_2, \ldots, x_r\}$$

we may construct the m-th Schläflian, a vector of powers and products,

$$x^{[m]} = \{x_1^m, x_1^{m-1} x_2, \ldots, x_r^m\}.$$

Then the relation which defines the m-th Schläflian matrix $B^{[m]}$ is

$$(Bx)^{[m]} = B^{[m]} x^{[m]}.$$

Hence we have $$(ABx)^{[m]} = (AB)^{[m]} x^{[m]},$$

but also $$(ABx)^{[m]} = A^{[m]}(Bx)^{[m]} = A^{[m]} B^{[m]} x^{[m]}.$$

It follows, exactly as before, that

$$(AB)^{[m]} = A^{[m]} B^{[m]},$$

a result* which can be extended to the product of any number of conformable rectangular matrices. In particular, for a square matrix A and its classical canonical form C we have

$$\begin{aligned} C^{[m]} &= H^{[m]} A^{[m]} (H^{-1})^{[m]} \\ &= H^{[m]} A^{[m]} [H^{[m]}]^{-1} \qquad (|H^{[m]}| \neq 0), \end{aligned}$$

$H^{[m]}$ being non-singular in consequence of Schläfli's theorem that $|H^{[m]}|$ is a power of $|H|$. Thus $C^{[m]}$ is similar to $A^{[m]}$, but once again it will be found that $C^{[m]}$ is not in general in a finally reduced classical shape.

(iii) *The multiplicative composite, or Zehfuss matrix, $M(A_j)_m$.*

Here we consider m different matrices A_j, $j = 1, 2, \ldots, m$, in general rectangular and not necessarily conformable, and we regard the row vectors of each matrix A_j as being associated, as in (ii) above, with linear forms in an arbitrary set of variables, there being in all m different sets of such variables, one for each matrix A_j. Taking one linear form from each set in all possible ways, and in each case constructing the m-ary product of the forms taken, we derive a set of multilinear forms ("direct product system") in corresponding multilinear variables. The matrix of the system so obtained will be denoted by $M(A_1, A_2, \ldots, A_m)$, or briefly by $M(A_j)_m$, or even by $M(A_j)$. The matrix occurs in group theory and in the multiplicative composition of vector spaces.

It is sufficient to consider the case $m = 2$, general results following by iterating the procedure. If then, for example, we take two arbitrary vectors

$$x = \{x_1, x_2, \ldots, x_p\}, \quad y = \{y_1, y_2, \ldots, y_q\},$$

where p and q are the respective numbers of columns in B_1 and B_2, matrices capable of multiplication by A_1 and A_2 respectively, we may construct a

* A. Hurwitz, *Math. Annalen*, 45 (1894), 381–404 (391).

multiplicative compound vector of pq elements,

$$M(x;\ y) = \{x_1 y_1,\ x_1 y_2,\ \ldots,\ x_p y_q\}.$$

Then the relation which defines $M(B_1,\ B_2)$ is

$$M(B_1 x;\ B_2 y) = M(B_1,\ B_2)\, M(x;\ y).$$

The multiplicative law, given by Hurwitz, *loc. cit.*, 389,

$$M(A_1 B_1,\ A_2 B_2) = M(A_1,\ A_2)\ M(B_1,\ B_2)$$

follows in the same way as before, with the consequence, for square matrices of given canonical form,

$$\begin{aligned} M(C_1,\ C_2) &= M(H_1 A_1 H_1^{-1},\ H_2 A_2 H_2^{-1}) \\ &= M(H_1,\ H_2)\ M(A_1,\ A_2)\ [M(H_1,\ H_2)]^{-1}, \end{aligned}$$

where, by a theorem of Zehfuss, $|M(H_1,\ H_2)|$, being a product of powers of $|H_1|$ and $|H_2|$, is not zero. Hence $M(C_1,\ C_2)$ is similar to $M(A_1,\ A_2)$, but once again it is not in general in classical shape, and must be further reduced.

If we make all the matrices A_j identical with A, we have a special case, considered from the point of view of latent roots by Burnside. By making certain sub-sets of the matrices A_j identical, or again by making certain sub-sets of the variables in the linear forms identical, one may construct various hybrid matrices intermediate between Schläflian, Burnside, and Zehfuss matrices. The value of the determinant of the most general of these has been found by Rosebrugh. The methods of the preceding paragraphs can be used to obtain multiplicative laws for these types, and the elementary divisors can be obtained by suitable modifications of the rules laid down in later sections. We shall not go into detail concerning such types; the appearance of generality would be specious, since even the most general of these hybrids is merely a Zehfuss matrix based on a set of Schläflian matrices.

The rule arrived at below for finding the elementary divisors of the m-th compound of a non-singular matrix was conjectured by the author some years ago, in connection with the solution of algebraic equations with multiple roots. A proof incomplete in certain respects has been given in a thesis* by Miss N. I. Calderwood, who has also verified the validity of the

* Submitted for the degree of Ph.D. (Edinburgh), April, 1931.

rule by actually carrying out the canonical reduction of all compounds $C^{(m)}$ of non-singular matrices C of every possible classical type up to the sixth order, with certain additional cases of higher order, in all a large number of cases.

2. *Compound matrices.*

Three illustrative examples, one of a compound, one of a Schläflian, one of a Zehfuss matrix, will show that these matrices are not, in general, in finally reduced classical form.

(i) If

$$C=\begin{bmatrix}\lambda_1 & 1 & & \\ & \lambda_1 & 1 & \\ & & \lambda_1 & 1 \\ & & & \lambda_1\end{bmatrix}, \quad \text{then} \quad C^{(2)}=\begin{bmatrix}\lambda_1^2 & \lambda_1 & . & 1 & . & . \\ & \lambda_1^2 & \lambda_1 & \lambda_1 & 1 & . \\ & & \lambda_1^2 & . & \lambda_1 & . \\ & & & \lambda_1^2 & \lambda_1 & 1 \\ & & & & \lambda_1^2 & \lambda_1 \\ & & & & & \lambda_1^2\end{bmatrix}.$$

(ii) If

$$C=\begin{bmatrix}\lambda_1 & 1 & & \\ & \lambda_1 & 1 & \\ & & \lambda_1 & \end{bmatrix}, \quad \text{then} \quad C^{[2]}=\begin{bmatrix}\lambda_1^2 & 2\lambda_1 & . & 1 & . & . \\ & \lambda_1^2 & \lambda_1 & \lambda_1 & 1 & . \\ & & \lambda_1^2 & . & \lambda_1 & . \\ & & & \lambda_1^2 & 2\lambda_1 & 1 \\ & & & & \lambda_1^2 & \lambda_1 \\ & & & & & \lambda_1^2\end{bmatrix}.$$

(iii) If

$$C_1=\begin{bmatrix}\lambda_1 & 1 \\ & \lambda_1\end{bmatrix}, \qquad C_2=\begin{bmatrix}\lambda_2 & 1 & \\ & \lambda_2 & 1 \\ & & \lambda_2\end{bmatrix},$$

then

$$M(C_1, C_2)=\begin{bmatrix}\lambda_1\lambda_2 & \lambda_1 & . & \lambda_2 & 1 & . \\ & \lambda_1\lambda_2 & \lambda_1 & . & \lambda_2 & 1 \\ & & \lambda_1\lambda_2 & . & . & \lambda_2 \\ & & & \lambda_1\lambda_2 & \lambda_1 & . \\ & & & & \lambda_1\lambda_2 & \lambda_1 \\ & & & & & \lambda_1\lambda_2\end{bmatrix}.$$

In none of these cases is the nature of the canonical form immediately apparent; and these examples are almost the simplest possible.

On the other hand, the matrices $C^{(m)}$, $C^{[m]}$, $M(C_j)$, having nothing but zeros below the diagonal, are *partially* reduced, and it only remains to complete the reduction. We can indeed ensure in all cases (§ 4 *infra*), by adopting a monotonic or "dictionary" order of precedence of compound umbral indices in forming these matrices, that not only are the elements below the diagonal zeros without exception, but in the diagonal equal latent roots λ_i are adjacent to each other. The matrices are, indeed, in a partially reduced canonical form employed by Jacobi.

It is known in regard to matrices of this shape that, if a_{ij} is a non-zero element above the diagonal, and if further $\lambda_i \neq \lambda_j$, then there exists a transformation HAH^{-1} which has the effect of replacing a_{ij} by zero. The matrix H in such a case is the unit matrix I, supplemented by a single non-zero element $h_{ij} = a_{ij}/(\lambda_i - \lambda_j)$. By the repetition of operations of this type with respect to elements a_{ij} taken in a suitable order, all elements aligned with distinct latent roots may be removed, *without altering the remaining elements of the matrix.*

When this is done, the submatrices corresponding to each distinct latent root are isolated from each other, and may be reduced independently. This reduction is facilitated, in the special case with which we are dealing, by the theorem of the next section.

3. *Maximal chains of positive non-diagonal elements.*

By means of the theorem about to be proved one can often discern, without further transformation, the elementary divisors of the type of characteristic determinant under consideration, namely $|\lambda I - A|$, where

$$A = [a_{ij}], \quad a_{ij} \geqslant 0; \quad a_{ij} = 0, \quad i \geqslant j,$$

i.e. where the only elements of A which are not zero are *above* the diagonal and are *positive.*

We shall denote a_{ij} by $\binom{i}{j}$, and minors of A also by umbral notation, *e.g.* $\binom{i_1 i_2 \cdots i_k}{j_1 j_2 \cdots j_k}$.

The theorem, which will be enunciated in due course, is concerned with *closed chains* of non-zero elements

$$\binom{i}{j}\binom{j}{k}\binom{k}{l}\binom{l}{m}, \quad \ldots \quad (i < j < k < l < m),$$

each element being linked to its successor by having an index in common with it in the manner shown. By "closed" it is meant, for example, that there exists no non-zero element $\binom{m}{n}$ in the m-th row which could be used to extend the above chain, nor any non-zero element $\binom{h}{i}$. The chain will often be denoted by $ijklm$, and the number of indices, in this case five, will be termed its *order*.

A chain, when the zero element $\binom{m}{i}$ is added to it, will be said to form a *cycle* of the same order as the chain. Diagonal elements $\binom{k}{k}$ will be termed cycles of order 1.

Chains and cycles possessing no index in common will be said to be *isolated*.

Of two chains such as $ijm, jklmn$, the second appears to be of higher order than the first; we may, however, unlink the jm of the first chain and replace it by the $jklmn$ of the second, so that the new chains are $ijklmn.jm$. In such a way as this two non-isolated chains can always be replaced, if necessary, by other chains so that, in the first chain, all partial chains between indices occurring also in the second chain are of order equal to or greater than the order of the corresponding partial chains in the second chain. We are therefore able, from among several non-isolated chains, to segregate chains of *maximal* order, which we shall call *maximal chains*; the order of such chains being equal to or greater than that of any other chain having indices in common with them.

Thus, ranging over all the elements $\binom{i}{j}$, we distinguish chains of maximal order, not necessarily unique nor isolated. These will be termed *first maximal chains*. Let their order be e_1.

When any first maximal chain has been set aside, there remain in general, isolated from it, certain residual chains, and from these can be formed in the same way other maximal chains. Let the highest order of these residual maximal chains be e_2. At this stage, then, we have certain first maximal chains of order e_1, and, isolated respectively from these, certain *second maximal chains* of order e_2, with $e_1 \geqslant e_2$.

In the same way, isolated from any pair of first and second maximal chains, there may remain non-zero elements forming third chains. Among these will be chains of maximal order e_3. Let these be called *third maximal chains*. Then $e_1 \geqslant e_2 \geqslant e_3$. Proceeding in this way we may come at last to single elements which, being determined by two indices, are counted as

chains of order 2; and if, proceeding in this successive chain-formation, we exhaust these also, we shall make the convention of completing the sequence e_i by mean of *units* until the condition $\Sigma e_i = n$ is satisfied.

The chains of elements which we obtain in this process of maximal chain formation are not necessarily, nor generally, unique, but the set of indices $e_1, e_2, e_3, \ldots$ is certainly unique, from the mode of its construction, and in regard to it we now enunciate the

THEOREM. *The indices e_i are the exponents of the elementary divisors of the characteristic determinant of the matrix A.*

COROLLARY. *The same is true for the case in which the non-zero elements above the diagonal of A are negative instead of positive.*

Proof. The value of the determinant $|\lambda I - A| = \Delta$ is λ^n.

Consider next the terms in the expansion of any first minor of Δ. We may suppose the elements comprising any term in the expansion of Δ to be arranged *in cycles*, that is, diagonal elements (cycles of order 1) and cycles of higher order. As we have seen, each cycle of higher order has one zero element. Deleting in Δ the row and column containing this element, we see that in the expansions of first minors of Δ only those terms are non-zero which arise either from diagonal elements alone, or from diagonal elements together with *not more than one chain*.

In the same way, deleting the zero elements from two cycles, we see that non-zero terms in second minors of Δ cannot involve more than two isolated chains; and generally that non-zero terms in r-th minors cannot involve more than r isolated chains.

It follows that in first minors the maximum number of non-diagonal elements in a non-zero term is $e_1 - 1$, the number of elements in a first maximal chain; that in second minors the maximum number of non-diagonal elements in a non-zero term is $e_1 + e_2 - 2$, the number of elements in a first and a second maximal chain together, and so on.

There remains, however, this possibility, that in the expansion of a first minor, and similarly in that of minors of lower order, the term containing all the elements of one maximal chain may be exactly cancelled by the term derived from some other maximal non-isolated chain.

It will be shown that this cannot arise. Consider first two maximal non-isolated chains such as *ijmq*, *iknp*, with the same initial, but a different final index. For the elements of *ijmq* to appear as factors in a term of a first minor we must, as was noted above, obtain the minor by deleting row

q. column i of Δ. But when we do this the cofactor, in this minor, of the elements of the other chain $iknp$ is of the form $\begin{pmatrix} \cdot\cdot jmp\ \cdot\cdot \\ \cdot\cdot jmq\ \cdot\cdot \end{pmatrix}$, which is zero, having a null row, since, because of the closure of $iknp$, all non-diagonal elements in row p are zero. In a similar way the elements of chains like $iknp$, $jkmp$ with the same final, but a different initial index, cannot both give a non-zero term in the same minor; nor, *a fortiori*, can elements of chains like $iknp$, $jkmq$ give such a term.

It remains then to consider terms arising from chains such as $ijmp$, $iknp$, which *begin and end with the same indices*. The diagonal factors $\begin{pmatrix} r \\ r \end{pmatrix}$, $\begin{pmatrix} s \\ s \end{pmatrix}$ in such terms contribute nothing to their determinantal sign in Δ. This is fixed, therefore, by the class of the cyclic permutations

$$\begin{pmatrix} ijmp \\ jmpi \end{pmatrix}, \quad \begin{pmatrix} iknp \\ knpi \end{pmatrix},$$

which is obviously the same for both chains. It follows that, when we delete row p and column i of Δ, the terms containing the elements of these maximal chains are of the same *determinantal* sign. They are also of the same *intrinsic* sign, since the assumption was that all non-zero super-diagonal elements were either all positive or all negative. It follows that terms of this kind do not cancel, but augment each other, and since there is at least one maximal chain of order e_2, there actually exists a non-zero term in the first minors which involves λ^{n-e_1}, and no term of lower degree in λ.

Hence, finally, the highest common factor in λ of the first minors of Δ is λ^{n-e_1}, from which it follows that e_1 is the exponent of the first elementary divisor of Δ.

An argument quite analogous, but applying to two chains and two deletions instead of one, shows that in the expansions of second minors there are non-zero terms involving elements of two, but not more than two chains, and that terms arising from two chains each beginning and ending with the same indices have the same sign, namely that of the compounding of two cycles, and so do not cancel. It follows that a first maximal chain and the second maximal chain associated with it together give rise to terms in $\lambda^{n-e_1-e_2}$, while other non-maximal chains lead to higher powers of λ.

Hence the highest common factor in λ of the second minors of Δ is $\lambda^{n-e_1-e_2}$, so that e_2 is the exponent of the second elementary divisor of Δ.

A similar argument, with necessary but obvious amplifications, can be applied to three or to any higher number of maximal chains. It follows in this way that the exponent of the r-th elementary divisor of Δ is e_r, the order of the r-th maximal chain, chains of unit order being introduced if

necessary according to the convention earlier mentioned. This completes the proof of the theorem.

Example. By inspection of $C^{(2)}$ in the illustration of § 2 (i), we see that, if $\lambda_1 \neq 0$, then 12356 is a maximal chain, the remaining chain being constituted by the single index 4. Hence the canonical form of $C^{(2)}$ here is

$$[C_5(\lambda_1^2), \quad C_1(\lambda_1^2)].$$

The chains 12456, 3 would have served as well. On the other hand, if $\lambda_1 = 0$, the chains are 146, 25, 3 and the canonical form of the singular matrix $C^{(2)}$ is in this case

$$[C_3(0), \quad C_2(0), \quad C_1(0)].$$

The canonical forms of the Schläflian and the Zehfuss matrix exemplified in § 2, (ii) and (iii), can easily be found in the same way.

4. *Application of chains to compound matrices.*

The problem of reducing compound and induced matrices to classical shape therefore comes to this: *to discover by what rules the non-zero elements above the diagonal in such matrices can be linked into isolated chains which shall be, in succession, of maximal orders in the sense of* § 3.

It is advantageous for compound matrices to employ an umbral notation involving compound indices. For example, denoting a matrix of the third order by (123), we shall denote its second compound by (12, 13, 23), indicating by paired indices, chosen in "dictionary" or, as we shall term it, *monotonic* order, the rows and columns from which the minors are taken. Written out in full the compound is

$$(12,\ 13,\ 23) = \begin{bmatrix} \binom{12}{12} & \binom{12}{13} & \binom{12}{23} \\ \binom{13}{12} & \binom{13}{13} & \binom{13}{23} \\ \binom{23}{12} & \binom{23}{13} & \binom{23}{23} \end{bmatrix},$$

where the elements are represented in umbral notation as minors of the determinant of the original matrix (123).

We shall consider first the m-th compound of a *simple* classical matrix $C = [c_{ij}]$, under the following conditions:

$$\text{(A)} \quad \begin{cases} \text{(i)} & c_{ii} = \binom{i}{i} = \lambda_1 \quad (\lambda_1 \neq 0). \\ \text{(ii)} & c_{ij} = \binom{i}{j} = 0 \quad (i > j, \text{ or } i < j-1). \\ \text{(iii)} & c_{i,i+1} = 1 \text{ or } -1, \text{ as } \lambda_1 \gtrless 0 \text{ or } < 0. \end{cases}$$

The case of zero latent roots will be reserved for special consideration.

In consequence of conditions (A) all non-zero elements of the m-th compound $C^{(m)}$, such as, let us say,

$$\binom{i_1\, i_2 \dots i_m}{j_1\, j_2 \dots j_m} \quad (i_1 < i_2 < \dots < i_m,\ j_1 < j_2 < \dots < j_m),$$

will have the same sign. For the elements of $C^{(m)}$ satisfy the following conditions:

$$\text{(B)} \quad \begin{cases} \text{(i)} & \text{If any } i_r > j_r, \text{ the element in question is zero.} \\ \text{(ii)} & \text{If any } i_r < j_r - 1, \text{ the element in question is zero.} \\ \text{(iii)} & \text{In all other cases the value of the element is} \\ & \qquad |\lambda_1|^{m-\Sigma(j_r - i_r)}, \\ & \text{with sign } (-)^m \text{ if } \lambda_1 \text{ is negative.} \end{cases}$$

These are consequences of conditions (A) above. Thus any minor of C for which (B i) is the case must be the product of two minors of lower order, one of which is zero. For example, if i_r is the *first* of the row indices that is greater than the corresponding j_r, then since each subsequent i_s is also greater than j_r all elements below the leading minor of order r in this minor must be zero, so that the minor falls into factors thus.

$$\binom{i_1\, i_2 \dots i_m}{j_1\, j_2 \dots j_m} = \binom{i_1\, i_2 \dots i_r}{j_1\, j_2 \dots j_r} \binom{i_{r+1} \dots i_m}{j_{r+1} \dots j_m}.$$

But again, since i_r exceeds j_r and every j_s preceding j_r, the last row of the first factor above consists entirely of zeros. Hence the element of $C^{(m)}$ in question vanishes.

Again, if (B ii) holds, we shall find that the minor of C in question has no non-zero elements above the diagonal, and at least one zero element in the diagonal. Hence it must vanish.

As for (B iii), it follows from the fact that whenever $i_r = j_r$ a factor λ_1 enters from the diagonal of C_1 but if $i_r = j_r - 1$, a positive or negative unit enters instead, according as λ_1 is positive or negative. The condition (A iii), therefore, produces through (B iii) the effect that all non-zero elements in $C^{(m)}$ are *of the same sign*, and so the theorem of § 3 on maximal chains can be applied.

Finally, if we arrange the compound indices in the umbral representation of $C^{(m)}$ in monotonic order, as, for example, (12, 13, 14, 23, 24, 34), it will follow from (A ii) that in $C^{(m)}$ there will be nothing but zeros below the diagonal.

Turning next to a compound $C^{(m)}$ of a classical canonical matrix C which is no longer simple, such as

$$C = \operatorname{diag}[C_3(\lambda),\ C_2(\mu)] = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & & \\ & & & \mu & 1 \\ & & & & \mu \end{bmatrix} \quad (\lambda \neq 0,\ \ \mu \neq 0),$$

we must note a new feature specifying the *isolation* of the constituent submatrices, namely that

$$\text{(A iv)} \qquad \binom{i_{(\lambda)}}{j_{(\mu)}} = 0,$$

by which is meant that any element having a row index belonging to one classical submatrix and a column index belonging to another is zero. (We shall say, "indices belonging to different categories".) The addition of this condition to the earlier ones (A, i, ii, iii) has the effect

(B iv). All non-diagonal elements in $C^{(m)}$, such as

$$\binom{i_1\, i_2 \cdots i_m}{j_1\, j_2 \cdots j_m},$$

in which any i_r is in a different category* from its j_r, must vanish. For again, just as in the case of (B i) above, let i_r and j_r be the first pair of indices that fall in different categories; then the minor is readily seen to break up, as before in (B i), into two factors, of which one is zero. The value of any

* The author visualizes indices of different categories as of different colours.

non-zero element of $C^{(m)}$, the compound of a *general* classical canonical matrix with submatrices corresponding to latent roots $\lambda_1, \lambda_2, \ldots, \lambda_k$, is of the form

$$(\text{B v}) \qquad (-)^{m_r+m_s+\ldots} |\lambda_1|^{m_1-\Sigma(j_{(1)}-i_{(1)})} |\lambda_2|^{m_2-\Sigma(j_{(2)}-i_{(2)})} \ldots |\lambda_k|^{m_k-\Sigma(j_{(k)}-i_{(k)})}$$

$$(\Sigma m_i = m),$$

the prefixed sign referring to any *negative* latent roots $\lambda_r, \lambda_s, \ldots$ of C. It follows as before that non-zero elements above the diagonal of a submatrix of $C^{(m)}$ belonging to a *particular* latent root of $C^{(m)}$ have the same sign, so that the theorem of § 3 on maximal chains is applicable.

For this general case we shall also extend our definition of monotonic order. If there are distinct latent roots $\alpha, \beta, \gamma, \ldots$ of C, we shall ensure (a) that no non-zero elements of $C^{(m)}$ lie below the diagonal, (b) that all equal latent roots of $C^{(m)}$ are adjacent to each other along the diagonal, by adopting the following convention of *compound monotonic* order of indices.

Let all umbral indices referring to α alone be arranged in m-ary groups, in simple monotonic order; then those referring to α and β together; then those referring to α, β, γ together, and so on; then those referring to α and γ, and so on, etc.: next, those referring to β; then those referring to β and γ; and so on. By this means, as may at once be verified from (B i) and (B iv), the consequences (a) and (b) above are assured.

The matrix $C^{(m)}$ is, therefore, in Jacobi's form, and so we can at once delete or disregard any non-zero elements lying outside the diagonal submatrices delimited by the sequences of equal latent roots of $C^{(m)}$ in the diagonal.

It will next be shown that for a *maximal* chain, over and above the condition, due to (B iv), that upper and lower indices in the constituent umbral elements, that is, consecutive compound indices in the links of the chain, must be in one-to-one correspondence of the same category, and the condition, due to (B i) and (B ii), that $i_r = j_r$ or $j_r - 1$, the following further condition must hold:

Principle of unit step. Each link in the chain (of compound umbral indices) must be obtained from the preceding one by increasing some one *index only by one unit.*

For example, supposing that the indices 12345 are all of the same category, then a chain such as

$$123,\ 124,\ 125,\ 135,\ 145,\ 245,\ 345$$

is in conformity with the principle.

[Chains of compound indices are an obvious extension of the chains of simple indices introduced in § 3. For example, the above chain is composed of the minors of $C^{(3)}$,

$$\binom{123}{124}, \quad \binom{124}{125}, \quad \binom{125}{135}, \quad \binom{135}{145}, \quad \binom{145}{245}, \quad \binom{245}{345}.]$$

For suppose that there was a chain of non-zero elements not obeying this rule of unit step, even though it was of consistent category and such that $i_r = j_r$ or $j_r - 1$; for example, the chain

123, 124, 135, 245:

it can evidently be extended by the insertion of intermediate links in unit step, thus,

123, 124, 134, 135, 235, 245, say,

and no further insertions are possible, while all the elements of the new chain are still non-zero, in accordance with (B); the chain is now maximal.

We proceed, therefore, to form orderings of m-ary groups of indices of given category (multiple category, if several submatrices of C are involved), consistent with the principle of unit step. Of these orderings there will be one, not necessarily nor generally unique, which yields a first maximal chain, then from the remaining indices a second maximal chain, and so on; and such an ordering will give the requisite canonical chains, the orders of the chains being the orders of the latent classical submatrices of $C^{(m)}$, that is, the exponents of the corresponding elementary divisors.

Example. The second compound of [(1234)], $(\lambda_1 \neq 0)$.

In unit step, we can form maximal chains 12, 13, 14, 24, 34, and 23. Hence the exponents of elementary divisors are 5 and 1, as was found in the example of § 3.

[*Note.* Sometimes the discrimination of maximal chains by unit step is complicated by alternatives. Let us take, for example, the third compound of [(123456)], $\lambda_1 \neq 0$, a matrix of the twentieth order. It might be supposed that the chains

123, 124, 125, 126, 136, 146, 156, 256, 356, 456;

134, 135, 235, 236, 246, 346; 145, 245, 345; 234,

of orders 10, 6, 3, 1 respectively, would provide the desired maximal orders.

Actually, with some rearrangement of the later triads of indices, we can substitute for the last three chains the two chains

$$134,\ 135,\ 145,\ 245,\ 345,\ 346;\quad 234,\ 235,\ 236,\ 246,$$

and the orders now given, 10, 6, 4, are maximal and are the correct* orders of the latent submatrices.]

The problem which we set out to solve passes over, therefore, from the calculus of matrices into combinatory analysis, into a study, namely, of the possible orderings under unit step of m-ads of indices formed from indices of different categories, and the discrimination of maximal chain successions among these orderings. For high orders of the generating matrix C the problem becomes complicated and direct formulation of the solution seems difficult; but for compounds of matrices which are of not too high an order the rule of unit step and consistent category furnishes a rapid and efficacious means of determining canonical shape. It leads in any case (§ 8 *infra*) to an explicit expression for the order of the *first* maximal chain associated with any latent root, from which the very important reduced characteristic function (R.C.F.) of any compound matrix can be completely determined.

5. *Induced or Schläflian matrices.*

Still deferring for later examination the case of zero latent roots in the generating matrices, let us next consider the induced or Schläflian matrix of C of the m-th class, $C^{[m]}$. It is convenient here to denote any element of a Schläflian by an umbral bracket of upper and lower indices, which may contain *repetitions*, the upper indices denoting the degrees of the powers or products of variables x_j (see the definition of a Schläflian in § 1), the lower the degrees of powers or products of the linear forms y_i. For example, the coefficient of $x_1^2 x_2$ in the expansion of $y_1 y_2^2$ would be a Schläflian element denoted by $\binom{112}{122}$. For Schläflian matrices themselves we shall therefore adopt an umbral notation similar to that which was used for compounds. In this notation the third Schläflian of $C = [(12)]$ would appear as

$$C^{[3]} = [(111,\ 112,\ 122,\ 222)].$$

Now, by examination of the procedure, § 1 (ii), by which Schläflians are constructed from the generating matrix C, it will be seen that, if compound

* Calderwood, *op. cit.*

monotonic order is adopted for the indices concerned, then conditions (A) bring about a distribution in $C^{[m]}$ of non-zero elements above the diagonal which is precisely similar to that already observed in the case of compound matrices, and which is actually more easily proved to hold; while the conditions (B) for non-diagonal elements are complied with in almost the same manner, the only difference being that a numerical factor involving multinomial coefficients is attached to elements corresponding to (B iii) and (B iv). (*Cf.* the illustrative Schläflian of § 2.)

Hence the reasoning already applied to compounds can be applied (in the non-singular case) equally well to Schläflians. *The only novel feature is that in forming chains by the unit step principle repetitions of an index are permitted.* For example, the canonical form of the second Schläflian $C^{[2]}$ of

$$\operatorname{diag}[C_2(\lambda_1),\ C_1(\lambda_2)] = [(12)_{\lambda_1}(3)_{\lambda_2}] \quad (\lambda_1, \lambda_2 \neq 0)$$

can be written down at once in double-index notation as

$$[(11,\ 12,\ 22)_{\lambda_1^2}(13,\ 23)_{\lambda_1\lambda_2}(33)_{\lambda_2^2}] = \operatorname{diag}[C_3(\lambda_1^2),\ C_2(\lambda_1\lambda_2),\ C_1(\lambda_2^2)].$$

Or again, to take the simple example of § 2 (ii), the maximal chains in the second Schläflian of [(123)] are 11, 12, 22, 23, 33, and 13, so that the exponents of elementary divisors are 5 and 1.

6. *Multiplicative composite or Zehfuss matrices.*

It will be sufficient here to consider the Zehfuss composite of two matrices A and B, namely $M(A, B)$, the case of m generating matrices* following at once by extension. For the time being we again suppose A and B to be non-singular.

We shall first elucidate the rule by a simple example. Let

$$A = [(123)(4)], \quad B = [(\alpha\beta)(\gamma)].$$

Then $M(A, B)$ is given canonically in double-index notation by

$$M(A, B) = [(1\alpha,\ 2\alpha,\ 3\alpha,\ 3\beta)(1\beta,\ 2\beta)(1\gamma,\ 2\gamma,\ 3\gamma)(4\alpha,\ 4\beta)(4\gamma)].$$

We choose indices from a bracket in A and a bracket in B and, preserving consistent category, form maximum chains by unit step. We designate

* The matrices A_j are taken to be square, in order that they may have a classical canonical form. The case of rectangular matrices may be treated by adding a border of zeros to make them square.

elements in $M(A, B)$ by umbral brackets in a manner precisely comparable with the Schläflian case, and we may readily verify that conditions (B) hold again in the same way [except that in (B v) the numerical factor attached to Schläflian elements does not now appear, being dispersed as a unit over each of many elements instead of being concentrated as a larger factor of one element], so that the procedure is justified, and may be extended at once to the case of any number of generating matrices A_j.

The rule for forming maximal chains for $M(A, B)$ is almost intuitive. Associating a bracket (1234) in A with $(\alpha\beta\gamma)$ in B, we construct the scheme

$$1\alpha \quad 2\alpha \quad 3\alpha \quad 4\alpha$$

$$1\beta \quad 2\beta \quad 3\beta \quad 4\beta$$

$$1\gamma \quad 2\gamma \quad 3\gamma \quad 4\gamma.$$

For a first maximal chain, the first row and last column of this scheme will obviously do; for a second chain, the rest of the second row and last column but one: and so on. Hence the theorem:

Elementary divisors $(\lambda-\lambda_i)^p$ *in* A *and* $(\lambda-\mu_j)^q$ *in* B *give rise to elementary divisors in* $M(A, B)$, *namely* $(\lambda-\lambda_i\mu_j)^h$, *where*

$$h = p+q-1, \quad p+q-3, \quad p+q-5, \quad \ldots.$$

The number of these is the smaller of p *and* q.

When all the matrices A_j are identical with A we derive the m-th Burnside matrix. The rule of chains will then be found to resemble the Schläflian rule, with yet one further extension, in the respect that, while using the unit step principle as before, we permit *not merely repetitions but also permutations* of indices. For example, the second Burnside matrix of

$$C = [(12)_{\lambda_1}(3)_{\lambda_2}] \quad (\lambda_1, \lambda_2 \neq 0)$$

can be written down immediately in canonical form as

$$[(11, 12, 22)_{\lambda_1^2}(21)_{\lambda_1^2}(13, 23)_{\lambda_1\lambda_2}(31, 32)_{\lambda_1\lambda_2}(33)_{\lambda_2^2}]$$
$$= \operatorname{diag}[C_3(\lambda_1^2), C_1(\lambda_1^2), C_2(\lambda_1\lambda_2), C_2(\lambda_1\lambda_2), C_1(\lambda_2^2)].$$

The hybrid cases obtained by basing a Zehfuss matrix on several Schläflians, or indifferently by constructing the Schläflian of several Zehfuss matrices, have been mentioned earlier. They introduce complications, but involve no new principle; and one sees easily how their properties,

their latent roots and canonical forms, may be derived by applying the rules. It is unnecessary to go into details.

7. *The case of singular submatrices.*

The restriction that no latent root is to be zero will now be removed.

As a preliminary, let us consider the matrix $C^{(2)}$ in the illustrative example § 2 (i), when $\lambda_1 = 0$. The only elements above the diagonal that now survive are the units in positions $\binom{1}{4}$, $\binom{2}{5}$, $\binom{4}{6}$, and these fall into two chains of orders 3 and 2, so that $C^{(2)}$ has the canonical form

$$[(123)]_0 \ (45)_0 \ (6)_0].$$

Now these units are the minors

$$\binom{12}{23}, \quad \binom{13}{24}, \quad \binom{23}{34}$$

of the determinant of C, and so the canonical form of $C^{(2)}$ may be written with compound indices thus,

$$[(12,\ 23,\ 34)_0 \ (13,\ 24)_0 \ (14)_0].$$

Here the rule of indices for zero latent roots is salient:

In forming chains not merely one, but all, of the indices corresponding to a zero latent root must move, and by a unit step each time.

The rule applies to compounds, Schläflians, and, multiplicative composites of generating matrices A_j which have any distribution of zero and non-zero latent roots. To establish the rule we merely observe a consequence of conditions (A), (B iii) and (B v) of § 4, namely that if a latent root is zero the only "power" of it that is not zero in an element of a compound, Schläflian, or Zehfuss matrix, is the zero-th power, the part of the umbral index of the element referring to that latent root being then of the form

$$\binom{i_1 \quad i_2 \quad \ldots i_k}{i_1+1 \ i_2+1 \ \ldots i_k+1},$$

in which *every* lower index exceeds the upper by a unit. Consequently, in any two consecutive links of a maximal chain, *all* the indices which refer to a zero latent root are bound to move *together* in unit step. For example,

the canonical form of the second compound of

$$[(123)_0\ (45)_{\lambda_1}] \qquad (\lambda_1 \neq 0)$$

is $$[(12,\ 23)_0\ (13)_0\ (14,\ 24,\ 34)_0\ (15,\ 25,\ 35)_0\ (45)_{\lambda_1{}^2}].$$

For Zehfuss matrices the rule of chains is again intuitive. Consider the scheme of § 6, but suppose that $\lambda_i = 0$, $\mu_j \neq 0$. Since the indices of (1234) *must* change from one link to the next, maximal chains are given by the *rows* of the scheme. Hence the elementary divisors arising in $M(A, B)$ are q in number, each being λ^p. If $\lambda_i \neq 0$, $\mu_j = 0$, we use columns.

On the other hand, suppose that both $\lambda_i = 0$, $\mu_j = 0$. Then *both* kinds of indices must change from link to link. The only movement possible in the scheme is therefore *diagonally downward*, and the chains are the diagonals of the scheme. Hence the elementary divisors arising in $M(A, B)$ are λ^q occurring $p-q+1$ times, and λ^{q-1}, λ^{q-2}, ..., λ^2, λ each twice. This is for $p \geqslant q$; for $p < q$ an analogous result holds.

8. *The reduced characteristic functions.*

In the compound case, as has been stated, the determination of the orders of *all* the maximum chains is a matter involving deep combinatorial investigation. The determination of the order of the *first* maximal chain corresponding to each distinct latent root is fortunately, however, a relatively simple matter. For example, in the m-th compound of the simple classical matrix

$$[(123\ldots n)_{\lambda_1}] \quad (\lambda_1 \neq 0),$$

we must begin the first chain with $123\ldots m$ and, proceeding by unit steps in one index at a time, must end with $n-m+1$, $n-m+2$, ..., n. The number of steps is thus evidently $m(n-m)$, and the order of the chain, which is the exponent of the first elementary divisor and the power to which the corresponding factor $\lambda-\lambda_1{}^m$ appears in the R.C.F. of $C^{(m)}$, is therefore $m(n-m)+1$, a result not altogether surprising, since this number is known to be the maximum number of linearly independent minors of order m that can be extracted from a general m-by-n array, $n > m$.

On the other hand, if the latent root λ_1 is zero, the chain must begin with links $123\ldots m$, $234\ldots m+1$ and must end with $n-m+1$, ..., n, all indices moving together. The number of steps is then seen to be $n-m$, and the order of the first chain is $n-m+1$.

These orders, then, are the exponents of the factors $\lambda-\lambda_1{}^m$, $\lambda_1 \neq 0$, or of λ^m, as the case may be, in the R.C.F. of the m-th compound matrix.

Consider next the maximal chain corresponding to a characteristic factor

$$\lambda-\lambda_1^{m_1}\lambda_2^{m_2}\dots\lambda_k^{m_k} \qquad (m_1+m_2+\dots+m_k=m)$$

in the m-th compound of

$$[(12\dots n_1)_{\lambda_1}(n_1+1,\ n_1+2,\ \dots,\ n_1+n_2)_{\lambda_2},\quad \dots.$$
$$(n_1+n_2+\dots+n_{k-1}+1,\ \dots,\ n_1+n_2+\dots+n_k)_{\lambda_k}]$$
$$(n_1+n_2+\dots+n_k=n).$$

By making the indices move by unit steps within each bracket, and bracket by bracket, we find without trouble the desired exponent, let us say

$$e(\lambda_1^{m_1}\lambda_2^{m_2}\dots\lambda_k^{m_k})=\sum_1^k m_j(n_j-m_j)+1,$$

provided that the latent root of $C^{(m)}$ in question is not zero. When zero latent roots occur, complications arise which are not easy to resolve by an unambiguous general formulation, though in simple cases they may be resolved easily enough by finding the appropriate maximal chain of umbral indices; thus for the third compound of

$$[(1234)_0\ (567)_0],$$

the chain 125, 236, 347 is maximal and points to the R.C.F. λ^3.

Example. In the compound $C^{(2)}$ of § 2 (1) we have $m=2$, $n=4$, so that, for $\lambda_1\neq 0$,

$$e_1=2(4-2)+1=5,$$

as was found earlier in other ways, and the R.C.F. of the compound is $(\lambda-\lambda_1{}^2)^5$.

For Schläflian matrices umbral indices can be repeated, and chains are consequently longer. It is readily seen that here, in the case of a simple classical matrix with non-zero latent root λ_1, we move from $111\dots1$, m indices, to $nnn\dots n$ by unit steps in one index at a time, that is, in $m(n-1)$ steps. In fact, we have in this simple case

$$e(\lambda_1{}^m)=m(n-1)+1,$$

and in the general case of several non-zero latent roots,

$$e(\lambda_1^{m_1}\lambda_2^{m_2}\dots\lambda_k^{m_k})=\sum_1^k m_j(n_j-1)+1,$$

and the powers of corresponding factors in the R.C.F. of Schläflians are thus determined.

For zero latent roots complications may occur, just as in the case of compound matrices. However, the R.C.F. of the m-th Schläflian of a non-singular matrix is explicitly and simply determined, the analogy with the case of a compound matrix being again noteworthy as well as helpful.

Example. In the Schläflian $C^{[2]}$ of § 2 (ii) we have $m = 2$, $n = 3$, so that, for $\lambda_1 \neq 0$,

$$e_1 = 2(3-1)+1 = 5,$$

as was earlier found in other ways.

Finally, for a Zehfuss matrix based on k matrices A_j, we have at once, from § 6, for non-zero latent roots λ_j with exponents n_j in the R.C.F. of A_j,

$$e(\lambda_1 \lambda_2 \dots \lambda_k) = \Sigma(n_j - 1) + 1.$$

In the case of zero latent roots, the degree of the characteristic factor in the R.C.F. corresponding to a zero latent root is the order of the classical submatrix of *greatest* order, corresponding to a zero latent root, among the C_j; unless the latent roots of the A_j are all zero, in which case the degree in question is the order of the classical matrix of *least* order among the C_j. The above facts follow readily from the conclusion of § 7.

Example. In the Zehfuss matrix of § 2 (iii) we have $n_1 = 2$, $n_2 = 3$. Hence, if $\lambda_1 \neq 0$, $\lambda_2 \neq 0$,

$$e_1 = (2-1)+(3-1)+1 = 4.$$

If $\lambda_1 = 0$, $\lambda_2 \neq 0$, $e_1 = n_1 = 2$. If $\lambda_1 \neq 0$, $\lambda_2 = 0$, $e_1 = n_2 = 3$.

Finally, if $\lambda_1 = 0$, $\lambda_2 = 0$, then $e_1 = \min(n_1, n_2) = n_1 = 2$.

That these results are true can be verified by substituting in the illustrative matrix and finding a maximal chain in the several cases.

It appears, therefore, that for non-singular matrices A or A_j of known canonical shape we are able to construct explicitly the R.C.F. of a prescribed compound, Schläflian, or Zehfuss matrix based upon them, and that an explicit rule exists also for the Zehfuss matrix in the singular case.

9. *Summary.*

The classical canonical form of compound matrices, of induced matrices ("power transformations"), and of multiplicative composite matrices ("product transformations") is investigated. A combinatory rule is

obtained, depending on the construction of successive maximal chains of index-groups under the principle of unit increase in one index at a time, with a modification for the case of zero latent roots. The rule leads to explicit constructions for the R.C.F. of these types of matrices in the non-singular case, and for the elementary divisors of Zehfuss matrices in all cases.

[*Added* 26 *November*, 1934.—A direct investigation of the elementary divisors of Zehfuss matrices has recently been given by W. E. Roth, *Bull. American Math. Soc.*, 40 (1934), 461–468. See especially pp. 463–465 which contain results given in §§ 6–8 of my paper *passim*. I am indebted to Prof. Roth for communicating to me the substance of his results before his paper was published.]

References.

In Muir's *History of determinants*, I, II, III, IV, and *Contributions to the history of determinants*, 1900–1920, which we shall refer to as V, much information is given concerning the early history of the matrices considered in this paper, from the point of view of the evaluation of their determinants, and their latent roots. The following passages are relevant:

I, 123–130, on the Binet-Cauchy theorem; also III, 13, ref. to Kronecker.

On the latent roots of compound determinants, IV, 215, 217, 218 (ref. Rados, Metzler), V, 224–5 (Petr). Also *Proc. Royal Soc. Edinburgh*, 47 (1927), 260 (Burnside), 265 (Petr), 275 (Whittaker).

On Schläfli's determinant, a list of references in V, 402, Subject Index. On the latent roots of Schläflians, *Proc. Royal Soc. Edinburgh*, 47 (1927), 256 (Franklin), 258 (Rados), 260 (Burnside), 275 (Whittaker), 276 (Metzler).

On Zehfuss' determinant, a list of references in V, 407, Subject Index. On the latent roots of Zehfuss matrices, IV, 72 (Stephanos), *Proc. Royal Soc. Edinburgh*, 47 (1927), 260 (Burnside), 275 (Whittaker), 276 (Metzler).

On generalized determinants of Schläfli-Zehfuss type, see T. R. Rosebrugh, *Bull. American Math. Soc.*, 33 (1927), 583–590.

On the multiplicative law of compound and Schläflian matrices, see Maclagan-Wedderburn, *Proc. Edinburgh Math. Soc.*, 27 (1908), 67–69, citing the dissertation of I. Schur, *Über eine Klasse von Matrizen* (Berlin, 1901).

On the canonical form of Jacobi and its reduction to classical shape, see Turnbull and Aitken, *Canonical matrices* (London and Glasgow, 1932), ch. VI, p. 67.

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SOME THEOREMS CONCERNING PRODUCTS OF HYPERGEOMETRIC SERIES

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1. In 1858, Cayley stated* without proof the theorem that, if

$$(1-x)^{\alpha+\beta-\gamma}\,{}_2F_1\left[\begin{matrix}2\alpha,\ 2\beta;\\ 2\gamma\end{matrix}\ x\right]=1+Bx+Cx^2+\ldots,$$

then

$${}_2F_1\left[\begin{matrix}\alpha,\ \beta;\\ \gamma+\frac{1}{2}\end{matrix}\ x\right]{}_2F_1\left[\begin{matrix}\gamma-\alpha,\ \gamma-\beta;\\ \gamma+\frac{1}{2}\end{matrix}\ x\right]=1+\frac{\gamma}{\gamma+\frac{1}{2}}\,Bx+\frac{\gamma(\gamma+1)}{(\gamma+\frac{1}{2})(\gamma+\frac{3}{2})}\,Cx^2+\ldots.$$

When $\gamma=\alpha+\beta$, this theorem reduces to Clausen's theorem† concerning the square of a certain hypergeometric series, namely

$$\left\{{}_2F_1\left[\begin{matrix}\alpha,\ \beta;\\ \alpha+\beta+\frac{1}{2}\end{matrix}\ x\right]\right\}^2={}_3F_2\left[\begin{matrix}2\alpha,\ 2\beta,\ \alpha+\beta;\\ 2\alpha+2\beta,\ \alpha+\beta+\frac{1}{2}\end{matrix}\ x\right].$$

It was not until 1899 that Cayley's theorem was proved by Orr‡, who gave some other results of a similar type. Since then several proofs have been given, but it is only within the last few years that proofs with any real claim to simplicity have been discovered§.

The simplest proof of Orr's theorems given by Whipple‖ depends on the transformation of a nearly-poised series ${}_3F_2$ of unit argument into a Saalschützian series ${}_4F_3$, the two series then being transformed into other series by known relations. He then derives Cayley's theorem by some

* Cayley, 3.
† Clausen, 4.
‡ Orr, 5.
§ Whipple, 7 and 8.
‖ Whipple, 8.

simple algebra. In this paper I give some further results of a similar nature concerning the product of two hypergeometric series.

2. In the first two theorems we use the transformation of a particular (Saalschützian) nearly-poised series ${}_4F_3$ into a Saalschützian ${}_4F_3$, and then transform these two Saalschützian series by the formula*

$$(2.1)\quad {}_4F_3\left[\begin{matrix}x, y, z, -n;\\ u, v, w\end{matrix}\right] = \frac{(u-x)_n (v-x)_n}{(u)_n (v)_n}\, {}_4F_3\left[\begin{matrix}x, w-y, w-z, -n;\\ 1-n+x-u, 1-n+x-v, w\end{matrix}\right],$$

where $u+v+w = x+y+z-n+1$, and n is a positive integer.

The transformation of nearly-poised series which we require is†

$$(2.2)\quad {}_4F_3\left[\begin{matrix}x, y, z, -n;\\ 1+x-y, 1+x-z, w\end{matrix}\right] = \frac{(w-x)_n}{(w)_n}\, {}_5F_4\left[\begin{matrix}1+x-w, \frac{1}{2}+\frac{1}{2}x, \frac{1}{2}x, 1+x-y-z, -n;\\ 1+x-y, 1+x-z, 1+\frac{1}{2}(x-w-n), \frac{1}{2}(1+x-w-n)\end{matrix}\right].$$

When the series on the left is Saalschützian, the series on the right reduces to a ${}_4F_3$. Both (2.1) and (2.2) can be proved in very simple ways‡.

THEOREM I. *If*

$$(1-x)^{\alpha+\beta-\gamma-\frac{1}{2}}\, {}_3F_2\left[\begin{matrix}2\alpha, 2\beta, \gamma;\\ 2\gamma, \alpha+\beta+\frac{1}{2}\end{matrix}\; x\right] = \sum_{n=0}^{\infty} a_n x^n,$$

then

$${}_2F_1\left[\begin{matrix}\alpha, \beta;\\ \alpha+\beta+\frac{1}{2}\end{matrix}\; x\right] {}_2F_1\left[\begin{matrix}\frac{1}{2}+\gamma-\alpha, \frac{1}{2}+\gamma-\beta;\\ 2\gamma-\alpha-\beta+\frac{1}{2}\end{matrix}\; x\right] = \sum_{n=0}^{\infty} \frac{(\gamma+\frac{1}{2})_n}{(2\gamma-\alpha-\beta+\frac{1}{2})_n}\, a_n x^n.$$

THEOREM II. *If*

$$(1-x)^{\alpha+\beta-\gamma-\frac{1}{2}}\, {}_3F_2\left[\begin{matrix}2\alpha, 2\beta, \gamma;\\ 2\gamma, \alpha+\beta+\frac{1}{2}\end{matrix}\; x\right] = \Sigma a_n x^n,$$

then

$${}_2F_1\left[\begin{matrix}\beta, \gamma-\beta;\\ \gamma+\frac{1}{2}\end{matrix}\; x\right] {}_2F_1\left[\begin{matrix}\alpha+\frac{1}{2}, \gamma-\alpha+\frac{1}{2};\\ \gamma+\frac{1}{2}\end{matrix}\; x\right] = \Sigma \frac{(\alpha+\beta+\frac{1}{2})_n}{(\gamma+\frac{1}{2})_n}\, a_n x^n.$$

* Whipple, **9** (10.11).

† Whipple, **10** (6.5).

‡ See Whipple, **9**, § 10A, and Bailey, **2**, § 2.

Comparing coefficients of x^n in Theorem I, we see that the identity to be proved is

$$(2.3)\quad {}_4F_3\left[\begin{matrix}2\alpha,\ 2\beta,\ \gamma,\ -n;\\ 2\gamma,\ \alpha+\beta+\tfrac{1}{2},\ \tfrac{1}{2}-n+\alpha+\beta-\gamma\end{matrix}\right]$$

$$=\frac{(\frac{1}{2}+\gamma-\alpha)_n\,(\frac{1}{2}+\gamma-\beta)_n}{(\gamma+\frac{1}{2})_n\,(\frac{1}{2}-\alpha-\beta+\gamma)_n}\,{}_4F_3\left[\begin{matrix}\alpha,\ \beta,\ \tfrac{1}{2}+\alpha+\beta-2\gamma-n,\ -n;\\ \tfrac{1}{2}+\alpha-\gamma-n,\ \tfrac{1}{2}+\beta-\gamma-n,\ \alpha+\beta+\tfrac{1}{2}\end{matrix}\right],$$

whereas the identity implied in Theorem II is

$$(2.4)\quad {}_4F_3\left[\begin{matrix}2\alpha,\ 2\beta,\ \gamma,\ -n;\\ 2\gamma,\ \alpha+\beta+\tfrac{1}{2},\ \tfrac{1}{2}-n+\alpha+\beta-\gamma\end{matrix}\right]$$

$$=\frac{(\alpha+\frac{1}{2})_n\,(\frac{1}{2}+\gamma-\alpha)_n}{(\alpha+\beta+\frac{1}{2})_n\,(\frac{1}{2}-\alpha-\beta+\gamma)_n}\,{}_4F_3\left[\begin{matrix}\beta,\ \gamma-\beta,\ \tfrac{1}{2}-n-\gamma,\ -n;\\ \gamma+\tfrac{1}{2},\ \tfrac{1}{2}-\alpha-n,\ \tfrac{1}{2}+\alpha-\gamma-n\end{matrix}\right].$$

The equality of the right-hand sides of (2.3) and (2.4) follows from (2.1). To prove (2.3) we have

$${}_4F_3\left[\begin{matrix}2\alpha,\ 2\beta,\ \gamma,\ -n;\\ 2\gamma,\ \alpha+\beta+\tfrac{1}{2},\ \tfrac{1}{2}-n+\alpha+\beta-\gamma\end{matrix}\right]$$

$$=\frac{(2\gamma-2\alpha)_n\,(\frac{1}{2}+\alpha-\beta+\gamma)_n}{(2\gamma)_n\,(\frac{1}{2}-\alpha-\beta+\gamma)_n}\,{}_4F_3\left[\begin{matrix}2\alpha,\ \tfrac{1}{2}+\alpha+\beta-\gamma,\ \tfrac{1}{2}+\alpha-\beta,\ -n;\\ \tfrac{1}{2}+\alpha-\beta+\gamma,\ \tfrac{1}{2}+\alpha+\beta,\ 1-n+2\alpha-2\gamma\end{matrix}\right]$$

$$=\frac{(\frac{1}{2}+\alpha-\beta+\gamma)_n}{(\frac{1}{2}-\alpha-\beta+\gamma)_n}\,{}_4F_3\left[\begin{matrix}\alpha,\ \alpha+\tfrac{1}{2},\ 2\gamma+n,\ -n;\\ \tfrac{1}{2}+\alpha-\beta+\gamma,\ \gamma+\tfrac{1}{2},\ \alpha+\beta+\tfrac{1}{2}\end{matrix}\right]$$

$$=\frac{(\frac{1}{2}+\gamma-\alpha)_n\,(\frac{1}{2}+\gamma-\beta)_n}{(\gamma+\frac{1}{2})_n\,(\frac{1}{2}-\alpha-\beta+\gamma)_n}\,{}_4F_3\left[\begin{matrix}\alpha,\ \beta,\ \tfrac{1}{2}+\alpha+\beta-2\gamma-n,\ -n;\\ \tfrac{1}{2}+\beta-\gamma-n,\ \tfrac{1}{2}+\alpha-\gamma-n,\ \alpha+\beta+\tfrac{1}{2}\end{matrix}\right].$$

In this analysis we first use (2.1), then (2.2), and finally (2.1) again.

3. The proof of the above two theorems follows the lines of Whipple's proof of Orr's theorems. We now give a result analogous to Cayley's theorem.

THEOREM III. *If*

$$(1-x)^{\alpha+\beta-\gamma}\,{}_3F_2\left[\begin{matrix}2\alpha,\ 2\beta,\ \gamma+\tfrac{1}{2};\\ 2\gamma,\ \alpha+\beta+\tfrac{1}{2}\end{matrix}\ x\right]=\Sigma\, b_n x^n,$$

then

$${}_2F_1\left[\begin{matrix}\alpha,\ \beta;\\ \alpha+\beta+\tfrac{1}{2}\end{matrix}\ x\right]{}_2F_1\left[\begin{matrix}\gamma-\alpha,\ \gamma-\beta;\\ 2\gamma-\alpha-\beta+\tfrac{1}{2}\end{matrix}\ x\right]=\Sigma\,\frac{(\gamma)_n}{(2\gamma-\alpha-\beta+\frac{1}{2})_n}\,b_n x^n.$$

The identity implied is

$$(3.1)\quad {}_4F_3\left[\begin{matrix}2\alpha,\ 2\beta,\ \gamma+\frac{1}{2},\ -n;\\ 2\gamma,\ \alpha+\beta+\frac{1}{2},\ 1-n+\alpha+\beta-\gamma\end{matrix}\right]$$
$$=\frac{(\gamma-\alpha)_n(\gamma-\beta)_n}{(\gamma)_n(\gamma-\alpha-\beta)_n}\,{}_4F_3\left[\begin{matrix}\alpha,\ \beta,\ \frac{1}{2}+\alpha+\beta-2\gamma-n,\ -n;\\ \alpha+\beta+\frac{1}{2},\ 1+\alpha-\gamma-n,\ 1+\beta-\gamma-n\end{matrix}\right].$$

Now

$${}_4F_3\left[\begin{matrix}2\alpha,\ 2\beta,\ \gamma-\frac{1}{2},\ -n;\\ 2\gamma-1,\ \alpha+\beta+\frac{1}{2},\ 1-n+\alpha+\beta-\gamma\end{matrix}\right]$$
$$=\sum_{r=0}^{n}\frac{(2\alpha)_r(2\beta)_r(\gamma+\frac{1}{2})_r(-n)_r}{r!\,(2\gamma)_r(\alpha+\beta+\frac{1}{2})_r(1+\alpha+\beta-\gamma-n)_r}\left(1-\frac{r}{2\gamma+2r-1}\right)$$
$$={}_4F_3\left[\begin{matrix}2\alpha,\ 2\beta,\ \gamma+\frac{1}{2},\ -n;\\ 2\gamma,\ \alpha+\beta+\frac{1}{2},\ 1+\alpha+\beta-\gamma-n\end{matrix}\right]$$
$$+\frac{\alpha\beta n}{\gamma(\alpha+\beta+\frac{1}{2})(1+\alpha+\beta-\gamma-n)}\,{}_4F_3\left[\begin{matrix}2\alpha+1,\ 2\beta+1,\ \gamma+\frac{1}{2},\ 1-n;\\ 2\gamma+1,\ \alpha+\beta+\frac{3}{2},\ 2+\alpha+\beta-\gamma-n\end{matrix}\right].$$

Thus the series on the left of (3.1) is equal to

$${}_4F_3\left[\begin{matrix}2\alpha,\ 2\beta,\ \gamma-\frac{1}{2},\ -n;\\ 2\gamma-1,\ \alpha+\beta+\frac{1}{2},\ 1-n+\alpha+\beta-\gamma\end{matrix}\right]$$
$$-\frac{\alpha\beta n}{\gamma(\alpha+\beta+\frac{1}{2})(1+\alpha+\beta-\gamma-n)}\,{}_4F_3\left[\begin{matrix}2\alpha+1,\ 2\beta+1,\ \gamma+\frac{1}{2},\ 1-n;\\ 2\gamma+1,\ \alpha+\beta+\frac{3}{2},\ 2+\alpha+\beta-\gamma-n\end{matrix}\right],$$

and, using (2.3) to transform both these series, we obtain

$$\frac{(\gamma-\alpha)_n(\gamma-\beta)_n}{(\gamma)_n(\gamma-\alpha-\beta)_n}\,{}_4F_3\left[\begin{matrix}\alpha,\ \beta,\ \frac{3}{2}+\alpha+\beta-2\gamma-n,\ -n;\\ 1+\alpha-\gamma-n,\ 1+\beta-\gamma-n,\ \alpha+\beta+\frac{1}{2}\end{matrix}\right]$$
$$-\frac{\alpha\beta n}{\gamma(\alpha+\beta+\frac{1}{2})(1+\alpha+\beta-\gamma-n)}\,\frac{(\gamma-\alpha+\frac{1}{2})_{n-1}(\gamma-\beta+\frac{1}{2})_{n-1}}{(\gamma+1)_{n-1}(\gamma-\alpha-\beta)_{n-1}}$$
$$\times{}_4F_3\left[\begin{matrix}\alpha+\frac{1}{2},\ \beta+\frac{1}{2},\ \frac{3}{2}+\alpha+\beta-2\gamma-n,\ 1-n;\\ \frac{3}{2}+\alpha-\gamma-n,\ \frac{3}{2}+\beta-\gamma-n,\ \alpha+\beta+\frac{3}{2}\end{matrix}\right].$$

The last series can be transformed by (2.1), and we obtain

$$\frac{(\gamma-\alpha)_n(\gamma-\beta)_n}{(\gamma)_n(\gamma-\alpha-\beta)_n}\left\{{}_4F_3\left[\begin{matrix}\alpha,\ \beta,\ \frac{3}{2}+\alpha+\beta-2\gamma-n,\ -n;\\ 1+\alpha-\gamma-n,\ 1+\beta-\gamma-n,\ \alpha+\beta+\frac{1}{2}\end{matrix}\right]\right.$$
$$+\frac{\alpha\beta n}{(\gamma-\alpha+n-1)(\gamma-\beta+n-1)(\alpha+\beta+\frac{1}{2})}$$
$$\left.\times{}_4F_3\left[\begin{matrix}\alpha+1,\ \beta+1,\ \frac{3}{2}+\alpha+\beta-2\gamma-n,\ 1-n;\\ \alpha+\beta+\frac{3}{2},\ 2+\alpha-\gamma-n,\ 2+\beta-\gamma-n\end{matrix}\right]\right\}.$$

Combining these series term by term we obtain the required identity.

4. We now note some particular cases of the above theorems.

In Theorem 1 put $\gamma = \alpha+\beta-\frac{1}{2}$, and we obtain the formula*

$$(4.1)\quad {}_2F_1\left[\begin{matrix}\alpha, \beta; \\ \alpha+\beta+\frac{1}{2}\end{matrix}\ x\right] {}_2F_1\left[\begin{matrix}\alpha, \beta; \\ \alpha+\beta-\frac{1}{2}\end{matrix}\ x\right] = {}_3F_2\left[\begin{matrix}2\alpha, 2\beta, \alpha+\beta; \\ \alpha+\beta+\frac{1}{2}, 2\alpha+2\beta-1\end{matrix}\ x\right],$$

while, if we put $\gamma = \alpha+\beta$, we derive†

$$(4.2)\quad {}_2F_1\left[\begin{matrix}\alpha, \beta; \\ \alpha+\beta+\frac{1}{2}\end{matrix}\ x\right] {}_2F_1\left[\begin{matrix}\alpha+\frac{1}{2}, \beta+\frac{1}{2}; \\ \alpha+\beta+\frac{1}{2}\end{matrix}\ x\right]$$
$$= (1-x)^{-\frac{1}{2}}\, {}_3F_2\left[\begin{matrix}2\alpha, 2\beta, \alpha+\beta; \\ \alpha+\beta+\frac{1}{2}, 2\alpha+2\beta\end{matrix}\ x\right],$$

an immediate consequence of Clausen's theorem.

When $\gamma = \alpha-\frac{1}{2}$, the theorem reduces to

$$(4.3)\quad {}_3F_2\left[\begin{matrix}2\alpha, 2\beta, \alpha-\frac{1}{2}; \\ 2\alpha-1, \alpha+\beta+\frac{1}{2}\end{matrix}\ x\right] = (1-x)^{-\beta}\, {}_2F_1\left[\begin{matrix}\beta, \alpha-\beta-\frac{1}{2}; \\ \alpha+\beta+\frac{1}{2}\end{matrix}\ x\right]$$

Similarly, from Theorems II and III, we can derive Clausen's theorem and other results given by Orr.

5. We now turn to the formula‡

$$(5.1)\quad {}_4F_3\left[\begin{matrix}-m, y, & z, & 1-m-u; \\ 1-m-y, & 1-m-z, & w\end{matrix}\right]$$
$$= \frac{\Gamma(w)\,\Gamma(w+u-1+2m)}{\Gamma(w+m)\,\Gamma(w-1+u+m)}$$
$$\times {}_5F_4\left[\begin{matrix}1-m-w, \frac{1}{2}(1-m), -\frac{1}{2}m, 1-m-y-z, 1-m-u; \\ 1-m-y, 1-m-z, \frac{1}{2}(3-u-w)-m, 1-\frac{1}{2}(u+w)-m\end{matrix}\right],$$

which is the form assumed by (2.2) when x is a negative integer and n is not necessarily a positive integer. From this formula we obtain

THEOREM IV. *If*

$${}_2F_1\left[\begin{matrix}\alpha, \beta; \\ \gamma\end{matrix}\ x\right] {}_2F_1\left[\begin{matrix}\alpha, \beta; \\ \delta\end{matrix}\ x\right] = \Sigma c_n x^n,$$

then

$${}_4F_3\left[\begin{matrix}\alpha, \beta, \frac{1}{2}(\gamma+\delta), \frac{1}{2}(\gamma+\delta-1); \\ \alpha+\beta, \gamma, \delta\end{matrix}\ 4x(1-x)\right] = \Sigma \frac{(\gamma+\delta-1)_n}{(\alpha+\beta)_n} c_n x^n.$$

* Orr, **5** (59′).

† Orr, **5** (54′).

‡ Whipple, **10** (6.6).

Similarly, changing y, u, w into $1-m-y, 1-m-u, 1-m-w$ in (5.1), we derive

THEOREM V. *If*

$$ {}_2F_1\left[\begin{matrix}\alpha, \beta; \\ \gamma\end{matrix} x\right] {}_2F_1\left[\begin{matrix}\alpha, \delta; \\ \gamma\end{matrix} x\right] = \Sigma d_n x^n, $$

then

$$ (1-x)^{-\alpha} {}_4F_3\left[\begin{matrix}\alpha, \beta, \delta, \gamma-\alpha; \\ \gamma, \frac{1}{2}(\beta+\delta), \frac{1}{2}(\beta+\delta+1)\end{matrix} -x^2/4(1-x)\right] = \Sigma \frac{(\gamma)_n}{(\beta+\delta)_n} d_n x^n. $$

6. From Theorem IV several interesting results can be derived. Thus, if $\delta = \alpha+\beta-\gamma+1$, we have

$$ (6.1) \quad {}_2F_1\left[\begin{matrix}\alpha, \beta; \\ \gamma\end{matrix} x\right] {}_2F_1\left[\begin{matrix}\alpha, \beta; \\ \alpha+\beta-\gamma+1\end{matrix} x\right] = {}_4F_3\left[\begin{matrix}\alpha, \beta, \frac{1}{2}(\alpha+\beta), \frac{1}{2}(\alpha+\beta+1); \\ \alpha+\beta, \gamma, \alpha+\beta-\gamma+1\end{matrix} 4x(1-x)\right], $$

valid inside the loop of the lemniscate $|4x(1-x)| = 1$ which surrounds the origin.

If $\gamma = \delta = \beta$, we obtain the quadratic transformation of Gauss,

$$ (6.2) \qquad {}_2F_1\left[\begin{matrix}\alpha, \beta; \\ \alpha+\beta+\frac{1}{2}\end{matrix} 4x(1-x)\right] = {}_2F_1\left[\begin{matrix}2\alpha, 2\beta; \\ \alpha+\beta+\frac{1}{2}\end{matrix} x\right]. $$

From (6.1) and (6.2) we have

$$ \left\{{}_2F_1\left[\begin{matrix}\alpha, \beta; \\ \alpha+\beta+\frac{1}{2}\end{matrix} 4x(1-x)\right]\right\}^2 = {}_3F_2\left[\begin{matrix}2\alpha, 2\beta, \alpha+\beta; \\ 2\alpha+2\beta, \alpha+\beta+\frac{1}{2}\end{matrix} 4x(1-x)\right], $$

which is equivalent to Clausen's theorem.

Similarly from Theorem V we obtain, when $\delta = \gamma-\beta$,

$$ (6.3) \quad {}_2F_1\left[\begin{matrix}\alpha, \beta; \\ \gamma\end{matrix} x\right] {}_2F_1\left[\begin{matrix}\alpha, \gamma-\beta; \\ \gamma\end{matrix} x\right] = (1-x)^{-\alpha} {}_4F_3\left[\begin{matrix}\alpha, \beta, \gamma-\alpha, \gamma-\beta; \\ \gamma, \frac{1}{2}\gamma, \frac{1}{2}(\gamma+1)\end{matrix} -x^2/4(1-x)\right], $$

and, when $\gamma = \delta = \beta$ (changing β into $\alpha+\beta$),

$$ (6.4) \qquad (1-x)^{-\alpha} {}_2F_1\left[\begin{matrix}\alpha, \beta; \\ \alpha+\beta+\frac{1}{2}\end{matrix} -x^2/4(1-x)\right] = {}_2F_1\left[\begin{matrix}2\alpha, \alpha+\beta; \\ 2\alpha+2\beta\end{matrix} x\right], $$

which is a known result*.

* Bailey, 1 (4.22).

7. The formula (6.3) can be derived from a result given by Watson*, namely

$$(7.1)\quad {}_2F_1\left[\begin{matrix}\alpha, \beta; \\ \gamma\end{matrix} z\right] {}_2F_1\left[\begin{matrix}\alpha, \beta; \\ \gamma\end{matrix} Z\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}(\gamma-\alpha)_m(\gamma-\beta)_m}{(\gamma)_m(\gamma)_{2m+n}\, m!\, n!} (zZ)^m (z+Z-zZ)^n.$$

If we take $z+Z-zZ=0$, we obtain

$$(7.2)\quad {}_2F_1\left[\begin{matrix}\alpha, \beta; \\ \gamma\end{matrix} z\right] {}_2F_1\left[\begin{matrix}\alpha, \beta; \\ \gamma\end{matrix} -z/(1-z)\right] = {}_4F_3\left[\begin{matrix}\alpha, \beta, \gamma-\alpha, \gamma-\beta; \\ \gamma, \tfrac{1}{2}\gamma, \tfrac{1}{2}\gamma+\tfrac{1}{2}\end{matrix} -z^2/4(1-z)\right].$$

Using Gauss's formula

$$(7.3)\quad {}_2F_1\left[\begin{matrix}\alpha, \beta; \\ \gamma\end{matrix} -z/(1-z)\right] = (1-z)^{\alpha}\, {}_2F_1\left[\begin{matrix}\alpha, \gamma-\beta; \\ \gamma\end{matrix} z\right],$$

we derive (6.3).

The formula (6.1) cannot be obtained from Watson's result, since the denominator parameters of the two series on the left are unequal. It does, however, suggest the possibility of the existence of another result of Watson's type†.

Further relations expressing a product of two hypergeometric series in terms of a series of the type ${}_4F_3$ can be obtained from (6.1) and (6.3). Thus, if we transform the series on the left of (6.1) by (7.3), and then put $x=-z/(1-z)$, we obtain a formula equivalent to

$$(7.4)\quad {}_2F_1\left[\begin{matrix}\alpha, \beta; \\ \gamma\end{matrix} z\right] {}_2F_1\left[\begin{matrix}\gamma-\beta, 1-\beta; \\ \alpha-\beta+1\end{matrix} z\right] = (1-z)^{\beta-\alpha-\gamma}\, {}_4F_3\left[\begin{matrix}\alpha, \gamma-\beta, \tfrac{1}{2}(\alpha+\gamma-\beta), \tfrac{1}{2}(\alpha+\gamma-\beta+1); \\ \alpha+\gamma-\beta, \gamma, \alpha-\beta+1;\end{matrix} -4z/(1-z)^2\right].$$

A result more general than this can be obtained by putting $z+n$ for z in

* Watson, **6**, 195.

† [*Added* 19 *October*, 1934.—Since this paper was written, the existence of such a result has been discovered. See W. N. Bailey, "A reducible case of the fourth type of Appell's hypergeometric functions of two variables", *Quart. J. of Math.* (Oxford), 4 (1933), 305–308, and W. N. Bailey, "On the reducibility of Appell's function F_4", *Quart. J. of Math.* (Oxford), 5 (1934), 291–292.]

(2.1). We thus derive

THEOREM VI. *If* $\gamma+\gamma'=\alpha+\alpha'+\beta+\beta'$, *and*

$$ {}_2F_1\left[\begin{matrix}\alpha, \beta; \\ \gamma\end{matrix}\ x\right] {}_2F_1\left[\begin{matrix}\alpha', \beta'; \\ \gamma'\end{matrix}\ x\right] = \Sigma e_n x^n, $$

then

$$ (1-x)^{1-\gamma-\gamma'} {}_4F_3\left[\begin{matrix}\alpha, \gamma-\beta, \frac{1}{2}(\gamma+\gamma'), \frac{1}{2}(\gamma+\gamma'-1); \\ \alpha+\alpha', \alpha+\beta', \gamma\end{matrix}\ -4x/(1-x)^2\right] $$

$$ = \Sigma \frac{(\gamma')_n (\gamma+\gamma'-1)_n}{(\alpha+\alpha')_n (\alpha+\beta')_n} e_n x^n. $$

When $\alpha'=\gamma-\beta$, $\beta'=1-\beta$, $\gamma'=1+\alpha-\beta$, this gives (7.4).

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SUBGROUPS OF ABELIAN GROUPS

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1. *Introduction.* The idea underlying this paper is that arbitrary subgroups of a fundamental Abelian group should be represented as matrices‡. This idea leads us to several new results, as well as providing a short-cut to some old ones.

In the first place, it permits us to classify completely all subgroups relative to any fixed choice of basis-elements of the fundamental group, in terms of three families of constants. In the second place, it leads to the determination of a new and more general class of relative invariants of subgroups; *i.e.*, invariants under automorphisms of the fundamental group.

It also compels us to correct the previously accepted results concerning characteristic subgroups, and permits us to complete the work of G. A. Miller in this direction. Finally, it yields various special results and interesting numerical formulae.

2. *Factorization.* Let $\mathfrak{G}$ be any Abelian group, and let $\mathfrak{A}_p$ denote the Sylow-subgroup of $\mathfrak{G}$ associated with the prime p; *i.e.*, the set of elements of $\mathfrak{G}$ whose order is a power of p. Let, further, $\mathfrak{S}$, $\mathfrak{S}'$, ... denote arbitrary subgroups of $\mathfrak{G}$.

The meet $(\mathfrak{S}, \mathfrak{A}_p)$ is evidently the Sylow-subgroup of $\mathfrak{S}$ associated with p; moreover, any Abelian group is the join, and hence the direct product, of its Sylow-subgroups. Further, since order is invariant, the Sylow-subgroups are characteristic. And lastly, each element as well as each subgroup of $\mathfrak{G}$ can be uniquely decomposed into its components in the various Sylow-subgroups.

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† I wish to express here my appreciation to Mr. P. Hall for many helpful criticisms and suggestions.

‡ *Cf.* especially A. Chatelet's *Les groupes Abéliens finis et les modules de points entiers* (Paris, 1925), for numerous other applications of this idea.

From these considerations it obviously follows that the problems of determining and characterizing the elements and subgroups of $\mathfrak{G}$ can be reduced to the corresponding problems for groups of prime-power order.

3. *Matrix representation.* We shall accordingly assume throughout that the orders of any set $A_1, \ldots, A_\omega$ of basis-elements of $\mathfrak{G}$ are powers of the same prime p, say $p^{h_1}, \ldots, p^{h_\omega}$. We shall so arrange the A_i that $h_i \geqslant h_{i+1}$, and consider them as fixed unless the contrary is expressly stated.

Any element of $\mathfrak{G}$ can be represented in the form $A^{k_1} \ldots A^{k_\omega}$; moreover, this representation is unique if we take k_i to modulus p^{h_i}. That is, we have

THEOREM 3.1. *The elements of $\mathfrak{G}$ are in* (1, 1) *correspondence with rows of ω integers, of which the i-th is a* least positive residue mod p^{h_i}. *The product of two elements of $\mathfrak{G}$ corresponds to the sum of the corresponding vectors in modular ω-space.*

If we take the rows as representing elements of $\mathfrak{G}$, we deduce

COROLLARY 3.1. *Any matrix of ω columns, the terms in whose j-th column are positive residues* mod p^{h_j}, *represents a subgroup $\mathfrak{S}$ of $\mathfrak{G}$; namely, that subset whose elements are linearly dependent on the rows of the matrix.*

Let us suppose that we are given any such matrix. We shall simplify further analysis by removing any factors p from each term $\Lambda_{i,j}$ of the matrix. In other words we adopt the notation $\Lambda_{i,j} = \tau_{i,j} p^{\lambda_{i,j}}$, where $0 \leqslant \lambda_{i,j} \leqslant h_j$, $0 < \tau_{i,j} \leqslant p^{h_j - \lambda_{i,j}}$, and $\tau_{i,j} \not\equiv 0 \pmod{p}$.

4. *Matrix transformation.* Let us consider the following five† ways of transforming a matrix, with respect to Corollary 3.1.

$\alpha_{i,j}$: the addition of the i-th column to the j-th $[i < j]$.

$\bar{\alpha}_{i,j}$: the addition of $p^{h_i - h_j}$ times the j-th column to the i-th $[i < j]$.

$\beta_{i,\sigma}$: the multiplication of the i-th column by the p-free constant σ.

$\bar{\beta}_{i,\sigma}$: the multiplication of the i-th row by the p-free constant σ.

$\gamma_{i,j}$: the addition of the i-th row to the j-th $[i \neq j]$.

Of these five transformations, $\alpha_{i,j}$ merely substitutes $A_j^{-1} A_i$ for A_i; $\bar{\alpha}_{i,j}$ substitutes $A_i^{-p^{h_i - h_j}} A_j$ for A_j; $\beta_{i,\sigma}$ substitutes $1/\sigma A_i$ for A_i as basis-elements of $\mathfrak{G}$ in §3; and $\bar{\beta}_{i,\sigma}$ and $\gamma_{i,j}$ perform exactly similar substitutions in the generators of $\mathfrak{S}$.

† The notation given here will be used throughout the paper without further explanation.

It may be verified that the first three transformations preserve (A) combination as addition, (B) reducibility of any term in any k-th column mod p^{h_k}, (C) distinctness, all with respect to the individual elements of $\mathfrak{G}$. By induction we can infer

THEOREM 4.1. *Any finite succession of* $\alpha_{i,j}$, $\bar{\alpha}_{i,j}$, *and* $\beta_{i,\sigma}$ *is an automorphism of* $\mathfrak{G}$.

COROLLARY 4.1. *If* $h_i = h_j$, *then* $\pi_{i,j}$, *the permutation of the* i*-th column and the* j*-th, is an automorphism of* $\mathfrak{G}$, *which can be effected by* $\alpha_{i,j}$, $-\bar{\alpha}_{i,j}$, $\alpha_{i,j}$, $\beta_{i,-1}$.

And, since any automorphism is merely a change in the choice of basis-elements, we have

THEOREM 4.2. *Any automorphism of* $\mathfrak{G}$ *can be effected by a finite succession of* $\alpha_{i,j}$, $\bar{\alpha}_{i,j}$, *and* $\beta_{i,\sigma}$.

It may be also verified that the last two transformations do not alter the set of vectors linearly dependent on the rows of the matrix.

THEOREM 4.3. *Any finite succession of* $\bar{\beta}_{i,\sigma}$ *and* $\gamma_{i,j}$ *transforms the generators of* $\mathfrak{S}$ *without altering* $\mathfrak{S}$.

With these preliminary assumptions and facts in mind, we shall apply a modification of the classical *reduction of matrices to standard form*.

Fixed basis-elements of the fundamental group.

5. *Partial reduction.* Let us in fact fix upon a particular subgroup $\mathfrak{S}$ of $\mathfrak{G}$. Consider the matrix of ω columns whose rows represent (by Theorem 3.1) the distinct elements of $\mathfrak{S}$, order being immaterial.

Let μ_1 be the maximum value assumed by $h_j - \lambda_{i,j}$, and let j_1 be the first column in which, for suitable i, $h_{j_1} - \lambda_{i,j_1} = \mu_1$. We can choose i so that not only $h_{j_1} - \lambda_{i_1,j_1} = \mu_1$, but also $\tau_{i_1,j_1} = 1$. Let us fix upon such an i_1, and let S_1 denote the element represented by the i_1-th row. We can then assert:

(1) Since $h_{j_1} - \lambda_{i,j_1} \leqslant \mu_1$, any element of $\mathfrak{S}$ is the product of a suitable power of S_1 and an element whose j_1-th component is $p^{h_{j_1}}$.

(2) The elements whose j_1-th component is $p^{h_{j_1}}$ form a subgroup $\mathfrak{S}_1$ of $\mathfrak{G}$.

(3) Since $h_j - \lambda_{i_1,j} \leqslant \mu_1$, E is the only power of S_1 lying in $\mathfrak{S}_1$.

(4) μ_1, j_1, and $\mathfrak{S}_1$ are fixed by $\mathfrak{S}$ and the basis of $\mathfrak{G}$.

We can now neglect the j_1-th column and proceed similarly with $\mathfrak{S}_1$ to form $\mu_2, j_2 \neq j_1, i_2, S_2$, and $\mathfrak{S}_2$, concerning which we can repeat assertions (1)–(4) but with altered subscripts. Proceeding by finite induction, we arrive at $\mathfrak{S}_\omega = E$. This gives us the temporary conclusion, writing $p^{\lambda_{jk}}$ for $p^{\lambda_{i_k, j_k}}$,

THEOREM 5.1. *$\mathfrak{S}$ may be represented in the sense of Corollary 3.1 as a square matrix $\Lambda^* = (\Lambda^*_{i,j})$ of ω rows and columns. The k-th row has $p^{h_{j_x}}$ in the j_x-th column if $x < k$, $p^{\lambda_{j_k}}$ in the j_k-th, and $\tau_{i_k, j_x} p^{\lambda_{i_k j_x}}$ in the j_x-th if $x > k$. Here $h_{i_k} - \lambda_{j_k} \geqq h_{i_k} - \lambda_{i_k, j_x}$ throughout, the equality sign being excluded if $j_x < j_k$. Moreover, $0 \leqq h_{j_{k+1}} - \lambda_{j_{k+1}} \leqq h_{j_k} - \lambda_{j_k}$, the equality sign being excluded if $j_{k+1} < j_k$.*

Finally, $\mathfrak{S} = \mathfrak{S}_1 \times \ldots \times \mathfrak{S}_\omega$, where $\mathfrak{S}_k$ denotes the cyclic group of order p^{μ_k} generated by S_k, and $\times$ denotes "direct product".

6. *Possible types.* Let us digress to consider the possible type of $\mathfrak{S}$, *i.e.*, by Theorem 5.1, $(p^{\mu_1}, \ldots, p^{\mu_\omega})$. $\mu_k \leqq h_{i_k}$ and $\mu_k \leqq \mu_{k-1}$, so that $\mu_k \leqq h_{i_1}, \ldots, h_{i_k}$. But, since $h_i \geqq h_{i+1}$, and $k \neq k'$ implies $i_k \neq i_{k'}$, this means that certainly $\mu_k \leqq h_k$. Conversely, given $\mu_1 \leqq h_1, \ldots, \mu_\omega \leqq h_\omega$ $[\mu_k \geqq \mu_{k+1}]$, the matrix $\Lambda^* : \Lambda^*_{i,i} \equiv p^{h_i - \mu_i}$ and $\Lambda^*_{i,j} \equiv p^{h_j}$ $[i \neq j]$ represents a subgroup of type $(p^{\mu_1}, \ldots, p^{\mu_\omega})$. Hence we have†

THEOREM 6.1. *The types of the subgroups of the Abelian group $\mathfrak{G}$ of type $(p^{h_1}, \ldots, p^{h_\omega})$ $[h_i \geqq h_{i+1}]$ are $(p^{\mu_1}, \ldots, p^{\mu_\omega})$, where $\mu_k \leqq h_k$ and $\mu_k \geqq \mu_{k+1}$.*

But if two sequences $\mu_1 \geqq \ldots \geqq \mu_\omega$ are term-by-term distinct, they are distinct in the aggregate. From this fact, if we write $\phi\{h_1, \ldots, h_\omega\}$ for the number of distinct subgroup-types in $\mathfrak{G}$ of Theorem 6.1, we can deduce the recurrence formula

$$\phi\{h_1, \ldots, h_\omega\} = \frac{(h_1+\omega)!}{h_1!\,\omega!} - \sum_{k=2}^{\omega} \frac{(h_1 - h_k + k)!}{(k+1)!\,(h_1 - h_k - 1)!}\, \phi\{h_{k+1}, \ldots, h_\omega\}.$$

From this and $\phi\{\,\} = 1$, we can easily calculate an explicit expression for small ω. For instance,

$$\phi\{h_1, h_2, h_3, h_4\} = \frac{(h_1+4)!}{h_1!\,4!} - \frac{(h_1 - h_4 + 3)!}{4!\,(h_1 - h_4 + 1)!} - \frac{(h_1 - h_3 + 2)!\,(h_4+1)}{3!\,(h_1 - h_3 - 1)!}$$

$$- \frac{(h_4 - h_3)(h_4 - h_3 + 1)}{2!} \left\{ \frac{(h_2+1)(h_2+2)}{2!} - \frac{(h_2 - h_1)(h_2 - h_1 + 1)}{2!} \right\}.$$

† A different proof is given by Burnside, *Theory of groups* (Cambridge, 1911), 107.

It also follows that $\phi\{h, \ldots, h\} = (h+\omega)!/(h!\,\omega!)$, where ω is the number of basis-elements†.

7. *Reciprocity.* Let the vectors (g) and (g') represent G and G' of $\mathfrak{G}$ as in Theorem 3.1. We write $G \perp G'$ (G is *orthogonal* to G') if, and only if,

$$\sum_{j=1}^{\omega} g_j g_j' p^{h_1-h_j} \equiv 0 \quad (p^{h_1}).$$

Evidently (1) if $G \perp G'$, then $G' \perp G$, (2) if $G \perp G'$ and $G \perp G''$, then $G \perp G'\,G''$.

Let $\mathfrak{S}'$ denote the subgroup [by (2)] of elements $G\colon (g)$ of $\mathfrak{G}$, which are orthogonal to every element of an arbitrary subgroup $\mathfrak{S}$ of $\mathfrak{G}$. Further, let $\mathfrak{S}$ be represented as in Theorem 5.1, but without the asterisk. From (2) we see that $G \perp$ every element of $\mathfrak{S}$ if and only if $G \perp S_1, \ldots, S_\omega$.

But if $k<i$, $\Lambda_{i,j_k} p^{h_1-h_{j_k}} \equiv 0$ (p^{h_1}). Further, if $k>i$, $\Lambda_{i,j_k} p^{h_1-h_{j_k}} \equiv 0$ $(p^{h_1-h_{j_i}+\lambda_{i,j_i}})$, since $h_{j_i}-\lambda_{i,j_i} \geqslant h_{j_k}-\lambda_{i,j_k}$. Finally, $\Lambda_{i,j_i} p^{h_1-h_{j_i}} = p^{h_1-h_{j_i}+\lambda_{j_i}}$. Consequently we have the

LEMMA. *Corresponding to any choice of $g_{j_{i+1}}, \ldots, g_{j_\omega}$, precisely $p^{\lambda_{i,j_i}}$ choices of g_{j_i} make $\Sigma g_j \Lambda_{i,j} p^{h_1-h_j} \equiv 0$ (p^{h_1}); i.e., $G \perp S_i$.*

That is, since $\mu_i = h_i - \lambda_{i,j_i}$, just $p^{h_\omega-\mu_\omega}$ choices of g_{j_ω} make $G \perp S_\omega$. Corresponding to any particular such choice of g_{j_ω}, just $p^{h_{\omega-1}-\mu_{\omega-1}}$ choices of $g_{j_{\omega-1}}$ make $G \perp S_{\omega-1}, \ldots$. Corresponding to any particular such choices of $g_{j_\omega}, \ldots, g_{j_2}$, just $p^{h_1-\mu_1}$ choices of g_{j_1} make $G \perp S_1$. Consequently we have

THEOREM 7.1. *The order $o\{\mathfrak{S}'\}$ of $\mathfrak{S}'$ is $p^{\Sigma h_i-\mu_i}$; $o\{\mathfrak{S}\}\,o\{\mathfrak{S}'\} = o\{\mathfrak{G}\}$.*

But by (1), if $\mathfrak{S}''$ denotes the subgroup of elements of $\mathfrak{G}$ orthogonal to every element of $\mathfrak{S}'$, then $\mathfrak{S}'' \supset \mathfrak{S}$. And, by Theorem 7.1, $o\{\mathfrak{S}''\} = o\{\mathfrak{S}\}$. Therefore $\mathfrak{S}'' = \mathfrak{S}$, and we have established a (1.1) correspondence between the subgroups of $\mathfrak{G}$ of given order and those of the same index. By §2, we can deduce‡

THEOREM 7.2. *The number of subgroups of order n in the Abelian group $\mathfrak{G}$ is equal to the number of subgroups of index n.*

† *Cf.* also H. Hilton, "On subgroups of a finite Abelian group", *Proc. London Math. Soc.* (2), 5 (1907), 1–5.

‡ This proof can also be based on the study of group characteristics. *Cf., e.g.,* Chatelet, *op. cit.*, 143. The only new result which I have obtained on reciprocity is that on characteristic subgroups in the case $p = 2$, claimed, I think erroneously, by Prof. Miller. Chatelet obtains only a little less; *cf.* 165, 168, where he only fails to show that the correspondence between *sous-groupes réciproques* is (1, 1) in the case of *sous-groupes caractéristiques au sens large.* Our correspondence inverts the operations of join and meet.

8. *Standard matrix.* Let us return to Theorem 5.1. Any representation of $\mathfrak{S}$ satisfying Theorem 5.1, and in addition the requirement that if $k > i$, then $0 < \Lambda^*_{i,j_k} \leqslant p^{\lambda_{j_k}}$, is known as a *standard matrix for* $\mathfrak{S}$.

Any $\mathfrak{S}$ has a standard matrix Λ^*. For, if we perform $\gamma_{i,i'}$ $[i > i']$ repeatedly in Theorem 5.1, we can make $0 < \Lambda^*_{i',j_i} \leqslant p^{\lambda_{j_i}}$. But if we do this in order of increasing i, and for fixed i in order of increasing i', we do not invalidate previously obtained results. This gives us *one* standard matrix.

Suppose that we have a second standard matrix M^*. The j_i, λ_{i,j_i}, and, if $k \leqslant i$, the Λ^*_{i,j_k} are fixed by the partial reduction of § 5. Consequently, since the i-th row $M^*_{i,j}$ of M^* represents an element of $\mathfrak{S}$, we have

$$M^*_{i,j} = \Lambda^*_{i,j} + \sum_{k>i} c_{i,k} \Lambda^*_{k,j}.$$

But, since $0 < M^*_{i,j_k} \leqslant p^{\lambda_{j_k}}$ $[k > i]$, if we let $k = i+1, \ldots, \omega$ successively, we see that (successively) $c_{i,i+1}\Lambda^*_{i+1,j} \equiv 0 \ (p^{h_j})$, $\ldots$, $c_{i,\omega}\Lambda^*_{\omega,j} \equiv 0 \ (p^{h_j})$. Thus $M^* = \Lambda^*$.

Applying Corollary 3.1 to pass back from Λ^* to $\mathfrak{S}$, we have the

FUNDAMENTAL THEOREM 8.1. *The subgroups $\mathfrak{S}$ of the Abelian group of type $(p^{h_1}, \ldots, p^{h_\omega})$ $[h_i \geqslant h_{i+1}]$ are in (1.1) correspondence with matrices of ω rows and columns, defined by three classes of numbers:*

(1) *$j_1, \ldots, j_\omega$ satisfying $1 \leqslant j_i \leqslant \omega$; $h \neq k$ implies $j_h \neq j_k$.*

(2) *$\lambda_1, \ldots, \lambda_\omega$ satisfying $h_{j_i} \geqslant h_{j_i} - \lambda_i \geqslant h_{j_{i+1}} - \lambda_{i+1} \geqslant 0$; $j_i > j_{i+1}$ implies $h_{j_i} - \lambda_i \neq h_{j_{i+1}} - \lambda_{i+1}$.*

(3) *$\tau_{i,i'}$ corresponding to $1 \leqslant i < i' \leqslant \omega$, and satisfying*

(a) *If $j_{i'} < j_i$, and $h_{j_{i'}} - h_{j_i} + \lambda_i \geqslant -1$, $0 < \tau_{i,i'} \leqslant p^{(h_{j_i}-\lambda_i)-(h_{j_{i'}}-\lambda_{i'})-1}$.*

(b) *If $j_{i'} < j_i$, and $h_{j_{i'}} - h_{j_i} + \lambda_i \leqslant -1$, $0 < \tau_{i,i'} \leqslant p^{\lambda_{i'}}$.*

(c) *If $j_{i'} > j_i$, then $h_{j_{i'}} - h_{j_i} + \lambda_i \geqslant 0$, and $0 < \tau_{i,i'} \leqslant p^{(h_{j_i}-\lambda_i)-(h_{j_{i'}}-\lambda_{i'})}$.*

The matrix corresponding to any such choice of j_i, λ_i, and $\tau_{i,i'}$ is:

$$\Lambda^*_{i,j} \equiv \left\{ \begin{array}{l} p^{h_{j_{i'}}} \text{ if } i > i', \quad p^{\lambda_i} \text{ if } i = i', \quad \tau_{i,i'} \text{ in case (3)(b)}, \\ \tau_{i,i'} p^{h_{j_{i'}} - h_{j_i} + \lambda_i + 1} \text{ in case (3)(a)}, \\ \tau_{i,i'} p^{h_{j_{i'}} - h_{j_i} + \lambda_i} \text{ in case (3)(c)} \qquad [1 \leqslant i, \ j \leqslant \omega]. \end{array} \right\}$$

$\mathfrak{S}$ is the subgroup generated by the elements corresponding under Theorem 3.1 to the rows of Λ^*.

Variable basis-elements of the fundamental group.

9. *Characteristic classes.* Let G be any element of $\mathfrak{G}$, and let $(\tau_i p^{\lambda_i})$ represent† G relative to suitably chosen basis-elements of $\mathfrak{G}$. Until $\lambda_i \geqslant \lambda_{i+1}$ and $h_{i+1}-\lambda_{i+1} \leqslant h_i-\lambda_i$, we can reduce λ_{i+1} and λ_i by $\alpha_{i,i+1}$ and $\bar{\alpha}_{i,i+1}$ respectively. Since $\lambda_i \geqslant 0$, after a finite number of reductions we get (by Theorem 4.1) basis-elements of $\mathfrak{G}$ relative to which the vector $(\tau_i p^{\lambda_i})$ representing G satisfies $\lambda_i \geqslant \lambda_{i+1}$ and $h_{i+1}-\lambda_{i+1} \geqslant h_i-\lambda_i$. But performing $\beta_{1,\sigma_1}, \ldots, \beta_{\omega,\sigma_\omega}$ $[\sigma_i \equiv 1/\tau_i \ (p^{h_i})]$, we get

THEOREM 9.1. *Under a suitable representation of* $\mathfrak{G}$, *the modular vector representing* G *is of the* characteristic form (p^{λ_i}), *where*

$$\lambda_{i+1} \leqslant \lambda_i \leqslant \lambda_{i+1}+(h_i-h_{i+1}).$$

But the fact that a vector is of the form $(c_i p^{\lambda_i})$, where c_i is any integer and $\lambda_{i+1} \leqslant \lambda_i \leqslant \lambda_{i+1}+(h_i-h_{i+1})$, is invariant under $\alpha_{i,j}$, $\bar{\alpha}_{i,j}$, and $\beta_{i,\sigma}$; therefore, by Theorem 4.2, it is invariant under any automorphism of $\mathfrak{G}$. But, since any automorphism carrying a vector *not* of this form into one having this form possesses an inverse, the fact that it is *not* of this form is also invariant. Thus we have

THEOREM 9.2. *If* $\lambda_{i+1} \leqslant \lambda_i \leqslant \lambda_{i+1}+(h_i-h_{i+1})$, *the fact that* G *is or is not representable as* $(c_i p^{\lambda_i})$ *is invariant under automorphisms of* $\mathfrak{G}$.

COROLLARY 9.2. *An element has only one characteristic form.*

We can prove Corollary 9.2 differently. Let y be called the *root-exponent*‡ of G if and only if y is the greatest number such that for suitable X in $\mathfrak{G}$, $X^{p^y} = G$. The root-exponents of G, G^p, G^{p^2}, ... are evidently invariants attached to G.

LEMMA. *If* G *and* G' *have the same root-exponents, they have the same characteristic forms* $(p^{\lambda_i}) = (p^{\lambda_i'})$; *consequently they are conjugate.*

For $\lambda_\omega = \lambda_\omega' =$ the root-exponent of G. And if $\lambda_\omega = \lambda_\omega', \ldots, \lambda_i = \lambda_i'$, then (1) if $h_i-\lambda_i = h_{i-1}-\lambda_{i-1}$, and $h_i-\lambda_i' = h_{i-1}-\lambda_{i-1}'$, certainly $\lambda_{i-1} = \lambda_{i-1}'$, (2) if $h_i-\lambda_i \neq h_{i-1}-\lambda_{i-1}$, and $h_i-\lambda_i' \neq h_{i-1}-\lambda_{i-1}'$, certainly

$$\lambda_{i-1}+(h_i-\lambda_i) = \lambda_{i-1}'+(h_i-\lambda_i) = \text{the root-exponent of } G^{p^{h_i-\lambda_i}},$$

† As always, τ_i is supposed to be prime to p, by choice of λ_i.

‡ The importance of this number was first recognized by H. Prüfer. *Cf.* his paper "Abzählbare primäre Abelsche Gruppen", *Math. Zeitschrift*, 17 (1923), 42, in which he calls it *Höhenexponent*.

(3) $h_i-\lambda_i < h_{i-1}-\lambda_{i-1}$ is incompatible with $h_i-\lambda_i' = h_{i-1}-\lambda_{i-1}'$, for the root-exponent of $G^{p^{h_i-\lambda_i}}$ would be $\lambda_{i-1}+(h_i-\lambda_i) < h_{i-1}$, while that of $G'^{p^{h_i-\lambda_i}}$ would be greater than or equal to $(h_i-\lambda_i')+\lambda_{i-1}' = h_{i-1}$, (4) similarly $h_i-\lambda_i = h_{i-1}-\lambda_{i-1}$ is incompatible with

$$h_i-\lambda_i' < H_{i-1}-\lambda_{i-1}'.$$

Thus, in any case, $\lambda_{i-1} = \lambda_{i-1}'$; whence the lemma. From this lemma we can deduce, besides Corollary 9.2,

THEOREM 9.3. *All the invariants of an element of an Abelian group can be expressed in terms of the root-exponents of its various powers.*

It remains to determine the characteristic forms which represent elements of $\mathfrak{G}$. Evidently it is necessary and sufficient that $0 \leqslant \lambda_i \leqslant h_i$, in addition to the conditions already stated. But, if $0 \leqslant \lambda_\omega \leqslant h_\omega$ and $\lambda_{i+1} \leqslant \lambda_i \leqslant \lambda_{i+1}+(h_i-h_{i+1})$ identically in i, then $0 \leqslant \lambda_i \leqslant h_i$. Hence we have

THEOREM 9.4. *The number of characteristic forms, and hence of characteristic classes, of $\mathfrak{G}$ is $\prod_{i=1}^{\omega}\{h_i-h_{i+1}+1\}$, where we take $h_{\omega+1}=0$.*

COROLLARY 9.4. *The number of abstractly distinct cyclic subgroups of $\mathfrak{G}$ is $\prod_{i=1}^{\omega}\{h_i-h_{i+1}+1\}$.*

For any two basis-elements of the same cyclic subgroup have the same characteristic form.

10. *Characteristic subgroups.* Inspection of the commutators under the automorphisms $\beta_{i,2}$ shows us that, if $p \neq 2$, any characteristic subgroup containing the element G whose invariants are λ_j contains the subgroup generated by the vectors $(\delta_{i,j}p^{\lambda_j})$† in the representation of $\mathfrak{G}$ which represents G as (p^{λ_j}).

But $\mathfrak{S}$ contains G, while, by Theorem 9.2, it is characteristic. Hence $\mathfrak{S}$ is the least characteristic subgroup of $\mathfrak{G}$ containing G. Moreover, the join of any two such subgroups is obviously such a subgroup. Hence any characteristic subgroup, being the join of the least characteristic subgroups containing its elements, is of this form. Consequently we have

THEOREM 10.1. *If $p \neq 2$, the characteristic subgroups of the Abelian group of type $(p^{h_1}, \ldots, p^{h_\omega})$ $[h_i \geqslant h_{i+1}]$ are the classes of elements representable*

† $\delta_{i,j}$ being the Kronecker δ, and i specifying the vector.

as $(c_i p^{\lambda_i})$, *where* $0 \leqq \lambda_\omega \leqq h_\omega$ *and* $\lambda_{i+1} \leqq \lambda_i \leqq \lambda_{i+1} + (h_i - h_{i+1})$. *The number of characteristic subgroups is* $\prod_{i=1}^{\omega} \{h_i - h_{i+1} + 1\}$.

The first half of Theorem 10.1 has been obtained essentially by G. A. Miller, who used other methods†; but his results in the case $p = 2$ are incomplete, contrary to his express assertion. He obtains, in fact, only the least characteristic subgroups containing single elements.

The complete result may, however, be stated as

THEOREM 10.2. *If* $p = 2$, *the characteristic subgroups of the Abelian group of type* $(p^{h_1}, \ldots, p^{h_\omega})$ $[h_i \geqq h_{i+1}]$ *are in* (1, 1) *correspondence with any* (λ), (j), *and* $\mathfrak{S}$ *satisfying*:

(1) (λ) *is a* ω*-vector,* $0 \leqq \lambda_\omega \leqq h_\omega$, *and* $\lambda_{i+1} \leqq \lambda_i \leqq \lambda_{i+1} + (h_i - h_{i+1})$.

(2) (j) *is a vector,* $j_i < j_{i+1}$, $1 \leqq j_i \leqq \omega$, *and*

$$\lambda_{j_i - 1} + (h_{j_i} - h_{j_i - 1}) \leqq \lambda_{j_i} - 1 \geqq \lambda_{j_i + 1}.$$

Let χ *denote the number of components in* (j).

(3) $\mathfrak{S}$ *is a subgroup of the Abelian group of type* (2, ..., 2) *and order* 2^χ, *whose standard matrix has* [*a*] *no column every term of which is* 2, *and* [*b*] *no row every term but one of which is* 2.

The characteristic subgroup corresponding to any choice of (λ), (j), and $\mathfrak{S}$ is the class of elements representable as $c_i p^{\lambda_i} + \sigma_i p^{\lambda_{j_i} - 1}$, where c_i is any integer, and (σ) represents any element of $\mathfrak{S}$.

By (2), these subgroups are transformed into themselves by $\alpha_{i,j}$, $\bar{\alpha}_{i,j}$, and $\beta_{i,\sigma}$; therefore they are characteristic. Moreover, the join of any two such subgroups, *i.e.*, the class of elements representable in the form $c_i p^{\lambda_i} + c_i' p^{\lambda_i'} + \sigma_i p^{\lambda_{j_i} - 1} + \sigma_i' p^{\lambda_{j_i'} - 1}$ is again such a subgroup; take the least λ_j such that $(\delta_{i,j} p^{\lambda_j})$ is so representable, and study the $(\sigma_i p^{\lambda_{j_i} - 1})$ which are so representable.

Finally, let $G: (p^{\lambda_1}, \ldots, p^{\lambda_\omega})$ be any element of $\mathfrak{G}$ in characteristic form. Inspection of the commutator under the automorphism $\beta_{1,3}$ of $\mathfrak{G}$ shows us that $(\delta_{i,j} p^{\lambda_j + 1})$ is contained in any characteristic subgroup $\mathfrak{E}$ of $\mathfrak{G}$ containing G. Further, if $\lambda_{j-1} = \lambda_j$ or $\lambda_{j+1} + (h_j - h_{j+1}) = \lambda_j$, inspection of the commutator under $\alpha_{j-1,j}$ and $\bar{\alpha}_{j,j+1}$ shows us that $\mathfrak{E} \supset (\delta_{i,j} p^{\lambda_j})$ also. But,

† "Determination of all the characteristic subgroups of an Abelian group", *Quart. J. of Math.*, 50 (1927), 54.

since $\mathfrak{E} \supset G$, we see that $\mathfrak{E}$ contains that characteristic subgroup of Theorem 10.2 for which $j_1, \ldots, j_\chi$ includes *every* j satisfying

$$\lambda_{j-1}+(h_j-h_{j-1}) \leqslant \lambda_j - 1 \geqslant \lambda_{j+1},$$

and $\mathfrak{S} = (1, \ldots, 1)+E$†. Hence the least characteristic subgroup containing any G of $\mathfrak{G}$ is one of the subgroups of Theorem 10.2.

But any characteristic subgroup is the join of the least characteristic subgroups containing its elements; whence, by the above paragraphs, we can conclude that *all* characteristic subgroups of $\mathfrak{G}$ are included in Theorem 10.2. To prove the theorem, we need only remark that the subgroups there defined are distinct.

Miller's proof of his theorem on reciprocity, namely

THEOREM 10.3. *The number of the characteristic subgroups of a given order m contained in any Abelian group is equal to the number of characteristic subgroups of index m contained in the same group,*

must, in the light of the above criticism, be regarded as incomplete. But, for reasons which are sufficiently clearly brought out in this paper, in order to prove Theorem 10.3 it is sufficient to show the existence of reciprocity in *caps* with respect to order.

LEMMA. *If the standard matrix of the subgroup $\mathfrak{S}$ of the Abelian group $\mathfrak{G}$ of type (2, ..., 2) satisfies [a] and [b] of Theorem 10.2, then so does the standard matrix of the orthogonal subgroup $\mathfrak{S}'$.*

For, if $\mathfrak{S}'$ did not satisfy [b], the column corresponding to the one term *not* 2 in a row of $\mathfrak{S}'$ would have to have *all* its terms even in $\mathfrak{S}$. And if $\mathfrak{S}'$ did not satisfy [a], then the element whose components were all 2 except the one in the exceptional column of $\mathfrak{S}'$ would be orthogonal to every element of $\mathfrak{S}'$, hence in $\mathfrak{S}$. But such an element would have to appear as a row in the standard matrix representing $\mathfrak{S}$.

This lemma, taken in connection with Theorem 7.1, establishes a (1, 1) correspondence between caps of reciprocal order; we can, therefore, assert the truth of Theorem 10.3.

11. *Characteristic matrix.* For the sake of uniformity, we shall ascribe to each subgroup of an Abelian group of prime-power order a *characteristic matrix*. The definition of this is only tentative, but it is nevertheless

† This is, in fact, Miller's *cap*; it is obviously only one of many possible caps.

illuminating. A probably eradicable defect of the definition is that not every characteristic matrix is standard†.

Let $\mathfrak{G}$ be any Abelian group of type $(p^{h_1}, \ldots, p^{h_\omega})$ $[h_i \geqslant h_{i+1}]$, and let $\mathfrak{S}$ be any subgroup of $\mathfrak{G}$. Under any choice of basis-elements of $\mathfrak{G}$, Theorem 8.1 assigns to $\mathfrak{S}$ a "standard matrix". But either $j_1 = 1$, or $\bar{\alpha}_{1, j_1}$ gives us a set of basis-elements of $\mathfrak{G}$ relative to which $j_1 = 1$. Similarly, consideration of $\bar{\alpha}_{2, j_2}$, $\bar{\alpha}_{3, j_3}$, ... gives us in the end a set of basis-elements of $\mathfrak{S}$ relative to which $j_i = i$, whence, in addition, *$i > j$ implies* $\Lambda_{i, j} = p^{h_j}$.

We shall define as the *characteristic matrix* of $\mathfrak{S}$ that one among the matrices satisfying the italicized condition and representing $\mathfrak{S}$ relative to suitably chosen basis-elements of $\mathfrak{G}$, which satisfies, subject to preceding ones, the following $\omega(\omega+1)$ conditions taken in order:

$$\begin{cases} \lambda_{i, i+j} \text{ is a minimum.} \\ \tau_{i, i+j} \text{ is a minimum.} \end{cases} \qquad [j = 0, \ldots, \omega-1;\ i = 1, \ldots, \omega-j.]$$

The $\lambda_{i, j}$ will be called the *type* invariants, and the $\tau_{i, j}$ the *residue* invariants of $\mathfrak{S}$, for reasons which will become evident.

Inherent in the definition is the trivial

THEOREM 11.1. *$\mathfrak{S}$ and $\mathfrak{S}'$ are conjugate in $\mathfrak{G}$ if, and only if, their characteristic matrices are identical.*

12. *Type invariants.* We shall show that the range of the type invariants is independent of p, provided only that $p > \omega$. More precisely, we shall prove

THEOREM 12.1. *If $p > \omega$, the type invariants of the subgroups of the Abelian group $\mathfrak{G}$ of type $(p^{h_1}, \ldots, p^{h_\omega})$ $[h_i \geqslant h_{i+1}]$ range over the sets of constants $\lambda_{i, j}$ $[i \leqslant j]$ in the "canonical range", i.e., satisfying*

$$\xi: \left\{ \begin{array}{l} \lambda_{i, j+1} \leqslant \lambda_{i, j} \leqslant \lambda_{i, j+1} + (h_j - h_{j+1}) \\ \lambda_{i, j} \leqslant \lambda_{i+1, j} \leqslant \lambda_{i, j} + (h_i - \lambda_i) - (h_{i+1} - \lambda_{i+1}) \end{array} \right\} \qquad [\lambda_i \equiv \lambda_{i, i}],$$

under the convention $0 = \lambda_{\omega+1} = h_{\omega+1}$.

For, if $\lambda_{i, j+1} > \lambda_{i, j}$, *any* $\tau\alpha_{j, j+1}$ $[\tau = 1, \ldots, \omega]$ would reduce $\lambda_{i, j+1}$, and *one* would not increase $\lambda_{1, j+1}$, ..., $\lambda_{i-1, j+1}$, $\lambda_{i+1, j+1}$, ..., $\lambda_{j, j+1}$. Or if $\lambda_{i, j} > \lambda_{i, j+1} + (h_j - h_{j+1})$, to every τ $[\tau = 1, \ldots, \omega]$ would correspond a τ' such

† The matrix $\begin{pmatrix} p^4 & p^3 & 2p^2 & 1 \\ 0 & p^3 & p^3 & 1 \\ 0 & 0 & p^2 & 1 \\ 0 & 0 & 0 & p \end{pmatrix}$ $h_1 = 8$, $h_2 = 6$, $h_3 = 4$, $h_4 = 2$, $p = 3$, is characteristic but not standard.

that the sequence $\tau\bar{\alpha}_{j,j+1}$, $\tau'\gamma_{j,j+1}$ would keep $\Lambda_{j+1,j}=p^{h_j}$. But *any* such sequence would reduce $\lambda_{i,j}$, and *one* would not increase† $\lambda_{i,j}$, ..., $\lambda_{i-1,j}$, $\lambda_{i+1,j}$, ..., $\lambda_{j,j}$ or $\lambda_{j+1,j+1}$, ..., $\lambda_{j+1,\omega}$.

Similarly, we could reduce $\lambda_{i,j}>\lambda_{i+1,j}$ or

$$\lambda_{i+1,j}>\lambda_{i,j}+(h_i-\lambda_i)-(h_{i+1}-\lambda_{i+1})$$

by $\tau\gamma_{i+1,i}$ or a sequence $\tau p^{(h_i-\lambda_i)-(h_{i+1}-\lambda_{i+1})}\gamma_{i.i+1}$, $\tau'\bar{\alpha}_{i+1,i}$ $[\tau 1, \ldots, \omega]$ without altering the fact that $i>j$ implied $\Lambda_{i,j}=p^{h_j}$, and without increasing other λ's. Therefore, if $p>\omega$, any system of type invariants satisfies ξ.

It remains to show that any system of type invariants satisfying ξ actually occurs in some characteristic matrix. In this case we can abandon the restriction $p>\omega$ and show that any system of type invariants in the canonical range is irreducible.

In fact, the elements represented by the rows of any matrix of $\mathfrak{S}$ satisfying ξ (and hence any characteristic matrix) are independent generators of $\mathfrak{S}$ arranged according to their orders. Hence, the order of any element of $\mathfrak{S}$ is the greatest of the orders of its components, whence we see by ξ‡ that any element of $\mathfrak{S}$ having the same order as the k-th independent generator satisfies the hypotheses of Theorem 9.2, the only change being λ_i is replaced by $\lambda_{k,i}$. But, by that theorem, this result cannot be altered by automorphisms of $\mathfrak{G}$. Or, remarking again that the order of the k-th independent generator is fixed by ξ, we see that any set of exponents in the canonical range is irreducible. The requirement $0=\lambda_{\omega+1}=h_{\omega+1}$, taken in connection with ξ, is merely a necessary and sufficient condition that $0\leqslant\lambda_{i,j}\leqslant h_j$.

It follows that the matrix $\Lambda_{i,j}=p^{\lambda_{i,j}}$ $[i\leqslant j]$ is characteristic, since neither the $\lambda_{i,j}$ nor the $\tau_{i,j}$ can be reduced. Thus we have proved Theorem 12.1, and in addition

THEOREM 12.2. *If $p\leqslant\omega$, the type invariants of the subgroups of $\mathfrak{G}$ in Theorem* 12.1 *range over the entire canonical range, and possibly a "singular" range associated with p as well.*

THEOREM 12.3. *If $p>\omega$, the $\lambda_{i,j}$ are invariants of $\mathfrak{S}$, in that $(p^{\lambda_{i,j}})$ is the characteristic form of an element of $\mathfrak{S}$ of order $p^{h_i-\lambda_i}$, and no element of $\mathfrak{S}$ of order $p^{h_i-\lambda_i}$ has a characteristic form whose j-th component is less than $p^{\lambda_{i,j}}$.*

† We are relying throughout on number theoretical consequences of the lemma: If σ_1 and σ_2 are prime to p, then the number of times that p divides $\sigma_1 p^{\mu_1}+\sigma_2 p^{\mu_2}$ is at most μ_1, unless $\mu_1=\mu_2$ and $\sigma_1\equiv-\sigma_2$ (p).

‡ Strictly, its consequence $\lambda_{i,j}\leqslant\lambda_{i+k,j}\leqslant\lambda_{i,j}+(h_i-\lambda_i)-(h_{i+k}-\lambda_{i+k})$ $[k\geqslant 1]$.

And, referring to Theorem 9.3, we have

COROLLARY 12.3. *If $p > \omega$, and we know the range of the root-exponents of the elements of each order in $\mathfrak{S}$, we know the type invariants of $\mathfrak{S}$.*

13. *Special cases.* Let us suppose that $\mathfrak{G}$ is of type $(p^h, \ldots, p^h)$ and of order $p^{h\omega}$. In this case, Corollary 4.1 permits us to interchange any two columns as well as any two rows. Hence, starting with any standard matrix of a given subgroup $\mathfrak{S}$ of $\mathfrak{G}$, permuting rows and columns so as to minimize successively $\lambda_{1,1}, \ldots, \lambda_{\omega,\omega}$, and multiplying the columns by suitable p-free constants, we obtain a matrix for $\mathfrak{S}$ in which $\Lambda_{i,i} = p^{\lambda_i}$ and $\lambda_i \leqslant \lambda_{i+h,i+k}$. If now we subtract successively suitable multiples of the 1st, 2nd, ..., $(\omega-1)$-th rows and columns from later rows and columns, we get a matrix of the form $\Lambda_{i,j} = p^h$ if $i > j$, $\Lambda_{i,j} = p^{\lambda_i}$ if $i \leqslant j$. But the type invariants of this are irreducible, by §12; the residue invariants are obviously irreducible; therefore we have the *characteristic matrix* of $\mathfrak{S}$. Thus we have

THEOREM 13.1. *Let $\mathfrak{S}$ be any subgroup of $\mathfrak{G}$, the Abelian group of type $(p^h, \ldots, p^h)$ and order $p^{h\omega}$, and let $(p^{h-\lambda_1}, \ldots, p^{h-\lambda_\omega})$ $[h-\lambda_i \geqslant h-\lambda_{i+1}]$ be the type of $\mathfrak{S}$. Then $\mathfrak{S}$ is characterized by its type, for its characteristic matrix is*

$$\Lambda_{i,j} = \left\{ \begin{array}{ll} p^h & \text{if } i > j \\ p^{\lambda_i} & \text{if } i \leqslant j \end{array} \right\} \quad [\lambda_i \leqslant \lambda_{i+1}].$$

From §6, we then obtain

COROLLARY 13.1. *The number of abstractly distinct subgroups of $\mathfrak{G}$ is $(h+\omega)!/h!\,\omega!$.*

Alternatively, let the type of $\mathfrak{G}$ be (p^{h_1}, p^{h_2}) $[h_1 \geqslant h_2]$, and let $\mathfrak{S}$ be any subgroup of $\mathfrak{G}$. Consider the characteristic matrix (Λ) of $\mathfrak{S}$. $\lambda_{1,2} \leqslant \lambda_1$, $\lambda_{1,2} \leqslant \lambda_2$, and $\lambda_1 \leqslant \lambda_{1,2} + (h_1 - h_2)$; otherwise we could reduce the λ's by $\alpha_{1,2}$, $\gamma_{2,1}$, or the sequence $\bar{\alpha}_{1,2}$, $\tau\gamma_{1,2}$. Likewise $\tau_{1,1} = \tau_{1,2} = \tau_{2,2} = 1$, or by some sequence $\bar{\beta}_{1,\sigma}$, $\beta_{2,\sigma}$, $\bar{\beta}_{2,\sigma}$ we could reduce the τ's. But any λ's which satisfy these equations and $0 \leqslant \lambda_{i,j} \leqslant h_j$ are irreducible, by §12; the $\tau_{i,j}$ are obviously irreducible. Consequently we have

THEOREM 13.2. *The characteristic matrices of the subgroups of the Abelian group $\mathfrak{G}$ of type (p^{h_1}, p^{h_2}) are of the form*

$$\begin{pmatrix} p^{\lambda_1} & p^{\lambda_{1,2}} \\ p^{h_1} & p^{\lambda_2} \end{pmatrix} \quad [0 \leqslant \lambda_2 \leqslant h_2;\ 0 \leqslant \lambda_{1,2} \leqslant \lambda_2;\ \lambda_{1,2} \leqslant \lambda_1 \leqslant \lambda_{1,2} + (h_1 - h_2)].$$

COROLLARY 13.2. *In Theorem 13.2, the number of abstractly distinct subgroups of $\mathfrak{G}$ is $\frac{1}{2}(h_2+1)(h_2+2)(h_1-h_2+1)$. Of these $(h_2+1)(h_1-h_2+1)$ are cyclic, and $\frac{1}{2}h_2(h_2+1)(h_1-h_2+1)$ are not.*

14. *Residue invariants.* It remains to define the range and nature of the residue invariants. Our results concerning these are fragmentary, but illuminating. We can only partially restrict their range by three types of reduction, and define a new class of invariants.

We may perform a simple *beta-type* reduction on any matrix, without altering the exponents of p. That is, we can so choose $\bar{\sigma}_1, \sigma_2, \ldots, \sigma_\omega, \bar{\sigma}_\omega$ that $\beta_{1,\bar{\sigma}_1}, \beta_{2,\bar{\sigma}_2}, \ldots, \beta_{\omega,\sigma_\omega}, \bar{\beta}_{\omega,\sigma_\omega}$ reduce $\tau_{1,1}, \tau_{1,2}, \ldots, \tau_{\omega-1,\omega}\ \tau_{\omega,\omega}$ successively to unity. Referring to §11, we have

THEOREM 14.1. *In any characteristic matrix,* $\tau_{i,i} = \tau_{i,i+1} = 1$. *Hence at most* $\frac{1}{2}(\omega-1)(\omega-2)$ *residue invariants can differ from unity.*

Or we may perform a simple *alpha-type* reduction. Repeated use of $a_{i,i+j}$ certainly reduces $\tau_{i,i+j}$ to a residue mod $p^{\lambda_i - \lambda_{i,i+j}}$, without affecting any preceding† coefficients. We may similarly employ the simple *gamma-type* reduction $\gamma_{i+j,i}$ to reduce $\tau_{i,i+j}$ to a residue mod $p^{\lambda_{i+j} - \lambda_{i,i+j}}$. Hence we obtain

THEOREM 14.2. *In any characteristic matrix, every* $\tau_{i,i+j}$ *is a residue* mod $p^{\min\{\lambda_i, \lambda_{i+j}\} - \lambda_{i,i+j}}$.

In connection with Theorem 12.1, Theorem 14.2 gives us

COROLLARY 14.2. *In any characteristic matrix, every* $\tau_{i,j}$ *is a residue* mod $p^{h_i - h_j}$ $[i < j]$.

Combination of $a_{i,i+1}$ or $\gamma_{i,i+1}$ and the beta-type reduction given above gives us even greater restrictions on the $\tau_{i,j}$. But, since the nature of the restrictions is more complicated, we shall not elaborate it, but shall only observe that in general the restrictions are like those of Theorems 14.1 and 14.2, with the addition of exceptional singularities.

Combination of $p^{(h_i - \lambda_i) - (h_j - \lambda_j)} \bar{a}_{i,j}$ with the reduction of §5 and the reductions just given will likewise in general not alter the $\lambda_{i,j}$, hence the restrictions of Theorems 14.1 and 14.2, and will often give us greater restrictions on the range of the $\tau_{i,j}$. But, since the results are complicated and inconclusive, we shall omit discussion of this reduction as well.

15. *New invariants.* Let now $\mathfrak{S}$ be any subgroup of $\mathfrak{G}$, and let S and T be any two elements of $\mathfrak{S}$. The characteristic classes of the $S^h T^k$, as elements of $\mathfrak{G}$ and as elements of $\mathfrak{S}$, are obviously invariant under automorphisms of $\mathfrak{G}$. Conversely, given a method of ascribing to preassigned

† Taken in the order of definitive importance established in § 11, namely

$$\tau_{1,1}, \ldots, \tau_{\omega,\omega}; \quad \tau_{1,2}, \ldots, \tau_{\omega-1,\omega}; \quad \ldots; \quad \tau_{1,\omega}.$$

$S^h T^k$ characteristic classes in $\mathfrak{G}$ and in $\mathfrak{S}$, the existence of a pair (S, T) of elements of $\mathfrak{S}$ such that the preassigned $S^h T^k$ are in the classes required, is a property of $\mathfrak{S}$ independent of the representation of $\mathfrak{G}$.

The transition from these invariant properties to invariant numbers is easy. For instance, given the characteristic classes of S, T, and ST in $\mathfrak{G}$ and in $\mathfrak{S}$, there may exist just one k mod p^c such that ST^k is in fourth given classes in $\mathfrak{G}$ and in $\mathfrak{S}$. k then becomes a numerical invariant attached to $\mathfrak{S}$. We shall give an indication of the generality of the invariant numbers which can be defined by these methods.

Let $\mathfrak{G}$ be as usual of type $(p^{h_1}, \ldots, p^{h_\omega})$ $[h_i \geqslant h_{i+1}]$, and let a matrix for $\mathfrak{S}$ have exponents within the canonical range, and coefficients satisfying the conditions of Theorem 14.1. Let further

$$(h_{i-1}-\lambda_{i-1})-(h_i-\lambda_i) \neq \lambda_{i,j}-\lambda_{i-1,j}, \quad \lambda_{i,j} \neq \lambda_{i+1,j}, \quad h_{i+2}-\lambda_{i,i+2} \neq h_{i+1}-\lambda_{i,i+1},$$

$$(h_i-\lambda_i)-(h_{i+1}-\lambda_{i+1}) \neq \lambda_{i+1,j}-\lambda_{i,j}, \quad \lambda_{i+1,j} \neq \lambda_{i+2,j},$$

$$h_{i+2}-\lambda_{i+1,i+2} \neq h_{i+1}-\lambda_{i+1,i+1}.$$

Let, finally, $r_k{}^*\{S\}$ denote the root-exponent of S^{p^k} in $\mathfrak{G}$, $\bar{r}_k{}^*\{S\}$ the root-exponent of S^{p^k} in $\mathfrak{S}$, and $o^*\{S\} \log_p$ (order of S).

We can choose S so that $o^*\{S\} = h_i - \lambda_i$, and $\bar{r}_0{}^*\{S\} = 0$. In this case [since, by Theorem 12.1, $k < i$ implies $(h_k-\lambda_k)-(h_i-\lambda_i)+\lambda_{k,j} > \lambda_{i,j}$, and $k > 1$ implies $\lambda_{k,j} > \lambda_{i,j}$] the vector representing S has the form $\sigma_1\{\hat{\tau}_j p^{\lambda_{i,j}}\}$ $[\hat{\tau}_j \equiv \tau_{i,j}(p)]$ so far as its i-th, ..., ω-th components are concerned, and the earlier components are irrelevant to the subsequent argument†. Similarly, we can so choose T that $o^*\{T\} = h_{i+1}-\lambda_{i+1}$, and $\bar{r}_o{}^*\{T\} = 0$. It will follow likewise that the vector representing T is of the form

$$\sigma_2\{\hat{\tau}_j{}' p^{\lambda_{i+1,j}}\} \quad [\hat{\tau}_j{}' \equiv \tau_{i+1,j}(p)]$$

so far as its $(i+1)$-th, ..., ω-th components are concerned, and that earlier components are irrelevant to the subsequent argument.

We can also require that $r^*_{h_{i+1}-\lambda_{i+1}-1}\{S^{p^{\lambda_{i+1}-\lambda_{i,i+1}} T^{-1}}\} \neq h_{i+1}-1$, by making $\sigma_1 \equiv \sigma_2(p)$. Moreover, this condition is necessary as well as sufficient.

But, if also $h_k-\lambda_{i,k} \neq h_{k+1}-\lambda_{i,k+1}$ and $h_k-\lambda_{i+1,k} \neq h_{k+1}-\lambda_{i+1,k+1}$, then a necessary and sufficient condition that

$$r^*_{h_k-\lambda_{i+1,k}-1}\{S^{p^{\lambda_{i+1,k}-\lambda_{i,k}} T^{-x}}\} \neq h_k-1$$

is that $\hat{\tau}_k \sigma_1 \equiv x \hat{\tau}_k{}' \sigma_2(p)$. That is, $x \equiv \hat{\tau}_k / \hat{\tau}_k{}'(p)$, *i.e.* $x \equiv \tau_{i,k}/\tau_{i+1,k}$.

† It may be deduced rigorously from Theorem 12.1 that $r_k{}^*\{S\}$ is unaltered if $k \leqslant h_{i+1}-\lambda_{i+1}$.

Since in a characteristic matrix $\tau_{k-1,k}=1$, it follows by induction that $\tau_{k-2,k}$, ..., $\tau_{1,k}$ are invariants mod p provided that the hypotheses stated above are satisfied for $i=1, \ldots, k-2$. This gives us the general result, which can be greatly extended to cases of weaker hypotheses.

THEOREM 15.1. *If* $i>j$ *implies* $\Lambda_{i,j}=p^h$, *and* $i\leqslant j$ *implies* $\tau_{i,i}=\tau_{i,i+1}=1$, $\tau_{i,j}<p$, $\lambda_{i,j+1}<\lambda_{i,j}<\lambda_{i,j+1}+(h_j-h_{j+1})$, *and*

$$\lambda_{i-1,j}<\lambda_{i,j}<\lambda_{i-1,j}+(h_{i-1}-\lambda_{i-1})-(h_i-\lambda_i),$$

then the matrix (Λ) *is characteristic.*

For by § 12, the λ's cannot be reduced, the $\tau_{i,i}$ and $\tau_{i,i+1}$ obviously cannot be reduced, and once these are fixed we have just seen that the other $\tau_{i,j}$ are invariants mod p and hence cannot be reduced.

COROLLARY 15.1. *The number of abstractly distinct kinds of subgroup in the Abelian group of type* (p^6, p^4, p^2) *tends to infinity with* p.

For the matrices

$$\begin{pmatrix} p^2 & p & \tau \\ p^6 & p^2 & p \\ p^6 & p^4 & p^2 \end{pmatrix}$$

$[0<\tau<p]$ are all characteristic. *Cf.* Theorem 11.1.

Let us now generate $\mathfrak{G}_p$ of Class 2 and order p^{22}, from elements A, B, C, D, F, and G, which are to be taken as independent and respectively of orders p^6, p^4, p^2, p^4, p^4, and p^2. Let A, B, and C commute with every element of $\mathfrak{G}_p$, let D commute with G, let $FD=DFA^{p^2}B^pC^\tau$, and $GD=DGB^{p^2}C^p$.

The centre of $\mathfrak{G}_p$ will be the subgroup generated by A, B, and C. The commutator-group will be a subgroup of the centre, and representable in it by the matrix of Corollary 15.1. We thus obtain

COROLLARY 15.2. *The number of kinds of groups of Class 2 and order* p^{22} *tends to infinity with* p.

(*Added 5 November*, 1934.)

16. *Ferrers' graph and type duality.* Let, as usual, $\mathfrak{G}$ be the Abelian group of order $p^{h_1+\ldots+h_\omega}$, with basis-elements of orders p^{h_1}, ..., p^{h_ω} $[h_1\geqslant\ldots\geqslant h_\omega]$. We can represent $\mathfrak{G}$ by a plane figure $F(\mathfrak{G})$ consisting of the sum of the ω rectangles $R_k: k-1\leqslant x\leqslant k$, $0\leqslant y\leqslant h_k$ $[k=1, \ldots, \omega]$.

It is geometrically obvious that $F(\mathfrak{G})$ is carried under the transformation $\tau : \tau(x) = y \ \ \tau(y) = x$, into the figure for a "type-dual" Abelian group $\mathfrak{G}^*$. If the basis of $\mathfrak{G}$ contains r_1 elements of order p^{x_1}, ..., r_a elements of order p^{x_a} $[x_1 > \ldots > x_a]$, then the basis of $\mathfrak{G}^*$ contains x_a elements of order $p^{r_1+\ldots+r_a}$, ..., $(x_a - x_{a-1})$ elements of order $p^{r_1+\ldots+r_{a-1}}$, ..., $(x_2 - x_1)$ elements of order p^{r_1}.

We observe now that (i) $(\mathfrak{G}^*)^* = \mathfrak{G}$, which justifies the term "type-dual", (ii) since the order of $\mathfrak{G}$ depends on the area of $F(\mathfrak{G})$, the order of $\mathfrak{G}$ is the order of $\mathfrak{G}^*$; (iii) since by Theorem (6.1) the figures for subgroups of $\mathfrak{G}$ are those contained in $F(\mathfrak{G})$, while interior figures are carried into interior figures, the types of subgroups of $\mathfrak{G}^*$ are the duals of the types of subgroups of $\mathfrak{G}$. Therefore

THEOREM 16.1. *There is a duality in the types of Abelian groups of each prime-power order, such that type-dual groups contain type-dual subgroups. Therefore type-dual groups contain the same number of types of subgroups.*

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ON THE GEOMETRY OF THE LINEAR REPRESENTATIONS OF THE SYMMETRIC GROUP

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Introduction.

A group is usually defined in one of three ways: as an abstract group (A), as a permutation group (P), or as a group of linear substitutions (L). The relations between these three methods of definition are fundamental, and as yet imperfectly known. Moore* has derived (A) from (P) in the case of the symmetric and alternating groups. Recently, however, this has been done in a much more elegant fashion by Todd†. He goes first to (L), implicitly making use of the analysis of Young, and then, with the help of the fundamental region, to (A). Coxeter‡ has extended Todd's results, though he is chiefly interested in the step from (L) to (A). (A) and (P) are linked to (L) by the theory of characteristics.

In the present paper we shall be concerned with the step from (P) to (L), which is completely determined in the case of the symmetric and hyper-octohedral groups by the analysis of Young, and in particular with the geometry of (L) when the group is written in orthogonal form. Certain general restrictions are found to apply to the configurations of points which may be generated by such a group, and it is conceivable that they might throw light on the problem of the order, or on that of its prime factors. It is interesting to see that the invariant configurations of the representations of the symmetric group which are obtained are projections of a regular simplex, and in the case of the hyper-octohedral group, of the cross-polytope, in a higher space.

* *Proc. London Math. Soc.* (1), 28 (1897), 357–366.

† *Proc. Camb. Phil. Soc.*, 27 (1931), 212–231.

‡ *Proc. London Math. Soc.* (2), 34 (1932), 126–189; *Annals of Math.*, 35 (1934), 588–620.

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1.1. *The symmetric group.*

The problem of the representation of the symmetric group as an irreducible group of linear transformations has been studied by Frobenius, Schur, and Young. Frobenius* determined the characteristics of the representations in a very elegant manner, but he did not obtain the actual matrices. A method of obtaining these matrices was given by Schur†, but that due to Young‡ is much more satisfactory and complete. Young's papers in these *Proceedings* will be referred to as Q.S.A., I to VI. Q.S.A., V, deals with the hyper-octohedral group which will be considered in § 3.

The whole of Young's theory is based on the arrangement of the n letters in the form of a tableau

$$F: \begin{matrix} a_{11}\, a_{12} \dots a_{1\alpha_1} \\ a_{21}\, a_{22} \dots a_{2\alpha_2} \\ \vdots \\ a_{h1}\, a_{h2} \dots a_{h\alpha_h}, \end{matrix}$$

where $$\alpha_1+\alpha_2+\dots+\alpha_h = n$$

and $$\alpha_1 \geqslant \alpha_2 \geqslant \alpha_3 \geqslant \dots \geqslant \alpha_h.$$

We shall denote by P the product of the h symmetric groups on the letters in the h rows of the tableau. Thus

$$P = \{a_{11}\, a_{12} \dots a_{1\alpha_1}\}\{a_{21}\, a_{22} \dots a_{2\alpha_2}\} \dots \{a_{h1}\, a_{h2} \dots a_{h\alpha_h}\};$$

these are called *positive* symmetric groups. A *negative* symmetric group is defined to be the sum of the even permutations minus the sum of the odd permutations. It will be written with a dash, and we shall obtain from the columns

$$N = \{a_{11}\, a_{21} \dots a_{h1}\}'\{a_{12}\, a_{22} \dots a_{h2}\}' \dots \{a_{1\alpha_1}\}'.$$

* *Berliner Sitzungsberichte* (1900), 516–543.

† *Ibid.* (1908), 664–678.

‡ *Proc. London Math. Soc.* (1), 33 (1900), 97–146; (1), 34 (1901), 361–397; (2), 28 (1927), 255–292; (2), 31 (1929), 254–272; (2), 31 (1929), 273–288; (2), 34 (1932), 196–231. *Journal London Math. Soc.*, 3 (1928), 14–19.

Thus with each tableau F_i we may associate a substitutional expression $P_i N_i$.

Amongst this set of tableaux corresponding to all possible arrangements of the n letters, having the same α's, we may define some to be *standard*, with regard to a given sequence of the letters. *A standard tableau is one in which in every row and in every column the letters appear in the order of the agreed sequence.* The number of standard tableaux belonging to the set, which we shall denote by $T_{\alpha_1, \alpha_2, \ldots, \alpha_h}$, or more briefly T_α, is

$$f_{\alpha_1 \alpha_2 \ldots \alpha_h} = n! \frac{\prod_{rs} (\alpha_r - \alpha_s - r + s)}{\prod_r (\alpha_r + h - r)!}.$$

This we may abbreviate to f_α.

The method of constructing the matrices of the linear transformations on the f_α variables, corresponding to the f_α standard tableaux of T_α, may be found in Q.S.A., III, §§ 1–7 (see also Q.S.A., VI, § 3). These transformations, as so obtained, are not orthogonal. In Q.S.A., VI, a rule is given for writing down the orthogonal from these non-orthogonal matrices. In what follows we shall be interested in the geometrical properties of the transformations, rather than in their precise algebraical form.

The total number of tableaux for a given set of α's is, of course, $n!$, but since we are choosing the subscript of P to refer to the column of the matrix and that of N to the row, we do not take into account the arrangement of the letters in the rows of F in the construction of the matrices. The number of tableaux which are different in this regard is evidently

$$\phi_{\alpha_1 \alpha_2 \ldots \alpha_h} = \binom{n}{\alpha_1 \alpha_2 \ldots \alpha_h} = \frac{n!}{\alpha_1! \alpha_2! \ldots \alpha_h!}.$$

This we may abbreviate to ϕ_α.

1.2. Let us consider the set of ϕ_α tableaux arranged in some order

$$F_1, F_2, \ldots, F_{\phi_\alpha}.$$

If S is any permutation of the symmetric group S_n, then

$$S^{-1} F_1 S, \quad S^{-1} F_2 S, \quad \ldots, \quad S^{-1} F_{\phi_\alpha} S$$

is the set in some new order, and the corresponding substitution may be denoted by

$$\binom{F}{S^{-1} F S}.$$

We thus obtain a representation of S_n as a substitution group on ϕ_α symbols. If all the α's are unity, we have Jordan's regular representation.

Such a representation is considered by Burnside*, but instead of the tableaux he uses a set of conjugate sub-groups. These may be taken to be those of the P's of the tableaux, but this breaks down when more than one of the α's are unity. Burnside proves† that the number of distinct representations of a group as a permutation group is equal to the number of sets of conjugate sub-groups. These are not all distinct, however, if we allow transformation by linear substitutions. In the case of the symmetric group the method given above enables us to form a base of ρ such representations, if ρ is the number of conjugate sets, in one-one correspondence with the irreducible linear representations.

1.3. In Q.S.A., VI, § 2, is obtained the substitutional equation

$$(1.31) \qquad \frac{1}{n!}\binom{n}{\alpha_1\,\alpha_2\,\ldots\,\alpha_h}\Gamma P_{\alpha_1\alpha_2\ldots\alpha_h} = \Sigma\,\Pi\, S_{rs}^{\lambda rs}\frac{n!}{f_{\alpha_1\alpha_2\ldots\alpha_h}}\,T_{\alpha_1\alpha_2\ldots\alpha_h},$$

where $P_{\alpha_1\ldots\alpha_h}$ is the product of the positive symmetric groups of any tableau of $T_{\alpha_1\ldots\alpha_h}$, and Γ is the operation of taking the sum of the $n!$ permutations of $P_{\alpha_1\ldots\alpha_h}$. The operation S_{rs} moves a letter from the s-th row up to the r-th row of the tableau. The sum and product signs give us all the T's preceding $T_{\alpha_1\ldots\alpha_h}$, each occurring as often as it can be formed in a different manner by means of the operations S_{rs} on T_α. Taking the coefficient of identity on each side of (1.31), we have‡

$$(1.32) \qquad \binom{n}{\alpha_1\,\alpha_2\,\ldots\,\alpha_h} = \sum_\beta S_\alpha{}^\beta f_\beta,$$

where the summation sign extends to all the T's derived as above described. If all the α's are unity, we have

$$n! = \Sigma f^2.$$

Thus, if we associate the tableaux F_i $(i = 1, 2, \ldots, \phi_\alpha)$ with variables x_i $(i = 1, 2, \ldots, \phi_\alpha)$, we may construct an orthogonal group G_{ϕ_α} of permutation matrices, and the equation (1.32) suggests that this group is reducible and has as irreducible components those corresponding to the T_β's preceding T_α, with multiplicity $S_\alpha{}^\beta$. We shall see that this is actually the case.

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* *Theory of groups* (Cambridge, 1911), 2nd edition, 242.

† *Ibid.*, § 181.

‡ *Cf.* "Note on an equation of quantitative substitutional analysis", *infra*, 414–416.

Of the ϕ_α tableaux f_α are standard and belong to T_α. It is possible to associate the remaining $\phi_\alpha - f_\alpha$ non-standard tableaux with the standard ones of preceding T's. If we take the corresponding expressions PNM and operate on them with suitably chosen positive symmetric groups*, we obtain $\phi_\alpha - f_\alpha$ linear equations which fall into sets, each of f equations, corresponding to the division of ϕ_α according to (1.32). These equations are sufficient and necessary for the construction of the matrices of T_α. How are they related to the groups G_{ϕ_α}?

Let† us consider the expressions obtained by putting in place of the P's the corresponding x's, according to the above association of the tableaux. These sets of linear sums of x's, or tableaux, behave with regard to the permutations of S_n in an exactly analogous manner to the standard tableaux of the corresponding T. Let us now choose as new variables for G_{ϕ_α} these sums of x's. First we have the f_α standard x's unaltered and then $\phi_\alpha - f_\alpha$ sums of which the last is the sum of all the x's and corresponds to T_n. The transforming matrix is easy to construct, and along the leading diagonal of the transformed group we have the irreducible components, according to (1.32), as they were originally obtained‡, with zeros below. Above these irreducible components will be certain non-zero elements which originally vanished in virtue of the linear equations above referred to. It is possible to remove these non-zero elements without altering the irreducible components. We are now in a position to apply the results of Q.S.A., VI, to each representation which appears and finally obtain G'_{ϕ_α} in a completely reduced orthogonal form. We shall write the combined transformation, which is necessarily orthogonal,

$$R: \begin{cases} x_r' = a_r^{(1)} x_1 + \ldots + a_r^{(\phi_\alpha)} x_{\phi_\alpha}, \\ x'_{f_\alpha + k} = B_k(x_i) \\ \qquad\quad \equiv b_k^{(1)} x_1 + \ldots + b_k^{(\phi_\alpha)} x_{\phi_\alpha} \end{cases}$$

$$(r = 1, 2, \ldots, f_\alpha;\ k = 1, 2, \ldots, \phi_\alpha - f_\alpha),$$

so that $$R G_{\phi_\alpha} R^{-1} = G'_{\phi_\alpha}.$$

The expressions $B_k(x_i)$ are linear functions of the sums of x's, and, when set equal to zero, define the invariant sub-space of f_α dimensions

* Theorem I, Q.S.A., III, 259 and 268; see also example in the following section.

† For a detailed account of the reduction of G_{ϕ_α} for $n = 4, 5$, *cf.* H. H. Ferns, *Trans. Royal Soc. of Canada* (3), 28 (1934), 35–60; also Young, *Proc. London Math. Soc.* (2), 37 (1934), 441–496, § VI.

‡ See (1.1).

of the representation corresponding to T_α. We shall fix our attention on this representation for the present.

Consider the simplex whose vertices are the points

$$\Pi_i : (0^{i-1},\ 1,\ 0^{\phi_\alpha - i}).$$

On account of the form of the matrices this is evidently an invariant configuration of G'_{ϕ_α}. Under R the coordinates of its vertices become

$$\Pi_i' : (a_1^{(i)},\ a_2^{(i)},\ \ldots,\ a_{f_\alpha}^{(i)},\ b_1^{(i)} \ldots b_{\phi_\alpha - f_\alpha}^{(i)}),$$

where $i = 1, 2, \ldots, \phi_\alpha$.

Since R is orthogonal, we must have

$$(1.33) \qquad \begin{cases} \Sigma\, a_r^{(j)2} = \Sigma\, b_k^{(j)2} = 1, \\ \Sigma\, a_r^{(j)} a_s^{(j)} = \Sigma\, a_r^{(j)} b_k^{(j)} = \Sigma\, b_k^{(j)} b_l^{(j)} = 0. \end{cases}$$

Let us now project the points Π_i' on to the space of f_α dimensions given by

$$B_k(x_i) = 0.$$

We obtain the points

$$P_i : (a_1^{(i)},\ a_2^{(i)},\ \ldots,\ a_{f_\alpha}^{(i)}),$$

and these points must constitute an invariant configuration of the irreducible component of G'_{ϕ_α}, which we shall denote by G_{f_α}. Evidently the points P_i are not linearly independent. From the orthogonality relations (1.33), putting $j = i$, we have

$$\Sigma\, a_r^{(i)} b_k^{(i)} = 0.$$

We may write these relations more briefly

$$B_k(P_i) = 0.$$

The significance of the remainder of (1.33) will be commented upon in § 4.

We may now state

THEOREM I. *There exists a configuration of ϕ_α points lying on the hypersphere in $[f_\alpha]$ which is left invariant by the transformations of the irreducible orthogonal representation corresponding to T_α. These points satisfy $\phi_\alpha - f_\alpha$ independent linear relations analogous to those among the expressions $P_u \sigma_{uv} N_v M_v$.*

If some of the α's are equal to each other, some of the linear relations take a particularly simple form.

E.g., in the case of $T_{4.1.1.1}$ from the tableau

$$a_1 a_2 a_3 a_4$$
$$a_5$$
$$a_6$$
$$a_7$$

we obtain $P_{567} N_{567}$; but, since $(a_5 a_6 a_7)$ is an even permutation,

$$P_{567} N_{567} = P_{675} N_{675} = P_{756} N_{756}.$$

We have the following

COROLLARY. (*a*) *If r of the a's are equal to a_s which is even, then the number of points is reduced to $\phi_a/r!$.* (*b*) *If a_s is odd, this becomes $\phi_a/\frac{1}{2}r!$. In this case, and only here, is the figure centrally symmetrical.*

1.4. We shall now see how the properties of these configurations may be completely determined from the linear relations

$$B_k(P_i) = 0.$$

We define the *product** of two points thus:

$$P_i P_j = \sum_{k=1}^{f} x_k^{(i)} x_k^{(j)}.$$

We shall always assume the configuration to have been radially projected on to the unit hyper-sphere, with centre O, so that

$$\cos P_i \hat{O} P_j = P_i P_j$$

and

$$P_i^2 = 1.$$

Sums and differences have been considered already. Division is not defined, since if a product is zero it does not follow that one of its factors must be zero.

THEOREM II. *If the points P_i $(i = 1, 2, \ldots, r)$ are interchangeable among themselves under a group of orthogonal transformations simply isomorphic with the symmetric group on r symbols, then they are the vertices of a regular simplex.*

---

* *Cf.* E. Study, *Einleitung in die Theorie der Invarianten linearer Transformationen auf Grund der Vectorenrechnung* (Vieweg, Braunschweig, 1923), §§ 1–3.

Any orthogonal transformation is a rotation possibly combined with a reflection. Such a motion will leave the relative distances between two points unaltered, *i.e.*

$$\sqrt{\{(P_i-P_j)^2\}} = \sqrt{\{(P_l-P_m)^2\}} \quad (i \neq j,\ l \neq m).$$

If we now define a regular simplex to be a set of r points equally distant from one another, the theorem is proved.

COROLLARY 1. *If* $\sum_{i=1}^{r} P_i = P$, *where* P *is unaltered by the group of the points* P_i, *then the centre of the simplex is the point* $(1/r)\,P$.

COROLLARY 2. *If* $\sum_{i=1}^{r} P_i = P$, *where* P *is unaltered by the group of the points* P_i, *and in addition*

$$P^2 = 1,$$

then the points P_i, $(-P)$ *are the vertices of a simplex centre the origin.*

We have $$\left(\sum_{i=1}^{r} P_i\right)^2 = 1 = r + r(r-1)\,P_1 P_2,$$

therefore $$P_1 P_2 = -\frac{1}{r}.$$

Again $$(-PP_1) = -1-(r-1)\,P_1 P_2 = -\frac{1}{r}.$$

Hence the corresponding angles must be equal, and the points equally distant from one another.

COROLLARY 3. *The angle* ϕ_r *subtended at the centre of the simplex* P_i, $(-P)$ *by the circumradius of a bounding simplex* P_i *is* $\cos^{-1} 1/r$.

From Corollary 2,

$$\cos\phi_r = PP_1 = \frac{1}{r}.$$

We shall illustrate the use of these results in the example of § 2.

1.5. We have projected the ϕ_a points Π_i' on to the space $[f_a]$ of the representation corresponding to T_a, but we could equally well have projected on to the space of any of the irreducible components. The projection of particular interest is that on to $[f_{n-1,1}]$. This representation appears one or more times for each T_a, and the configurations of ϕ_a points which we shall obtain may vary, as different projections correspond to different manners of obtaining $T_{n-1,1}$ from T_a, according to (1.32).

In another paper* the author has considered the fundamental region of the linear group corresponding to $T_{n-1.1}$, and the various configurations which may arise. The position of the initial point was there determined by means of the operations of Mrs. Stott. In the paper cited the importance of some "regularity condition" is apparent. Could this be taken to be "that the figure remain connected when its points are joined by lines of shortest length"?

2.1. *Example* $T_{3.2}$.

As an illustration, we shall consider $T_{3.2}$ from the geometrical standpoint developed above.

$$\phi_{3.2} = 10,$$

and with the tableau

$$a_1 a_2 a_3$$

$$a_4 a_5$$

we shall associate the variable x_{45}, where

$$x_{45} = x_{54},$$

which will be followed by $x_{35}, x_{34}, x_{25}, x_{24} : x_{15}, x_{14}, x_{13}, x_{12} : x_{23}$. On these variables we may construct the orthogonal group $G_{\phi_{3.2}}$.

The linear sums of x's are easily seen to be

$$(2.11) \qquad \begin{cases} x_{15}+x_{25}+x_{35}+x_{45} \\ x_{14}+x_{24}+x_{34}+x_{54} \\ x_{13}+x_{23}+x_{43}+x_{53} \\ x_{12}+x_{32}+x_{42}+x_{52} \end{cases}$$

and $$x_{45}+x_{35}+\ldots+x_{15}+x_{14}+\ldots+x_{23}.$$

If we transform $G_{\phi_{3.2}}$ so that these sums replace the variables x_{15}, x_{14}, etc., the square diagonal matrices belong to $T_{3.2}$, $T_{4.1}$, and T_5. By a further transformation we may remove the non-zero elements elsewhere and then, by the rule of Q.S.A., VI, we arrive at the completely reduced orthogonal form $G'_{\phi_{3.2}}$.

* *Journal London Math. Soc.*, 6 (1931), 70–75. It has been shown by Ferns (*loc. cit.*) that the configurations of ϕ_a points in $[f_{n-1.1}]$ are *not* necessarily Stott figures. His conclusions regarding the definition of the fundamental region by these configurations are interesting.

2.2. Let us first consider the configuration of ten points in [5]. We shall obtain the desired linear relations by replacing the x_{ij} of (2.11) by P_{ij} and setting the sums equal to zero. These relations may be written in the form

$$P_{12} = P_{45} + P_{35} + P_{34},$$

and from Theorem II, Corollary 2, we know that $(-P_{12})$, P_{45}, P_{35}, P_{34} are the vertices of a tetrahedron centre the origin. We shall have ten relations of this type. It is easy to see that the shortest distance between any two points will be

$$\sqrt{\{(P_{12}-P_{34})^2\}} = \sqrt{(P_{12}^2+P_{34}^2-2P_{12}P_{34})} = 2/\sqrt{3},$$

from Corollary 2. The lines of this length meeting at P_{12} are mutually perpendicular.

Algebraically,

$$(P_{34}-P_{12})(P_{35}-P_{12}) = P_{34}P_{35}-P_{34}P_{12}-P_{35}P_{12}+P_{12}^2 = 0,$$

but it is evident geometrically since they are the edges of a cube meeting in P_{12}. A triad of such mutually perpendicular lines radiates from each point of the configuration.

Instead of projecting on to the space of $G_{f_{3.2}}$, we could equally well have projected on to the space of $G_{f_{4.1}}$. The linear relations are not so obvious and involve the transformation R. We have the points arranged by fours about the vertices of the simplex; this follows from (2.11). They are the vertices* of the semi-regular polytope ce_1a_4.

3. *The hyper-octohedral group.*

The group is most simply defined as the group of symmetries of the regular polytope β_n, having $2n$ vertices in $[n]$. In applying the method of the tableaux to determine the irreducible representations† it was found convenient to associate them by pairs, and construct ${}_{\alpha_1\alpha_2\ldots\alpha_h}T_{\beta_1\beta_2\ldots\beta_k}$, where

$$\alpha_1+\alpha_2+\ldots+\alpha_h = n_1,$$

$$\beta_1+\beta_2+\ldots+\beta_k = n_2,$$

and

$$n_1+n_2 = n.$$

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* *Cf.* Ferns, *loc. cit.*, 48, § 4.2.

† Q.S.A., V, §§ 1–4.

We may now construct an equation analogous to (1.32), merely by multiplication:

$$(3.1) \qquad \binom{n_1+n_2}{n_1}\binom{n_1}{\alpha}\binom{n_2}{\beta} = \binom{n_1+n_2}{n_1}\Sigma f(n_1)f(n_2),$$

and each product of f's on the right gives the degree of an irreducible representation of the group. The invariant configuration will here consist of

$$(3.2) \qquad 2\binom{n_1+n_2}{n_1}\phi_\alpha\phi_\beta = 2m$$

points, projected from β_m in $[m]$. The process is analogous to that adopted in the case of the symmetric group. The exact form of the irreducible orthogonal matrices has not yet been determined, but this will not affect the linear relations for the projection on to $\left[\binom{n_1+n_2}{n_1} f_\alpha f_\beta\right]$, nor the validity of any other projection.

The properties of the configuration could be determined as before, only here the cross polytope would have the importance previously assigned to the simplex.

4. *Generalities.*

The geometrical point of view from which we have considered the symmetric and the hyper-octohedral groups may be adopted for the consideration of any linear group.

Let us take the general case and assume that the group is in unitary form on n variables. If it is not orthogonal it becomes so when written on $2n$ variables. Denote this orthogonal group by G_f. Choosing an *arbitrary* point on the hyper-sphere in $[2n]$ and applying the transformations of G_f, we generate an invariant configuration of N points, which defines a *transitive* permutation group on N symbols. As in the case of the symmetric group, we may write this as an orthogonal permutation matrix group G_ϕ on N variables. Among the N points on the hyper-sphere in $[2n]$ there will be

$$N-2n$$

independent linear relations which are transformed among themselves under the group. To each of these relations will correspond a prime in $[N]$ passing through the origin, and as a set they determine an invariant sub-space of $2n$ dimensions. Making the suitable transformation of

variables, the orthogonality relations are satisfied as before, and we may state

THEOREM III. (*a*) *If the configuration of points* $P_i(a_r^{(i)})$ *is left invariant by a group of orthogonal substitutions, then we must have*

$$\Sigma a_r^{(i)2} = 1, \quad \Sigma a_r^{(i)} a_s^{(i)} = 0.$$

(*b*) *The linear relations* $B_k(P_i) = 0$ *must satisfy the conditions*

$$\Sigma b_k^{(i)2} = 1, \quad \Sigma b_r^{(i)} b_k^{(i)} = 0.$$

The question of the connectedness of the configuration again arises, and it would seem possible that some modification of the tableau representation could be set up for the general group.

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NOTE ON AN EQUATION OF QUANTITATIVE SUBSTITUTIONAL ANALYSIS

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The equation

$$\frac{1}{n!}\binom{n}{a_1\, a_2 \dots a_h}\Gamma P_{a_1 a_2 \dots a_h} = \Sigma\,\Pi\, S_{rs}^{\lambda_{rs}} \frac{n!}{f_{a_1 a_2 \dots a_h}}\, T_{a_1 a_2 \dots a_h} \qquad (1)$$

is obtained by Young in a recent paper*.

$$P_{a_1 a_2 \dots a_h} = \{a_1 a_2 \dots a_{a_1}\}\{a_{a_1+1} \dots a_{a_1+a_2}\} \dots,$$

the letters appearing in the order of their subscripts. Γ represents the operation of taking the sum of the $n!$ permutations of the letters in P_a. S_{rs} is a symbol which represents the process of raising a letter from the r-th row to the s-th row of the tableau $T_{a_1 a_2 \dots a_h}$. The tableaux T_β so obtained may appear more than once, the same tableaux being given by different combinations of S's. If T_β appears $S_a{}^\beta$ times, then it is the values of these integers which we calculate explicitly in this note.

Consider the conjugate set, the sum of whose $n_{\gamma_1 \gamma_2 \dots}$ members we denote by $t_{\gamma_1 \gamma_2 \dots}$, any permutation being composed of γ_1 cycles of one letter, γ_2 cycles of two letters, and so on. If the number of these permutations which are to be found in P_a is $n_\gamma^{(a)}$, then†

$$n_\gamma\, B_a{}^\gamma = \binom{n}{a_1\, a_2 \dots a_h} n_\gamma^{(a)}, \qquad (2)$$

where $B_a{}^\gamma$ is the coefficient of $x_1^{a_1} x_2^{a_2} \dots x_h^{a_h}$ in the expansion of

$$W_\gamma = (x_1 + x_2 + \dots + x_n)^{\gamma_1} (x_1{}^2 + x_2{}^2 + \dots + x_n{}^2)^{\gamma_2} \dots.$$

* *Proc. London Math. Soc.* (2), 34 (1932), 196–230, see middle of p. 199. A summation sign should be inserted as above.

† Frobenius, *Berliner Sitzungberichte* (1900), 516–534 (517).

Consider one of the $n_\gamma^{(\alpha)}$ members of t_γ contained in P_α. Operating on it with Γ, we obtain $(n!/n_\gamma)t_\gamma$. Thus

$$\binom{n}{\alpha_1 \alpha_2 \dots \alpha_h} \Gamma P_{\alpha_1 \alpha_2 \dots \alpha_h} = \binom{n}{\alpha_1 \alpha_2 \dots \alpha_h} \sum_\gamma \frac{n!}{n_\gamma} n_\gamma^{(\alpha)} t_\gamma,$$

i.e.

$$\frac{1}{n!} \binom{n}{\alpha_1 \alpha_2 \dots \alpha_h} \Gamma P_{\alpha_1 \alpha_2 \dots \alpha_h} = \sum_\gamma B_\alpha{}^\gamma t_\gamma. \tag{3}$$

In a previous paper*, Young has proved that

$$\frac{n!}{f_\alpha} T_\alpha = \sum_\gamma \chi_\gamma{}^\alpha t_\gamma, \tag{4}$$

and combining (1), (3), (4) we see that

$$\Sigma \Pi S_{rs}^{\lambda_{rs}} \sum_\gamma \chi_\gamma{}^\alpha t_\gamma = \sum_\gamma B_\alpha{}^\gamma t_\gamma,$$

or explicitly, for each t_γ,

$$\sum_\beta S_\alpha{}^\beta \chi_\gamma{}^\beta = B_\alpha{}^\gamma, \tag{5}$$

where the summation applies to all the tableaux T_β obtainable by means of the operations S from T_α, T_β appearing with a multiplicity $S_\alpha{}^\beta$.

To effect the solution of these equations we multiply each side by $\chi_\gamma{}^\beta$ and sum over the operations of the group. Thus, in virtue of the equations

$$\sum_\gamma \chi_\gamma{}^\beta \chi_\gamma{}^{\beta'} n_\gamma = 0, \qquad \sum_\gamma (\chi_\gamma{}^\beta)^2 n_\gamma = n!,$$

we have

$$S_\alpha{}^\beta = \frac{1}{n!} \sum_\gamma B_\alpha{}^\gamma \chi_\gamma{}^\beta n_\gamma = \frac{1}{\alpha_1! \, \alpha_2! \dots \alpha_h!} \sum_\gamma \chi_\gamma{}^\beta n_\gamma^{(\alpha)}, \tag{6}$$

$n_\gamma^{(\alpha)}$ being the number of members of t_γ contained in P_α.

Taking the coefficient of the identical substitution on each side of (1), we have†

$$\binom{n}{\alpha_1 \alpha_2 \dots \alpha_h} = \sum_\beta S_\alpha{}^\beta f_\beta, \tag{7}$$

and in the case of $\alpha_i = 1$, $h = n$, $n_{1.1\dots}^{1.1\dots} = 1$ while all other $n_\gamma^{(\alpha)}$'s are zero, $\chi_{1.1\dots}^\beta = f_\beta$, so that

$$S_\alpha{}^\beta = f_\beta,$$

* *Proc. London Math. Soc.* (2), 31 (1929), 253–272; corollary to Theorem I, p. 256.

† *Cf.* Robinson, *Proc. London Math. Soc.* (2), 38 (1935), 402–413 for a further discussion of this equation.

and (7) becomes $$n! = \sum_{\beta} f_{\beta}^2. \qquad (8)$$

It is of considerable interest that the equation (8), which holds for any group, is here one of a series (7), each member of which is associated uniquely with a conjugate set.

[*Added in proof*, 27 *November*, 1934. In a recent paper*, D. E. Littlewood and A. R. Richardson have developed a method of calculating the characters of the symmetric group which depends essentially on an application of the equation (7).]

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* *Phil. Trans. Royal Soc.* (A), 233 (1934), 99–141. *Cf.* pp. 103–4.

ON THE DIRECTIONS OF BOREL OF ANALYTIC FUNCTIONS

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Introduction.

1.1. The object of this paper is to prove certain theorems about the directions of Borel of a function which is regular and of finite order in a certain angle, and, in particular, about the relation of the directions of Borel to Lindelöf's function $h(\theta)$.

Suppose that $f(z)$ is regular for $\alpha \leqslant \arg z \leqslant \beta$, $|z| \geqslant l > 0$, and let

$$M(r, \alpha, \beta) = \max_{\alpha \leqslant \theta \leqslant \beta} |f(re^{i\theta})|.$$

Then $f(z)$ is said to be of order ρ, where ρ is positive and finite, if

$$\overline{\lim_{r\to\infty}} \frac{\log M(r, \alpha, \beta)}{r^k}$$

is equal to 0 if $k > \rho$ and equal to ∞ if $k < \rho$. Let $\rho(r)$ be a positive function, differentiable for $r > l$ except at isolated points at which $\rho'(r-0)$ and $\rho'(r+0)$ exist, and such that

$$\lim_{r\to\infty} \rho(r) = \rho, \quad \lim_{r\to\infty} r\rho'(r) \log r = 0. \tag{1.11}$$

We say that $f(z)$ is of *proximate order* $\rho(r)$ in the sector, if

$$\overline{\lim_{r\to\infty}} \frac{\log M(r, \alpha, \beta)}{r^{\rho(r)}} = A, \tag{1.12}$$

where $0 < A < \infty$. In this case *Lindelöf's function* is

$$h(\theta) = \overline{\lim_{r\to\infty}} \frac{\log |f(re^{i\theta})|}{r^{\rho(r)}}. \tag{1.13}$$

We can define directions of Borel as follows: Let $n(r, \alpha', \beta', a)$ denote the number of zeros of $f(z)-a$ for which

$$l \leqslant |z| \leqslant r, \ \alpha < \alpha' \leqslant \arg z \leqslant \beta' < \beta;$$

and suppose that

$$\overline{\lim_{r\to\infty}} \frac{n(r, \gamma-\delta, \gamma+\delta, a)}{r^{\rho(r)}} = \eta(\delta) > 0 \quad (\alpha < \gamma < \beta) \tag{1.14}$$

for every $\delta > 0$, and every value of a, except perhaps one. Then $\arg z = \gamma$ is said to be *a direction of Borel of proximate order* $\rho(r)$ of $f(z)$. If, as will be the case throughout this paper, (1.12) is satisfied, then we say that $\arg z = \gamma$ is *a direction of Borel of the maximum kind.*

Valiron* has shown that, *if* $h(\gamma) = 0$ *and* $h(\theta) > 0$ *for* $\gamma < \theta \leqslant \gamma + \pi/\rho$, *then* $\arg z = \gamma$ *is a direction of Borel of the maximum kind*; and I have shown elsewhere† that *every angle greater than* π/ρ *in which* $h(\theta) > 0$ *for some direction* $\arg z = \theta$ *contains a direction of Borel of the maximum kind.*

In §§ 3.1 and 3.2 we make these results more precise by using a narrower definition of direction of Borel. The main part of the paper begins in § 4.1, where we prove the basic theorems about functions which have comparatively few zeros in the angle concerned. Roughly speaking, the theorems show that, if $f(z)$ has very few zeros, then the zeros of $f(z)-a$ must lie in small circles (*cercles de remplissage*‡) whose centres lie on the borderlines between the regions where $|f(z)|$ is mostly large and the regions where $|f(z)|$ is small. Further, if $|f(z)|$ is neither very large nor very small in a certain circle, then it remains about the same in any larger circle in which the zeros of $f(z)$ are few. We use these results in §§ 5.1–5.6 to prove some theorems connecting $h(\theta)$ with directions of Borel; and from these we make deductions about the relation between the directions of Borel of

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

where $f(z)$ is an integral function of order 1 and mean type, and the Borel polygon of

$$\phi(z) = \sum_{n=0}^{\infty} n! a_n z^n.$$

For it is well known§ that the Borel polygon consists of points $re^{i\theta}$ such that

$$r \leqslant \frac{1}{h^+(\theta)},$$

where $h^+(\theta) = h(\theta)$ if $h(\theta) > 0$ and 0 otherwise, $h(\theta)$ being defined by (1.13).

* See G. Valiron, *Comptes rendus*, 194 (1932), 1305.

† M. L. Cartwright, *Comptes rendus*, 194 (1932), 1889–1891.

‡ See H. Milloux, *Acta Math.*, 52 (1928), 189.

§ See E. Borel, *Leçons sur les séries divergentes* (Paris, 1928), 152.

A short account of some of these results has been published in *Comptes rendus**; the other theorems which I stated there about directions of Borel will be reserved for another paper.

Preliminary lemmas.

2.1. We frequently need the following lemma, which is due, in its present form, to V. Bernstein†.

LEMMA 1. *Suppose that* $F(z)$ *is regular for* $|z| \leqslant R$, *and that*

$$\log|F(0)| > -A,$$

$$\log|F(z)| < A \quad \text{for} \quad |z| \leqslant R.$$

Then (1) $n(r) \leqslant 2A/\log(R/r)$, *where* $n(r)$ *is the number of zeros of* $F(z)$ *for which* $|z| \leqslant r < R$, (2) *given any* $\zeta > 0$, $k > 1$, *we can choose* $K = K(\zeta, k)$, *independently of* z *and* A, *so that*

$$\log|F(z)| > -KA$$

for $0 < |z| < R/k$ *except for values of* z *which can be enclosed in circles, the sum of whose radii is less than* ζR.

2.2. If $\rho(r)$ is any function satisfying the conditions stated in §1.1, then there is a function $V(z)$, regular for $\alpha \leqslant \arg z \leqslant \beta$, $|z| > l$, and such that, as $r \to \infty$,

$$V(re^{i\theta}) \sim r^{\rho(r)} e^{i\theta\rho} \tag{2.21}$$

uniformly for $\alpha \leqslant \theta \leqslant \beta$‡. Also, as $r \to \infty$,

$$\frac{V(kr)}{V(r)} \sim k^{\rho} \tag{2.22}$$

for every $k > 1$. For from (2.21) it follows that

$$\frac{V(kr)}{V(r)} \sim \frac{(kr)^{\rho(kr)}}{r^{\rho(r)}} = k^{\rho(kr)} r^{\rho(kr)-\rho(r)}. \tag{2.23}$$

Since $\rho(kr) \to \rho$, $k^{\rho(kr)} \to k^{\rho}$, and

$$\{\rho(kr)-\rho(r)\}\log r = (k-1) r\rho'(\xi r)\log r, \tag{2.24}$$

* M. L. Cartwright, *Comptes rendus*, 194 (1932), 1889, 2280.

† V. Bernstein, *Annali di Mat.* (4), 12 (1934), 173–196 (179).

‡ See M. L. Cartwright, *Proc. London Math. Soc.* (2), 38 (1934), 158–179. I shall refer to this paper in future as H. See also G. Valiron, *Lectures on the general theory of integral functions* (1923), 128–130.

where $1 < \xi < k$. It follows from (1.11) that the right-hand side of (2.24) tends to 0 as $r \to \infty$ and so (2.22) follows from (2.23).

It is sometimes convenient to write $V(r)$ in place of $r^{\rho(r)}$ in order to avoid complicated indices.

2.3. The properties of $h(\theta)$ have been discussed by Phragmén and Lindelöf*, and also by Pólya†. We shall use the following results. We suppose in each case that $f(z)$ is regular and of finite proximate order $\rho(r)$ for $\alpha \leqslant \arg z \leqslant \beta$, $|z| \geqslant l$.

LEMMA 2. *If* $h(\theta) \leqslant A\cos\theta\rho + B\sin\theta\rho$ *for* $\theta = \alpha$, $\theta = \beta$, *where* $\beta - \alpha < \pi/\rho$, *then* $h(\theta) \leqslant A\cos\theta\rho + B\sin\theta\rho$ *for* $\alpha \leqslant \theta \leqslant \beta$.

Most of the intrinsic properties of $h(\theta)$ can be deduced from this fundamental result of Phragmén and Lindelöf‡, in particular the following, which is also due to them.

LEMMA 3. *Every interval in which* $h(\theta)$ *is negative is bounded on each side by intervals in which* $h(\theta)$ *is positive, or by* α *or* β. *An interval in which* $h(\theta) < 0$ *is not greater than* π/ρ. *Every interval in which* $h(\theta) > 0$, *and which is bounded by values of* θ *for which* $h(\theta) = 0$, *is greater than or equal to* π/ρ. *If* $h(\theta)$ *is finite for* $\alpha \leqslant \theta \leqslant \beta$, *then the interval can be divided into not more than* $(\beta - \alpha)\rho/\pi$ *sub-intervals such that in each* $h(\theta)$ *keeps its sign unchanged, or remains equal to* 0.

In order to connect $h(\theta)$ more closely with $f(z)$ we also need

LEMMA 4. *For every* $\epsilon > 0$, *we have*

$$\log|f(re^{i\vartheta})| < \{h(\theta) + \epsilon\}V(r)$$

uniformly in θ *for* $r > r_0(\epsilon)$.

Finally we shall use a result due to Pólya.

LEMMA 5. *If* $h(\theta)$ *is finite for* $\alpha \leqslant \theta \leqslant \beta$, *then* $h(\theta)$ *is differentiable on the right and on the left for* $\alpha < \theta < \beta$.

* E. Phragmén and E. Lindelöf, *Acta Math.*, 31 (1908), 381–406.

† See G. Pólya, *Math. Zeitschrift*, 29 (1929), 549.

‡ See E. Phragmén and E. Lindelöf, *loc. cit.*; also G. Pólya, *loc. cit.*, and M. L. Cartwright, H.

The properties of $h(\theta)$ are closely analogous to properties of convex functions. This aspect has been considered by Pólya.

The alternative definition of directions of Borel.

3.1. Let $\Pi(z)$ be an integral function of *proximate order* $\rho(r)$ *and minimum type*; that is to say

$$\log|\Pi(re^{i\theta})| < \epsilon r^{\rho(r)}$$

for every $\epsilon > 0$ and every $r > r_0(\epsilon)$. Let $Z_1, Z_2, \ldots, Z_m = R_m e^{i\Theta_m}, \ldots$ be a sequence of points such that $\lim_{m\to\infty} \Theta_m = \gamma$, $\lim_{m\to\infty} R_m = \infty$, and let $N(Z_m, \delta, \Pi)$ denote the number of zeros of $f(z) - \Pi(z)$ for which

$$|z - Z_m| \leqslant \delta R_m.$$

Suppose that

$$\varliminf_{m\to\infty} \frac{N(Z_m, \delta, \Pi)}{V(R_m)} = \xi(\delta) > 0 \tag{3.11}$$

for every $\delta > 0$ and every $\Pi(z)$ except perhaps one. Then we say that $\arg z = \gamma$ is *a direction of Borel of proximate order* $\rho(r)$; and if $f(z)$ is of proximate order $\rho(r)$ in the angle considered, then we say that $\arg z = \gamma$ is *a direction of Borel of the maximum kind*. In this definition $\Pi(z)$ may be a polynomial or any constant including 0. This definition is equivalent to the one given in § 1.1, as the following theorem shows.

THEOREM I. *If $f(z)$ is regular for $|\arg z| \leqslant \alpha$, $|z| \geqslant l$, and if*

$$\varliminf_{r\to\infty} \frac{n(r, -\delta, \delta, a)}{r^{\rho(r)}} = \zeta(\delta) > 0 \tag{3.12}$$

for every δ such that $0 < \delta < \alpha$, and every value of a except perhaps one, then $\arg z = 0$ is a direction of Borel of proximate order $\rho(r)$ by the definition just given.

We need a theorem of Milloux* and a theorem of Rauch†.

MILLOUX'S THEOREM. *Suppose that $F(z)$ is meromorphic for $|z| < 1$, and let $d < 1$, $d' < 1$. Suppose that, in $|z| < 1$, $F(z)$ does not take the three values a_1, a_2, a_3, whose spherical distances are greater than d, more than n times. Then the number of zeros of $F(z) - a$ for which $|z| < \frac{1}{2}\frac{1}{0}$ is less than*

$$K_1 n + K_2 \log(1/d) + K_3 \log(1/d') + K_4,$$

* H. Milloux, *Acta Math.*, 52 (1928), 202.

† A. Rauch, *Comptes rendus*, 192 (1931), 1189, also *Thèse* (Paris, 1933), 21.

where d' denotes the spherical distance of a from a certain possible exceptional value a_0, and K_1, K_2, K_3, K_4 are numerical constants.

RAUCH'S THEOREM. *Suppose that $F(z)$, $\Pi_1(z)$, $\Pi_2(z)$, $\Pi_3(z)$ are meromorphic for $|z|<1$, and let $d<1$, $d'<1$. Let N be the greatest number of zeros of $F-\Pi_1$, $F-\Pi_2$, $F-\Pi_3$, and n the number of zeros and poles of the set of functions Π_1, Π_2, Π_3, $\Pi_1-\Pi_2$, $\Pi_2-\Pi_3$, $\Pi_3-\Pi_1$ in $|z|<1$. If*

$$(3.13)\quad \frac{1}{\pi}\iint \log^+\left\{|\Pi_1|+|\Pi_2|+|\Pi_3|+\frac{1}{|\Pi_1-\Pi_2|}+\frac{1}{|\Pi_2-\Pi_3|}+\frac{1}{|\Pi_3-\Pi_1|}\right\}d\sigma<\log\frac{1}{d},$$

where the integral is taken over the unit circle, then the number of zeros of $F(z)-a$ in $|z|<\frac{1}{20}$ is less than

$$K_1N+K_2n+K_3\log(1/d)+K_4\log(1/d')+K_5,$$

where d' denotes the spherical distance from a possible exceptional value a_0 and K_1, K_2, K_3, K_4, K_5 are numerical constants.

This theorem, of course, includes the previous one. If $f(z)$ is of proximate order $\rho(r)$ in an angle greater than π/ρ, it is possible to use the methods developed in Theorems II and III of this paper, and so obtain the required result without Milloux's or Rauch's theorem. However, the method is cumbersome, and the work becomes extremely complicated when we attempt to apply it to the general case.

Proof of Theorem I. Let $Z_m=R_0k^m$, where $R_0>l, k>1$; and suppose that, given any $\epsilon>0$, $\delta_1>0$, we can choose $m_0(\epsilon)$ so that $f(z)$ does not take the values $a_1(m)$, $a_2(m)$ more than $\epsilon V(R_m)$ times in the circle

$$|z-Z_m|\leqslant\delta_1R_m<\sin\alpha R_m,\quad\text{for}\quad m>m_0.$$

Suppose also that

$$-\log|a_1(m)-a_2(m)|<\epsilon V(R_m),$$

$$\log|a_1(m)|<\epsilon V(R_m),\quad \log|a_2(m)|<\epsilon V(R_m)\quad\text{for}\quad m>m_0.$$

Then, applying Milloux's theorem to $f(Z_m+z\delta_1R_m)$ with $a_1=a_1(m)$, $a_2=a_2(m)$, $a_3=\infty$, we have

$$N\left(Z_m,\ \frac{\delta_1}{20},\ a\right)<K\epsilon V(R_m),$$

where K is a numerical constant, except perhaps for values of a such that $\log|a-a_0(m)|<-\epsilon V(R_m)$, provided that $m>m_0$.

Now, given $\delta<\frac{1}{40}\sin\alpha$, and any $r>r_0>R_0k^{m_0}$, we can choose k, δ_1, and m so that the circles $|z-Z_s|\leqslant\frac{1}{20}\delta_1 R_s$, $s=m_0, m_0+1, \ldots, m$ cover the whole region $|\arg z|\leqslant\delta$, $r_0\leqslant|z|\leqslant r$, and also $R_{m-1}<r\leqslant R_m$. It is, in fact, sufficient to put $k-1=\frac{1}{20}\delta_1=2\delta$. Hence

$$(3.14)\quad n(r,-\delta,\delta,a)-n(r_0,-\delta,\delta,a)$$

$$<K\epsilon\sum_{s=m_0}^{m}V(R_s)<2K\epsilon V(R_m)\left\{1+\frac{1}{k^\rho}+\frac{1}{k^{2\rho}}+\ldots+\frac{1}{k^{(m-m_0)\rho}}\right\}$$

$$<\frac{2K\epsilon}{1-k^{-\rho}}V(R_m)<\frac{3K\epsilon k^\rho V(r)}{1-k^{-\rho}},$$

except perhaps for values of a such that $\log|a-a_0(s)|<-\epsilon V(R_s)$, where $s=m_0, m_0+1, \ldots, m$. The total measure of these excepted sets is less than

$$\sum_{s=m_0}^{m}\exp\{-\epsilon V(R_s)\}<\sum_{s=m_0}^{m}\exp\{-2\epsilon k^{(s-m_0)\rho}V(R_{m_0})\}$$

$$<\frac{\exp\{-2\epsilon V(R_{m_0})\}}{1-\exp\{-2\epsilon(k^\rho-1)VR_{m_0}\}}.$$

It follows from (3.14) that, unless a belongs to the excepted set, (3.12) is false; and, given $\epsilon>0$ and $k>1$, we can choose m_0 so large that the excepted set is of area less than 1. But, by our hypothesis, (3.12) holds for every value of a, except perhaps one, and so we have a contradiction in both cases. Hence, given any $\epsilon>0$, $\delta>0$, we can find a subsequence Z_m' such that

$$(3.15)\qquad N(Z_m',\delta,a)>\zeta(\delta)V(R_m'),$$

except perhaps for values of a such that $\log|a|>\epsilon V(R_m')$ and

$$\log|a-a_0(m)|<-\epsilon V(R_m').$$

Now apply Rauch's theorem to

$$F(z)=\frac{1}{f(Z_m'+z\delta R_m')},\quad \Pi_s(z)=\frac{1}{\Pi_s'(Z_m'+z\delta R_m')}\quad(s=1,2,3),$$

where $\Pi_s'(z)$ is an integral function of proximate order $\rho(r)$ and minimum type for $s=1$, 2, and $\Pi_3(z)\equiv0$. Let $\log(1/d')<\epsilon V(R_m')$, and suppose that $N<\epsilon V(R_m')$ for all sufficiently large m. It is easy to see that $n<\epsilon V(R_m')$ for all large m, and also that we may take $\log(1/d)$ in (3.13)

to be less than $\epsilon V(R_m')$ for large m. For it follows from Jensen's formula that

$$(3.16)\quad \frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{|\Pi_1'(Z_m''+re^{i\theta})|}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\log^+|\Pi_1'(Z_m''+re^{i\theta})|\,d\theta-\log|\Pi_1'(Z_m'')|,$$

and, by Lemma 1, given $\epsilon>0$, $\eta>0$, we can find a sequence Z_m'' such that $|Z_m'-Z_m''|\leqslant\eta R_m'$ and $\log|\Pi_1'(Z_m'')|>-\epsilon V(R_m')$ for all sufficiently large m. But

$$\frac{1}{\pi}\iint_{D_1}\log^+|\Pi_1(z)|\,d\sigma=\frac{1}{\pi(\delta R_m')^2}\iint_{D_2}\log^+\frac{1}{|\Pi_1'(z)|}\,d\sigma < \frac{1}{\pi(\delta R_m')^2}\iint_{D_3}\log^+\frac{1}{|\Pi_1'(z)|}\,d\sigma,$$

where D_1 is the circle $|z|\leqslant 1$, D_2 the circle $|z-Z_m'|\leqslant\delta R_m'$, D_3 the circle $|z-Z_m''|\leqslant(\delta+\eta)R_m'$; and so, by (3.16),

$$\frac{1}{\pi}\iint_{D_1}\log^+|\Pi_1(z)|\,d\sigma < \frac{1}{\pi(\delta R_m')^2}\iint_{D_3}\log^+|\Pi_1'(z)|\,d\sigma+\frac{1}{\pi(\delta R_m')^2}\iint_{D_3}d\sigma\log^+|\Pi_1'(Z_m'')| < K\epsilon V(R_m').$$

A similar argument applies to $\Pi_2(z)$ and the other expressions involved in (3.13). Hence Rauch's theorem gives a contradiction of (3.15). It follows that the number of zeros of either $F(z)-\Pi_2(z)$ or $F(z)-\Pi_3(z)$ must be greater than $\epsilon V(R_m')$, but then the number of zeros of either $f(z)-\Pi_1'(z)$ or $f(z)-\Pi_2'(z)$ must be greater than $\epsilon V(R_m')$, where $\Pi_1'(z)$ and $\Pi_2'(z)$ are any integral functions of proximate order $\rho(r)$. Hence

$$\varliminf_{m\to\infty}\frac{N\{Z_m',\delta,\Pi(z)\}}{V(R_m')}=\zeta(\delta)>0$$

for every $\delta>0$ and every $\Pi(z)$ except perhaps one.

3.2. It follows from Theorem I that the results of Valiron* and myself† hold with the stricter definition of §3.1.

* See G. Valiron, *Comptes rendus*, 194 (1932), 1306.

† M. L. Cartwright, *Comptes rendus*, 194 (1932), 1889.

THEOREM II. *Every integral function of finite positive order has at least one direction of Borel of the maximum kind.*

Most of the other theorems are only significant for integral functions when $\rho > \frac{1}{2}$.

THEOREM III. *If $f(z)$ is of finite proximate order $\rho(r)$ for*

$$\alpha \leqslant \arg z \leqslant \beta, \quad |z| \geqslant l,$$

where $\beta - \alpha > \pi/\rho$, then there is at least one direction of Borel of the maximum kind for which $\alpha < \arg z < \beta$.

It follows from the definition of proximate order that $h(\theta) > 0$ for some θ such that $\alpha \leqslant \theta \leqslant \beta$; and the method is the same as for an integral function.

COROLLARY 1. *Every integral function of order $\rho > \frac{1}{2}$ has at least two directions of Borel of the maximum kind.*

COROLLARY 2. *If Ω is the least angle containing all the directions of Borel of the maximum kind of an integral function of order ρ, then*

$$\Omega \geqslant \min(\pi/\rho;\ 2\pi - \pi/\rho).$$

THEOREM IV. *If $f(z)$ is of proximate order $\rho(r)$ for $\alpha \leqslant \arg z \leqslant \beta$, $|z| \geqslant l$, and if $h(\gamma) = 0$, where $\alpha < \gamma < \beta$, $h(\theta) > 0$ for $\gamma < \theta < \beta$, then $\arg z = \gamma$ is a direction of Borel of the maximum kind.*

If $f(z)$ is an integral function of proximate order $\rho(r)$, we can state the theorem in the form given by Valiron: *the sides of the angles in which $h(\theta)$ is positive are directions of Borel of the maximum kind.*

It is easy to deduce Theorem IV from Theorem VI of this paper. Theorem III requires, in addition, an argument of the Phragmén-Lindelöf type, similar to that which I used in Theorem II of H, and it seems on the whole simpler to follow the method used previously and deduce it from Milloux's theorem.

4.1. We now proceed to develop a new method of dealing with directions of Borel and *cercles de remplissage*; we first prove a theorem about functions which have very few zeros in a certain sequence of circles. This theorem is the key to those which follow. We require the following lemma.

LEMMA 6. *Let $u_m(z)$ be a sequence of real harmonic functions, regular for $|z| \leqslant \lambda_1 R_m$, where $\lambda_1 > 0$, and $R_1 < R_2 < \ldots < R_m \to \infty$. Suppose*

that $u_m(0)=0$, *and that*

$$\max_{0\leqslant\theta\leqslant 2\pi} u_m(\lambda_1 R_m e^{i\theta}) \leqslant AV(R_m), \tag{4.11}$$

where $A=A(\lambda_1)$ *is independent of* m. *Then*

$$u_m(\lambda R_m e^{i\theta}) = \sum_{\mu=1}^{\infty} (\lambda/\lambda_1)^\mu B_\mu(m)\cos\{\mu\theta+\sigma_\mu(m)\}V(R_m), \tag{4.12}$$

where $\sigma_\mu(m)$ *and* $B_\mu(m)$ *are positive numbers independent of* λ *and* θ, *for* $\lambda<\lambda_1$, *and also*

$$B_\mu(m)\leqslant 2A \tag{4.13}$$

for all μ *and* m.

This is obtained by constructing the analytic function $F_m(z)$, of which $u_m(z)$ is the real part, and for which $F_m(0)=0$. Applying Taylor's theorem and taking the real part of both sides, we have (4.12); and (4.13) follows from Borel's inequality*. If m is fixed and $\lambda\to 0$ the series in (4.12) is dominated by the first non-vanishing term, *e.g.*, if

$$B_1(m)=B_2(m)=\ldots=B_{\mu-1}(m)=0, \quad B_\mu(m)>0,$$

we have

$$u_m(\lambda R_m e^{i\vartheta}) \sim (\lambda/\lambda_1)^\mu B_\mu(m)\cos\{\mu\theta+\sigma_\mu(m)\}V(R_m) \tag{4.14}$$

as $\lambda\to 0$. If $f(z)$ has no zeros, we can write $\log|f(z+Z_m)|-\log|f(Z_m)|$ in place of $u_m(z)$, and use the result. We shall prove that something like this holds for small values of λ when $m\to\infty$, provided that the number of zeros of $f(z)$ is sufficiently small. We have, however, to distinguish between two cases when $m\to\infty$. In the first case the coefficients $B_\mu(m)\to 0$ as $m\to\infty$ for all μ, and so the series is of the form $o\{V(R_m)\}$. In the second case the series is dominated by one of the early terms, as in (4.14).

We write $N(Z_m,\lambda)=N(Z_m,\lambda,0)$ and $P_m(z,\lambda)=\prod_n(1-z/z_n)$, where z_n is a zero of $f(z)$, and the product is taken over zeros for which $|z_n-Z_m|\leqslant\lambda R_m$.

THEOREM V. *Suppose that there is a number* $\lambda_1>0$ *and a sequence of points* $Z_1, Z_2, \ldots, |Z_m|=R_m\to\infty$, *such that* $f(z)$ *is regular for*

$$|z-Z_m|\leqslant\lambda_1 R_m<R_m,$$

and that

$$\log|f(Z_m)/P_m(Z_m,\lambda_1)|>-AV(R_m), \tag{4.15}$$

$$\log|f(z)|<AV(R_m) \quad \textit{for} \quad |z-Z_m|\leqslant\lambda_1 R_m. \tag{4.16}$$

* G. Valiron, *General theory of integral functions* (Toulouse, 1923), 18.

Suppose also that

$$\lim_{m\to\infty} N(Z_m, \lambda_1)/V(R_m) = 0. \tag{4.17}$$

Then the sequence $\{Z_m\}$ *consists of two kinds of subsequences* $\{Z_{m'}\}$ *and* $\{Z_{m''}\}$, *one of which may be non-existent, having the following properties*: (1) *There is a* $\lambda_2 > 0$ *such that, given any* $\epsilon > 0$, $\delta > 0$, *we can choose* $m_0(\epsilon, \delta)$ *so that*

$$\left|\log|f(z)| - \log|f(Z_{m'})/P_{m'}(Z_{m'}, \lambda_1)|\right| < \epsilon V(R_{m'}) \tag{4.18}$$

for $|z - Z_{m'}| \leqslant \lambda_2 R_{m'}$, *except in circles the sum of whose radii is less than* $\delta\lambda_2 R_{m'}$ *provided that* $m > m_0(\epsilon, \delta)$.

(2) *There are positive numbers* Q *and* B *such that, given* $\epsilon > 0$, $\delta > 0$, $\eta > 0$, *we can choose* $\lambda_0 = \lambda_0(\epsilon) < \lambda_1$, $m_0(\epsilon, \delta, \eta, \lambda_0)$ *so that*

$$\begin{aligned}\Big|\log|f(Z_{m''} + \lambda R_{m''} e^{i\theta})| - \log|f(Z_{m''})/P_{m''}(Z_{m''}, \lambda_1)| \\ - (\lambda/\lambda_1)^{\mu(m'')} B(m'') \cos\{\mu(m'')\theta + \sigma(m'')\} V(R_{m''})\Big| \\ < \epsilon(\lambda/\lambda_1)^{\mu(m'')} V(R_{m''}),\end{aligned} \tag{4.19}$$

where $\mu(m'')$ *is an integer less than* Q, $B < B(m) < 6A$, *for* $\eta\lambda_0 < \lambda < \lambda_0$, *except perhaps for values of* $\lambda R_{m''} e^{i\theta}$ *lying in circles the sum of whose radii is less than* $\delta\lambda_0 R_{m''}$.

There may be an enumerable infinity of subsequences of the second kind, corresponding to an increasing sequence of numbers, Q, and a decreasing sequence of numbers, B.

Of course, the excepted regions are only the regions near the zeros of $f(z)$, and, if $N(Z_m, \lambda_1) = 0$, (4.18) and (4.19) hold uniformly in the regions defined without excepting any circles. It should be observed that *every integral function* $f(z)$ *of proximate order* $\rho(r)$ *satisfies* (4.15) *and* (4.16) *for every sequence of points* $Z_1, Z_2, \ldots, |Z_m| \to \infty$ *for some* A. For (4.16) is satisfied by the definition of proximate order $\rho(r)$ and the inequalities from which (4.15) is easily deduced occur as intermediate steps in the proof of Lemma 1 and its weaker forms. It is also possible to show that, *if* $f(z)$ *is regular and of proximate order* $\rho(r)$ *for* $|z| > l$, $\alpha \leqslant \arg z \leqslant \beta$, where $\beta - \alpha > \pi/\rho$, *then* (4.15) and (4.16) *are satisfied for every sequence* $Z_1, Z_2, \ldots, |Z_m| \to \infty$ *such that* $\alpha + \delta \leqslant \arg Z_m \leqslant \beta - \delta < \beta$ *with some* $A = A(\delta)$. As in the case of integral functions, (4.16) is satisfied by definition, and (4.15) can be proved by the method of Theorem II of H. The proof is cumbersome, and, in all the cases which we shall consider, the

condition is satisfied for other reasons. If we know only that $f(z)$ is of proximate order $\rho(r)$ in an angle less than π/ρ, we know that (4.15), (4.16) must be satisfied for some sequence $Z_1, Z_2, \ldots, |Z_m| \to \infty$, but there may be sequences for which (4.15) is not satisfied.

The remarkable point about the result is that in (4.19) the values of θ are divided into $2\mu(m)$ equal parts, in $\mu(m)$ of which $\cos\{\mu(m)\theta+\sigma(m)\}$ is positive, and in the rest $\cos\{\mu(m)\theta+\sigma(m)\}$ is negative, the negative values exactly balancing the positive. Hence, if $\log|f(z)|$ increases rapidly as z moves away from Z_m in one direction, then there is at least one direction in which $\log|f(z)|$ decreases with approximately equal rapidity.

4.2. *Proof of Theorem V.* We observe first that it follows from (4.17) that

$$\log|P_m(z, \lambda_1)| < \epsilon_1 V(R_m), \tag{4.21}$$

where $|z| < 2R_m$, for every $\epsilon_1 > 0$ and every $m > m_0(\epsilon_1)$. Hence, by Lemma 1, given any $\epsilon_1 > 0$, $\delta_1 > 0$, we can choose $m_0(\epsilon_1, \delta_1)$ so that

$$\log|P_m(z, \lambda_1)| > -\epsilon_1 V(R_m) \tag{4.22}$$

for $|z| \leqslant \frac{3}{2}R_m$, $m > m_0(\epsilon_1, \delta_1)$, except perhaps for circles the sum of whose radii is less than $\delta_1 R_m$.

Let
$$F_m(z) = \frac{f(z+Z_m)\,P_m(Z_m, \lambda_1)}{f(Z_m)\,P_m(z+Z_m, \lambda_1)},$$

then $F_m(0) = 1$ and $F_m(z)$ has no zeros for which $|z| \leqslant \lambda_1 R_m$. Suppose that for some subsequence $\{Z_{m'}\}$ there is a $\lambda_3 > 0$ such that

$$\log|F_{m'}(z)| < \epsilon_2 V(R_{m'}),$$

where $|z| \leqslant \lambda_3 R_{m'}$, for every $\epsilon_2 > 0$ and every $m' > m_0(\epsilon_2)$. Then it follows from Carathéodory's theorem that

$$\log|F_{m'}(z)| > -2\epsilon_2 V(R_{m'})$$

for $|z| \leqslant \frac{1}{2}\lambda_3 R_{m'}$, $m' > m_0(\epsilon_2)$. Using (4.21) and (4.22), and putting $\lambda_2 = \frac{1}{2}\lambda_3$, we have (4.18) for this subsequence.

Now consider the remaining subsequence for which no such $\lambda_3 > 0$ can be found. Given any $\lambda > 0$, we can extract a subsequence $\{Z_{m''}\}$ such that

$$\max_{0 \leqslant \theta \leqslant 2\pi} \log|F_{m''}(\lambda R_{m''} e^{i\theta})| > \zeta V(R_{m''}), \quad \zeta = \zeta(\lambda) > 0, \tag{4.23}$$

for all $m'' > m_0(\zeta)$. It is easy to see that $\log|F_m(z)|$ satisfies the hypotheses

of Lemma 6, with $3A$ in place of A, and we have

$$(4.24)\quad \log|F_{m''}(\lambda R_{m''}e^{i\theta})| = \sum_{\mu=1}^{\infty}(\lambda/\lambda_1)^{\mu}B_{\mu}(m'')\cos\{\mu\theta+\sigma_{\mu}(m'')\}V(R_{m''}),$$

where $B_{\mu}(m'')<6A$. Suppose that, given any $\epsilon_3>0$ and any integer μ_0, we can choose $m_0(\epsilon_3, \mu_0)$ so that

$$B_{\mu}(m'')<\epsilon_3$$

for $\mu<\mu_0$, $m''>m_0(\epsilon_3, \mu_0)$. Then

$$\log|F_{m''}(\lambda R_{m''}e^{i\theta})|<\frac{\epsilon_3\lambda}{\lambda_1-\lambda}V(R_{m''})+\frac{6A(\lambda/\lambda_1)^{\mu_0-1}\lambda}{\lambda_1-\lambda}V(R_{m''}),$$

and so, given any $\lambda<\lambda_1$, and any $\epsilon_2>0$, we can choose ϵ_3 and μ_0 so that

$$\log|F_{m''}(\lambda R_{m''}e^{i\theta})|<\epsilon_2 V(R_{m''})$$

for $m''>m_0(\epsilon_3, \mu_0)=m_0(\epsilon_2, \lambda)$, which is contrary to our hypothesis (4.23).

Let $\mu_0(\epsilon_3, m)$ be the upper bound of integers μ such that $B_{\mu}(m)<\epsilon_3$ for $\mu<\mu_0$. Then we have shown that

$$\overline{\lim_{m\to\infty}}\,\mu_0(\epsilon_3, m)=Q(\epsilon_3)$$

is finite. But $Q(\epsilon_3)$ is positive and does not increase as $\epsilon_3\to 0$. Hence $Q(\epsilon_3)\to Q$ as $\epsilon_3\to 0$. Further, since $Q(\epsilon_3)$ only takes integral values, we can choose $B>0$ so small that $Q(B)=Q$. It follows that we can divide the sequence $Z_{m''}$ into not more than Q subsequences $Z_{m(q)}$, $q=1, 2, \ldots, Q_1\leqslant Q$, to each of which belongs a number $\mu(q)\leqslant Q+1$ such that

$$\underline{\lim_{m(q)\to\infty}}\,B_{\mu(q)}\{m(q)\}\geqslant B>0,$$

and also

$$\lim_{m(q)\to\infty}B_{\mu}\{m(q)\}=0,$$

for $\mu=1, 2, \ldots, \mu(q)-1$.

For one of these subsequences, given $\epsilon_4>0$, we have

$$(4.25)\quad \left|\sum_{\mu=1}^{\mu(q)-1}(\lambda/\lambda_1)^{\mu}B_{\mu}\{m(q)\}\cos[\mu\theta+\sigma_{\mu}\{m(q)\}]V(R_{m(q)})\right|<\epsilon_4\mu(q)V(R_{m(q)})$$

and

$$(4.26)\quad \left|\sum_{\mu(q)+1}^{\infty}(\lambda/\lambda_1)^{\mu}B_{\mu}\{m(q)\}\cos[\mu\theta+\sigma_{\mu}\{m(q)\}]V(R_{m(q)})\right| < \frac{6A(\lambda/\lambda_1)^{\mu(q)}\lambda}{\lambda_1-\lambda}V(R_{m(q)})$$

for $m(q) > m_0(\epsilon_4)$. But, given $\epsilon > 0$, we can choose λ_0 so that

$$6A\lambda_0/(\lambda_1 - \lambda_0) < \tfrac{1}{2}\epsilon;$$

and then, given $\eta > 0$, we can choose ϵ_4 so that

$$\epsilon_4 \mu(q) < \tfrac{1}{2}\epsilon(\eta\lambda_0/\lambda_1)^{\mu(q)}.$$

It follows from (4.24), (4.25), and (4.26) that

$$\left|\log|F_{m(q)}(\lambda R_{m(q)}e^{i\theta})| - B_{\mu(q)}(m)(\lambda/\lambda_1)^{\mu(q)}\cos\{\mu(q)\theta + \sigma_{\mu(q)}(m)\}V(R_{m(q)})\right|$$
$$< \epsilon(\lambda/\lambda_1)^{\mu(q)}V\{R_{m(q)}\}$$

for $\eta\lambda_0 < \lambda < \lambda_0$, $m > m_0(\epsilon, \eta, \lambda_0)$. Using (4.21) and (4.22), we have (4.19).

4.3. We next show that any point of the second kind such that $|f(Z_{m''})/P_{m''}(z, \lambda_1)|$ is neither very large nor very small is the centre of a *cercle de remplissage.*

THEOREM VI. *Suppose that the conditions of Theorem V hold. Suppose also that*

$$\lim_{m\to\infty} \frac{\log|f(Z_m)/P_m(Z_m, \lambda_1)|}{V(R_m)} = 0, \tag{4.31}$$

and that all the points belong to the second kind of sequence considered in Theorem V. Then, for every $\lambda > 0$,

$$\lim_{m\to\infty} \frac{N(Z_m, \lambda, \Pi)}{V(R_m)} = \zeta(\lambda) > 0, \tag{4.32}$$

where $\Pi(z)$ *is any integral function of proximate order* $\rho(r)$ *and minimum type which is not identically zero.*

Conversely, if the conditions of Theorem V hold, and if (4.32) *holds for every* $\lambda > 0$ *and every integral function* $\Pi(z) \not\equiv 0$ *of proximate order* $\rho(r)$ *and minimum type, then* (4.31) *holds and all the points belong to the second kind of subsequence in Theorem V.*

We may express the conclusion loosely as follows. If $f(z)$ has very few zeros in a certain sequence of circles, then the necessary and sufficient condition that they should be *cercles de remplissage* is that their centres should lie on the border line between the regions where $|f(z)|$ is chiefly large and those where $|f(z)|$ is small. For near points of the first kind $|f(z)|$ remains approximately the same size; and, if (4.31) is not satisfied, we can find a number $\lambda_3 > 0$ such that either $\log|f(z)| < -\xi V(P_m)$, or $\log|f(z)| > \xi V(R_m)$, where $\xi > 0$, for $|z - Z_m| \leqslant \lambda_3 R_m$ for points of both kinds, provided that $m > m_0$. That is, $|f(z)|$ is either large in some neighbourhood of Z_m or

small in some neighbourhood of Z_m. If (4.31) holds, and also the points are of the second kind, then both large and small values of $|f(z)|$ occur near Z_m.

4.4. *Proof of Theorem VI.* Consider $F(z)=f(z)-\Pi(z)$, where $\Pi(z)$ is an integral function of proximate order $\rho(r)$ and minimum type, and $\Pi(z)\not\equiv 0$. We observe first that, given any $\epsilon_1>0$, $\delta_1>0$, we can choose $m_0(\epsilon_1, \delta_1)$ so that

$$\log|\Pi(z)|<\epsilon_1 V(R_m) \tag{4.41}$$

for $|z|<2R_m$, and also

$$\log|\Pi(z)|>-\epsilon_1 V(R_m) \tag{4.42}$$

for $|z|<2R_m$, $m>m_0(\epsilon_1, \delta_1)$, except in circles the sum of whose radii is less than $\delta_1 R_m$.

Let B, Q, ϵ, δ, η, λ_0, etc., have the meanings attached to them in the enunciation of Theorem V; let $\zeta>0$, and let $D_1=D_1(\zeta, \eta, \delta, \lambda_0, m)$ be the set of points for which (4.19) holds and also $\cos\{\mu(m)\theta+\sigma(m)\}>\zeta$. We have

$$\log|f(Z_m+\lambda R_m e^{i\vartheta})|>(B\zeta-\epsilon)(\eta\lambda_0/\lambda_1)^{\mu(m)}V(R_m) \tag{4.421}$$

in D_1. Choosing $\epsilon<\frac{1}{3}B$ and ζ so that $2\epsilon<B\zeta<3\epsilon$, and then

$$\epsilon_1<\epsilon(\eta\lambda_0/\lambda_1)^{\mu(m)}<\epsilon(\eta\lambda_0/\lambda_1)^{Q}, \tag{4.43}$$

we have

$$\epsilon_1-(B\zeta-\epsilon)(\eta\lambda_0/\lambda_1)^{\mu(m)}<-(B\zeta-2\epsilon)(\eta\lambda_0/\lambda_1)^{\mu(m)}<-\epsilon(\eta\lambda_0/\lambda_1)^{\mu(m)}<-\epsilon,$$

and so, by (4.421) and (4.41),

$$\left|\frac{\Pi(Z_m+\lambda R_m e^{i\vartheta})}{f(Z_m+\lambda R_m e^{i\vartheta})}\right|<\exp\{-\epsilon V(R_m)\}.$$

Hence

$$\left|\log\left|1-\frac{\Pi(Z_m+\lambda R_m e^{i\vartheta})}{f(Z_m+\lambda R_m e^{i\vartheta})}\right|\right|<2\exp\{-\epsilon V(R_m)\}<\epsilon(\lambda/\lambda_1)^{\mu(m)}V(R_m)$$

in D_1, provided that $m>m_0(\epsilon, \lambda)=m_0(\epsilon, \eta)$. It follows that, in D_1,

$$\begin{aligned}\log|F(Z_m+\lambda R_m e^{i\vartheta})| &= \log|f(Z_m+\lambda R_m e^{i\vartheta})|+\log\left|1-\frac{\Pi(Z_m+\lambda R_m e^{i\vartheta})}{f(Z_m+\lambda R_m e^{i\vartheta})}\right| \\ &= B(m)(\lambda/\lambda_1)^{\mu(m)}\cos\{\mu(m)\theta+\sigma(m)\}V(R_m)+\epsilon(m)(\lambda/\lambda_1)^{\mu(m)}V(R_m),\end{aligned} \tag{4.44}$$

where $|\epsilon(m)|<2\epsilon$ for $m>m_0(\epsilon, \eta)$.

Let $D_2 = D_2(\zeta, \eta, \delta, \lambda_0, \delta_1, m)$ be the set of points for which (4.19) and (4.42) hold, and also $\cos\{\mu(m)\theta+\sigma(m)\} < -\zeta$.

Then

$$\log|f(Z_m+\lambda R_m e^{i\theta})| < -(B\zeta-\epsilon)(\eta\lambda_0/\lambda_1)^{\mu(m)} V(R_m)$$

in D_2; and, choosing ϵ_1 to satisfy (4.43), we have, as above,

$$\begin{aligned}(4.45)\quad &\log|F(Z_m+\lambda R_m e^{i\theta})| \\ &= \log|\Pi(Z_m+\lambda R_m e^{i\theta})| + \log\left|1-\frac{f(Z_m+\lambda R_m e^{i\theta})}{\Pi(Z_m+\lambda R_m e^{i\theta})}\right| \\ &> -2\epsilon(\lambda/\lambda_1)^{\mu(m)} V(R_m)\end{aligned}$$

in D_2, provided that $m > m_0(\epsilon, \eta)$.

It follows from (4.44) and our choice of ϵ that

$$\log|F(Z_m')| - \log|F(Z_m+\eta\lambda_0 R_m e^{-i\sigma(m)})| > 0$$

for $m > m_0(\epsilon, \eta)$. Also $\log|F(z)| < (A+\epsilon_1) V(R_m)$ for $|z-Z_m| \leqslant \lambda_1 R_m$, $m > m_0(\epsilon_1)$. Hence it follows from Lemma 1 that, given $k > 1$, $\delta_2 > 0$, we can choose $H = H(k, \delta_2)$ so that

$$(4.46)\qquad \log|F(z)| > -HV(R_m)$$

for $|z-Z_m'| \leqslant (\lambda_1-\eta\lambda_0) R_m/k$, $m > m_0(\epsilon, \eta)$, except perhaps in circles the sum of whose radii is less than $\delta_2 R_m$.

We have to show that (4.32) holds for every $\lambda > 0$. Since $N\{Z_m, \lambda, \Pi(z)\}$ is a non-decreasing function of λ, we may suppose that $\lambda < \lambda_0$. A difficulty arises from zeros near Z_m. We can either apply Jensen's formula in a circle with centre Z_m' including the circle $|z-Z_m| = \lambda R_m$, or else apply Jensen's formula in the circles

$$|z-Z_m| = \lambda' R_m, \quad |z-Z_m| = \lambda'' R_m,$$

where

$$(4.47)\qquad \tfrac{1}{4}\lambda < \lambda' < \tfrac{1}{2}\lambda, \quad \tfrac{3}{4}\lambda < \lambda'' < \lambda,$$

and take the difference. We use the latter method here, and we have

$$\begin{aligned}(4.48)\quad &\int_{\lambda'}^{\lambda''} \frac{N(Z_m, \lambda)}{\lambda}\, d\lambda \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log|F(Z_m+\lambda'' R_m e^{i\vartheta})| - \frac{1}{2\pi}\int_0^{2\pi} \log|F(Z_m+\lambda' R_m e^{i\vartheta})|\, d\theta \\ &= I_1 + I_2.\end{aligned}$$

Then

$$(4.49)\quad \int_{\lambda'}^{\lambda''} \frac{N(Z_m, \lambda)}{\lambda}\, d\lambda \leqslant N(Z_m, \lambda'') \log(\lambda''/\lambda') < N(Z_m, \lambda) \log 4.$$

Given λ, we can certainly choose η and δ_2 so that (4.46) holds for some λ', λ'' satisfying (4.47). Let E_1, E_2 denote the sets of values of θ for which (4.44) and (4.45) hold respectively and E_3 the remaining set. Then

$$(4.491)\quad \frac{1}{2\pi}\int_{E_2} \log|F(Z_m+\lambda'' R_m e^{i\theta})|\, d\theta > -2\epsilon(\lambda''/\lambda_1)^{\mu(m)} V(R_m);$$

and, since the measure of E_3 is less than $2\zeta\mu(m)+\delta_1+\delta < 2\zeta Q+\delta_1+\delta$, given ϵ_3, we have

$$(4.492)\quad \frac{1}{2\pi}\int_{E_3} \log|F(Z_m+\lambda'' R_m e^{i\theta})|\, d\theta > -\frac{H}{2\pi}(2\zeta Q+\delta_1+\delta) V(R_m) > -2\epsilon_3(\lambda''/\lambda_1)^{\mu(m)} V(R_m),$$

provided that ζ, δ_1, and δ are sufficiently small and m is sufficiently large. Also it is easy to see that

$$(4.493)\quad \frac{1}{2\pi}\left\{\int_{E_2}+\int_{E_3}\right\} \log|F(Z_m+\lambda' R_m e^{i\theta})|\, d\theta < 4\epsilon_3(\lambda'/\lambda_1)^{\mu(m)} V(R_m),$$

provided that ζ and ϵ_1 are sufficiently small. Hence, combining (4.48) with (4.49), and using (4.44), (4.491), (4.492), (4.493), we have

$$N(Z_m, \lambda)\log 4 \geqslant \frac{1}{2\pi} B(m)\left\{\left(\frac{\lambda''}{\lambda_1}\right)^{\mu(m)} - \left(\frac{\lambda'}{\lambda_1}\right)^{\mu(m)}\right\} V(R_m)\int_{E_1} \cos\{\mu(m)\theta+\sigma(m)\}\, d\theta + \epsilon'(m)\left(\frac{\lambda''}{\lambda_1}\right)^{\mu(m)} V(R_m),$$

where $|\epsilon'(m)| < 6\epsilon_3+2\epsilon$ for $m > m_0(\epsilon_3)$.

We may obviously suppose that $\zeta < \frac{1}{2}$, and we have

$$N(Z_m, \lambda) > \frac{B(\lambda/\lambda_1)^{\mu(m)}}{2\pi\log 4}\{(\tfrac{3}{4})^{\mu(m)} - (\tfrac{1}{2})^{\mu(m)}\} V(R_m)\int_0^{\frac{1}{3}\pi} \cos\{\mu(m)\theta\}\, d\theta - (6\epsilon_3+2\epsilon)(\lambda/\lambda_1)^{\mu(m)}.$$

Since $B > 0$, $\mu(m) < Q$, the required result follows from a suitable choice of ϵ_3 and ϵ.

4.5. *Proof of the converse part of Theorem VI.* This is the easier part, and we shall only sketch the proof in the simple case in which $N(Z_m, \lambda_1) = 0$,

$\Pi(z) = a$. The rather tiresome subsidiary complications arising in the general case can be dealt with by means of (4.17) and Lemma 1; the details are similar to those which have been given for the more difficult positive theorem. We suppose then that $N(Z_m, \lambda_1) = 0$, and so we may write

$$P_m(z, \lambda_1) \equiv 1, \quad \log|f(Z_m)| = \xi(m)\, V(R_m).$$

If $\xi(m) > \xi > 0$ for $m > m_0(\xi)$, it follows from (4.18) and (4.19) that there are numbers λ_3 and $m_0(\xi, \lambda_3)$ such that $\log|f(z)| > \frac{1}{2}\xi V(R_m)$ for $|z - Z_m| \leqslant \lambda_3 R_m$, $m > m_0(\xi, \lambda_3)$. But then $\log|f(z) - a| \sim \log|f(z)|$ for $|z - Z_m| \leqslant \lambda_3 R_m$ as $m \to \infty$ for every a; and since $|f(z)| \to \infty$, $N(Z_m, \lambda, a) = 0$ for $\lambda < \lambda_3$, $m > m_0(\xi, a, \lambda_3)$.

If $\xi(m) < -\xi < 0$ for $m > m_0(\xi)$, we can choose λ_3 and $m_0(\xi, \lambda_3)$ so that $\log|f(z)| < -\frac{1}{2}\xi V(R_m)$ for $|z - Z_m| \leqslant \lambda_3 R_m$, $m > m_0(\xi, \lambda_3)$. Hence

$$\log|f(z) - a| \sim \log|a| \quad \text{for} \quad |z - Z_m| \leqslant \lambda_3 R_m$$

and every $a \neq 0$ as $m \to \infty$. It is easy to see that $N(Z_m, \lambda_3, a) = 0$ for every $a \neq 0$ and all large m.

If $\xi(m) \to 0$ as $m \to \infty$ and the sequence is of the first kind, then either (4.18) holds with $f(z) - a$ in place of a for every a, or else there is a number a_0 and a number $\xi > 0$ such that

$$\log|f(z) - a_0| < -\xi V(R_m)$$

for $|z - Z_m| < \lambda_3 R_m$, where $\lambda_3 = \lambda_3(\xi) > 0$, $m > m_0(\xi)$. In the first case it follows from Lemma 1 that

$$N(Z_m, \tfrac{1}{2}\lambda_2, a) = o\{V(R_m)\} \quad \text{as} \quad m \to \infty.$$

In the second,

$$\log|f(Z_m) - a| > \tfrac{1}{2}\log|a|$$

for every $a \neq a_0$ and all $m > m_0(a, \xi)$. Hence $N(Z_m, \frac{1}{2}\lambda_2, a) = 0$ for every $a \neq a_0$. Hence (4.32) is not satisfied unless (4.31) holds; and if (4.31) holds and the sequence is of the first kind, then (4.32) only holds for a single function $\Pi_0(z)$. The required result follows.

4.6. The next theorem shows that, if $f(z)$ behaves like an exponential, $\exp(A - iB)z^\rho$, say, in a certain sequence of circles, then $f(z)$ also behaves like $\exp(A - iB)z^\rho$ in larger circles concentric with these, provided that the number of zeros in the larger circles is small.

THEOREM VII. *Suppose that there is a number $\lambda_1 > 0$, and a sequence of points $Z_1, Z_2, \ldots, Z_m, \ldots$; $R_m = |Z_m| \to \infty$, such that $f(z)$ is regular for*

$|z-Z_m| \leqslant \lambda_1 R_m$, *and that*

$$\lim_{m\to\infty} \frac{\log|f(Z_m)\exp\{Ce^{i\gamma\rho}V(Z_m)\}|}{V(R_m)} = 0, \tag{4.61}$$

$$\log|f(z)\exp\{Ce^{i\gamma\rho}V(z)\}| \leqslant AV(R_m) \quad \textit{for} \quad |z-Z_m| \leqslant \lambda_1 R_m, \tag{4.62}$$

$$\lim_{m\to\infty} \frac{N(Z_m, \lambda_1)}{V(R_m)} = 0. \tag{4.63}$$

Suppose also that there is a positive number $\lambda_2 < \lambda_1$ *such that*

$$\log|f(z)\exp\{Ce^{i\gamma\rho}V(z)\}| < \epsilon V(R_m), \tag{4.64}$$

where $|z-Z_m| \leqslant \lambda_2 R_m$, *for every* $\epsilon > 0$ *and every* $m > m_0(\epsilon)$.

Then, given $\epsilon_1 > 0$, $\delta > 0$, $k > 1$, *we can choose* $m_0(\epsilon_1, \delta, k)$ *so that*

$$\Big|\log|f(z)\exp\{Ce^{i\gamma\rho}V(z)\}|\Big| < \epsilon_1 V(R_m) \tag{4.65}$$

for $|z-Z_m| \leqslant R_m\lambda_1/k$, *except in circles the sum of whose radii is less than* δR_m, *for* $m > m_0(\epsilon, \delta, k)$.

Proof. We first observe that, applying Lemma 1 to

$$F(z) = f(z)\exp\{Ce^{i\gamma\rho}V(z)\}$$

in the circles $|z-Z_m| \leqslant \lambda_2 R_m$, we obtain the conclusion with λ_2 in place of λ_1. That is $f(z)$ behaves like $\exp\{-Ce^{i\gamma\rho}V(z)\}$ in $|z-Z_m| < \lambda_2 R_m$. Also, applying Lemma 1 to $F(z)$ in the circle $|z-Z_m| \leqslant \lambda_1 R_m$, we see that $F(z)$ satisfies the conditions of Theorem V for any λ_3 and any sequence of points Z_m' such that $0 < \lambda_3 < \lambda_1$, $|Z_m'-Z_m| \leqslant (\lambda_1-\lambda_3) R_m$.

Let $\lambda_4(\epsilon)$ be the upper bound of numbers λ_2 such that (4.64) holds for $|z-Z_m| \leqslant \lambda_2 R_m$, $m > m_0(\epsilon)$, and, since $\lambda_4(\epsilon)$ decreases as $\epsilon \to 0$, we may put $\Lambda_4 = \lim \lambda_4(\epsilon)$. Then, given $\epsilon_1 > 0$, $k > 1$, $\delta > 0$, we can choose $m_0(\epsilon_1, \delta, k)$ so that

$$\Big|\log|F(z)|\Big| < \epsilon_1 V(R_m) \tag{4.66}$$

for $|z-Z_m| \leqslant \Lambda_4 R_m/k$, $m > m_0(\epsilon_1, \delta, k)$, except in circles the sum of whose radii is less than δR_m.

Suppose that $\Lambda_4 < \lambda_1$ and apply Theorem V to $F(z)$ in the circles $|z-Z_m'| \leqslant (\lambda_1-\Lambda_4) R_m$, where $|Z_m'-Z_m| = \Lambda_4 R_m$. Let $P_m(z)$ refer to the zeros of $F(z)$ in the circle $|z-Z_m'| \leqslant (\lambda_1-\Lambda_4) R_m$, and suppose that

$$\overline{\lim_{m'\to\infty}} \frac{\log|F(Z_{m'}')/P_{m'}(Z_{m'}')|}{V(R_{m'})} = -\xi < 0$$

for some subsequence $\{Z'_{m'}\}$. Then it follows, from (4.18) and (4.19), that there is a number $\lambda_5 > 0$ and a corresponding number $\xi(\lambda_5) > 0$ such that

$$(4.67) \quad \log|F(z)| < -\xi(\lambda_5)V(R_{m'}) \text{ for } |z-Z'_{m'}| \leqslant \lambda_5 R_{m'} \quad [m > m_0(\lambda_5)].$$

But, by suitable choice of ϵ_1, k, and δ, we can make (4.66) contradict (4.67). Similarly, if

$$\varliminf_{m'\to\infty} \frac{\log|F(Z'_{m'})/P_{m'}(Z_{m'})|}{V(R_{m'})} = \xi > 0$$

for some subsequence, we obtain a contradiction of (4.66). Hence

$$\lim_{m\to\infty} \frac{\log|F(Z_m')/P_m(Z_m')|}{V(R_m)} = 0$$

for every sequence $\{Z_m'\}$ such that $|Z_m - Z_m'| = \Lambda_4 R_m$. But now, if any sequence or subsequence is of the second kind considered in Theorem V, we can find $\lambda_0 > 0$, $\xi = \xi(B) > 0$, $\zeta = \zeta(Q) > 0$ so that for this subsequence either

$$\log|F(z)| < -\xi V(R_m)$$

for $\frac{1}{2}\lambda_0 R_m \leqslant |z-Z_m'| \leqslant \lambda_0 R_m$, $|\arg(z-Z_m')-\tau(m)| \leqslant \zeta$, where $\tau(m)$ depends on $\sigma(m)$ and is chosen so that this region lies inside $|z-Z_m| \leqslant (\Lambda_4 - \frac{1}{4}\lambda_0) R_m$, or else

$$\log|F(z)| > \xi V(R_m)$$

in a similar region, except in circles the sum of whose radii is less than δR_m. Choosing $\epsilon_1 < \xi$, $k < \Lambda_4/(\Lambda_4 - \frac{1}{4}\lambda_0)$ in (4.66), we obtain a contradiction. Hence every sequence $\{Z_m'\}$ consists entirely of points of the first kind. But then it follows that (4.66) is satisfied for $|z-Z_m| \leqslant \lambda_6 R_m$, where $\lambda_6 > \Lambda_4$, except in circles the sum of whose radii is less than δR_m, and again we have a contradiction. Hence $\Lambda_4 = \lambda_1$, and we have the required result.

5.1. The previous results can be applied so as to yield various results about $h(\theta)$ and directions of Borel. For if $f(z)$ is regular and of proximate order $\rho(r)$ for $\alpha \leqslant \arg z \leqslant \beta$, $|z| > l$ and $\alpha < \gamma < \beta$, then either

$$(5.11) \qquad \varlimsup_{r\to\infty} \frac{n(r, \gamma-\delta, \gamma+\delta, a)}{V(r)} - \xi(\delta) > 0$$

for every $\delta > 0$ and every a, or else

$$\lim_{r\to\infty} \frac{n(r, \gamma-\delta, \gamma+\delta, a)}{V(r)} = 0$$

for some $\delta > 0$ and some α. In the first case, $\arg z = \gamma$ is obviously a direction of Borel, and in the second we can apply Theorem V to $f(z)$ in any sequence of circles $|z-Z_m| \leqslant \lambda_1 R_m$, $m = 1, 2, \ldots$, such that (4.15) is satisfied. That is, in any sequence of circles in which $|f(z)|$ is not too small. For instance, if the hypotheses of Theorem IV are satisfied, *i.e.* $h(\gamma) = 0$, $h(\theta) > 0$ for $\gamma < \theta < \beta$, it is not difficult to see that either (5.11) holds or else there is a sequence of circles for which the hypotheses of Theorem VI are satisfied. Hence $\arg z = \gamma$ is a direction of Borel of the maximum kind.

5.2. If

$$h(\theta) = C \cos(\theta-\gamma)\rho \tag{5.21}$$

for $\alpha-\delta \leqslant \theta \leqslant \alpha+\delta$, where $\delta > 0$, we say that $h(\theta)$ is a *sinusoid* at $\theta = \alpha$. Similarly, if (5.21) holds for $\alpha \leqslant \theta \leqslant \beta$, we say that $h(\theta)$ is a sinusoid in that interval. V. Bernstein* has suggested that for integral functions the directions of Borel correspond to points at which $h^+(\theta)$ is not a sinusoid. This is not true without some modification, as I shall show later†. The following quite general result was proved in H.

THEOREM VIII. *Suppose that $f(z)$ is regular and of proximate order $\rho(r)$ for $|z| \geqslant l$, $0 \leqslant \arg z \leqslant \alpha$, and also that*

$$h(\theta) = C \cos(\theta-\gamma)\rho,$$

where $C \geqslant 0$, $0 \leqslant |\gamma| \leqslant \pi/\rho$, for $0 \leqslant \theta \leqslant \alpha$. Then there is a sequence $R_1, R_2, \ldots, R_m \to \infty$ such that (1) for every $\delta > 0$, $\eta > 0$,

$$\lim_{m\to\infty} \frac{n(R_m, \delta, \alpha-\delta) - n(\eta R_m, \delta, \alpha-\delta)}{V(R_m)} = 0; \tag{5.22}$$

(2) given $\epsilon > 0$, $\delta > 0$, $\eta > 0$, $\zeta > 0$, we can choose $m_0(\epsilon, \delta, \eta, \zeta)$ so that

$$\left|\log|f(re^{i\theta})| - C\cos(\theta-\gamma)\rho V(r)\right| < \epsilon V(r) \tag{5.23}$$

for $\delta \leqslant \theta \leqslant \alpha-\delta$, $\eta R_m \leqslant r \leqslant R_m$, $m > m_0(\epsilon, \delta, \eta, \zeta)$, except for values of $re^{i\theta}$ lying in circles the sum of whose radii is less than ζR_m.

If $h(\theta) > 0$ for $0 \leqslant \theta \leqslant \alpha$, then

$$\overline{\lim_{r\to\infty}} \frac{\log|f(re^{i\theta})-a|}{V(r)} = h(\theta) \quad \text{for} \quad 0 \leqslant \theta \leqslant \alpha.$$

Applying this theorem to $f(z)-a$, we see that *if $h^+(\theta)$ is a sinusoid for*

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* V. Bernstein, *Annali di Mat.* (4), 12 (1934), 173–196. See also § 6.1 of this paper.

† See § 5.9.

$0 < \theta < \alpha$ *no line inside the angle* $0 < \arg z < \alpha$ *is effectively a direction of Borel for the sequence of values* $|z| = R_1, R_2, \ldots$. For

$$\lim_{m\to\infty} \frac{n(R_m, \delta, \alpha-\delta, a) - n(\eta R_m, \delta, \alpha-\delta, a)}{V(R_m)} = 0$$

for all values of a. If the line $\arg z = \gamma + \frac{1}{2}\pi/\rho$ lies inside the angle $0 \leqslant \arg z \leqslant \alpha$, then it is obviously a direction of Borel in virtue of Theorem IV. The number of zeros of $f(z)$ for which $R_{m-1} \leqslant |z| \leqslant \eta R_m$ may be very large unless some further condition is imposed. Combining Theorem VIII with Theorem IX, we have the following result.

THEOREM IX. *Suppose that the hypotheses of Theorem VIII are satisfied, and that* $f(z)$ *is regular and of proximate order* $\rho(r)$ *for* $|z| \geqslant l$, $0 \leqslant \arg z \leqslant \beta$, *where* $\beta > \alpha$. *Then either there is a sequence* $R_1', R_2', \ldots$ *such that* $\lim_{m\to\infty} R_m'/R_m = 1$, *and the conclusions of Theorem VIII hold with* R_m' *in place of* R_m *and* β *in place of* α, *or else*

$$(5.24)\quad \overline{\lim_{m\to\infty}} \frac{n(R_m, \alpha-\delta, \beta-\delta') - n(\eta R_m, \alpha-\delta, \beta-\delta')}{V(R_m)} = \xi(\delta) > 0$$

for every $\delta > 0$ *and some* $\delta' > 0$.

COROLLARY. *If, in addition,* $C > 0$ *and* $\gamma - \frac{1}{2}\pi/\rho < 0 < \gamma + \frac{1}{2}\pi/\rho < \beta$, *then either there is a direction of Borel of the maximum kind without an exceptional value in the angle* $\alpha \leqslant \arg z < \gamma + \frac{1}{2}\pi/\rho$, *or else* $\arg z = \gamma + \frac{1}{2}\pi/\rho$ *is a direction of Borel of the maximum kind.*

Proof. Suppose first that $\beta \leqslant 2\alpha$, and that (5.24) does not hold for some $\delta > 0$, every $\delta' > 0$. Then it follows from (5.22) that

$$\lim_{m\to\infty} \frac{n(R_m, \delta, \beta-\delta) - n(\eta R_m, \delta, \beta-\delta)}{V(R_m)} = 0$$

for every $\delta > 0$. Consider the line $\arg z = \alpha - \delta_1$, and divide the interval $\eta_1 R_m \leqslant |z| \leqslant R_m$ into $1/\zeta_1$ intervals $I_j(m)$, of length less than $\zeta_1 R_m$. Then it follows from Theorem VIII that we can find a point $Z_m^{\,j}$ in each interval $I_j(m)$ such that

$$\lim_{m\to\infty} \frac{\log|F(Z_m^{\,j})|}{V(R_m)} = \lim_{m\to\infty} \frac{\log|f(Z_m^{\,j}) \exp\{-Ce^{i\gamma\rho} V(Z_m^{\,j})\}|}{V(R_m)} = 0.$$

Hence we may apply Theorem VII to $f(z)$ in the sequence of circles $|z - Z_m^{\,j}| \leqslant \lambda_1 |Z_m^{\,j}| = \lambda_1 R_m^{\,j}$, where $\lambda_1 = \sin(\beta-\alpha)$, with $\lambda_2 = \sin\delta_1$. It

follows that (5.23) holds for $|z-Z_m{}^j| \leqslant \lambda_1 R_m{}^j/k$, where $k>1$, except in circles the sum of whose radii is less than $\frac{1}{2}\zeta_1{}^2 R_m$, provided that

$$m > m_0(\epsilon, k, \zeta_1).$$

Now the circle $|z-Z_m{}^j| \leqslant \lambda_1 R_m{}^j/k$ includes an interval of

$$\arg z = \beta - 2\delta_1$$

of length

$$2\left\{\frac{\lambda_1{}^2}{k^2} - \sin^2(\beta-\alpha-\delta_1)\right\}^{\frac{1}{2}} R_m{}^j$$

$$= 2\left\{\frac{\sin^2(\beta-\alpha)}{k^2} - \sin^2(\beta-\alpha)\{1+O(\delta_1)\}\right\}^{\frac{1}{2}} R_m{}^j$$

$$= 2\sin(\beta-\alpha)\{O(k-1)-O(\delta_1)\}^{\frac{1}{2}} R_m{}^j > 2K\sin(\beta-\alpha)\,\delta_1 R_m{}^j$$

$$> 2K\sin(\beta-\alpha)\,\delta_1\eta_1 R_m,$$

provided that $(k-1)/\delta_1 \to \infty$ as $\delta_1 \to 0$. By choosing $\zeta_1 < 2K\sin(\beta-\alpha)\,\delta_1\eta_1$, we can make these circles cover the whole interval of $\arg z = \beta-2\delta$, from $(\eta_1+\zeta_1)R_m$ to $(1-\zeta_1)R_m$, and *a fortiori* the region

$$\alpha-\delta_1 \leqslant \arg z \leqslant \beta-2\delta_1,\ (\eta_1+\zeta_1)R_m \leqslant |z| \leqslant (1-\zeta_1)R_m.$$

Putting $\delta = 2\delta_1$, $R_m = R_m(1-\zeta_1)$, $\eta = (\eta_1+\zeta_1)/(1-\zeta_1)$, and $\zeta = 2\zeta_1$, we have the required result.

If $\beta > 2\alpha$, we apply the theorem with $\beta = 2\alpha$, and then repeat the argument, considering the line $\arg z = 2\alpha-\delta_1$, instead of $\alpha-\delta_1$.

Proof of the Corollary. We have $h(\theta) > 0$ for $0 < \theta < \gamma+\frac{1}{2}\pi/\rho$. If $\gamma+\frac{1}{2}\pi/\rho \leqslant \alpha$, we can apply Theorem IV, and we see that $\arg z = \gamma+\frac{1}{2}\pi/\rho$ is a direction of Borel of the maximum kind. Suppose that $\alpha < \gamma+\frac{1}{2}\pi/\rho$; then for every a we have

$$\lim_{r\to\infty} \frac{\log|f(re^{i\theta})-a|}{V(r)} = h(\theta)$$

for $0 < \theta < \alpha$; and so $f(z)-a$ satisfies the hypotheses of the theorem for every a. Suppose that for some a and some β', such that $\alpha < \beta' \leqslant \beta$,

$$\lim_{m\to\infty} \frac{n(R_m, \alpha-\delta, \beta'-\delta, a) - n(\eta R_m, \alpha-\delta, \beta'-\delta, a)}{V(R_m)} = 0 \tag{5.25}$$

for every $\delta > 0$, $\eta > 0$. Then it follows from the theorem that the conclusions of Theorem VIII hold with β' in place of α, $f(z)-a$ in place of $f(z)$, and $R_m{}'$ in place of R_m, where $\lim R_m{}'/R_m = 1$. Since $C\cos(\theta-\gamma)\rho > 0$

for $0<\theta<\gamma+\frac{1}{2}\pi/\rho$, it follows that, if $\beta'\leqslant\gamma+\frac{1}{2}\pi/\rho$, then $f(z)-a$ satisfies the inequalities (5.23) for every a. Hence, applying Lemma 1 to $\{f(z)-a\}\exp\{-Ce^{-i\gamma\rho}V(z)\}$, we see that, for every a,

$$\lim_{m\to\infty}\frac{n(R_m',\delta,\beta'-\delta,a)-n(\eta R_m',\delta,\beta'-\delta,a)}{V(R_m)}=0,$$

where $\lim R_m'/R_m=1$ for every $\delta>0$, $\eta>0$. If $\beta'>\gamma+\frac{1}{2}\pi/\rho$, we can apply Theorem V to a sequence of circles with centres $\frac{1}{2}R_m e^{i(\gamma+\frac{1}{2}\pi/\rho)}$, and we see that $\arg z=\gamma+\frac{1}{2}\pi/\rho$ is a direction of Borel of the maximum kind. If, for some a, (5.25) holds for $\beta'\leqslant\beta''\leqslant\gamma+\frac{1}{2}\pi/\rho$, but not for any $\beta'>\beta''$, then, by the argument just given, (5.25) holds for every a with $\beta'\leqslant\beta''$, but cannot hold for any a with $\beta'>\beta''$. That is

$$\overline{\lim_{m\to\infty}}\frac{n(R_m,\beta''-\delta,\beta''+\delta,a)-n(\eta R_m,\beta''-\delta,\beta''+\delta,a)}{V(R_m)}=\xi(\delta)>0$$

for every a and every $\delta>0$, and $\arg z=\beta''$ is a direction of Borel of the maximum kind without an exceptional value.

5.3. We now consider integral functions, and introduce an extra hypothesis under which the conjecture of Bernstein mentioned in §5.2 is true. We denote by $n(r,a)$ the number of zeros of the integral function $f(z)-a$ for which $|z|\leqslant r$, and we write $n(r,0)=n(r)$.

THEOREM X. *Suppose that $f(z)$ is an integral function of proximate order $\rho(r)$ and that*

$$\lim_{r\to\infty}\frac{n(r)}{\rho V(r)}=\frac{1}{2\pi}\int_{-\pi}^{\pi}h(\theta)\,d\theta=H. \tag{5.31}$$

Then (1) *$f(z)$ has a direction of Borel of the maximum kind without an exceptional value, $\arg z=\alpha$, corresponding to every point $\theta=\alpha$ at which $h(\theta)$ is not a sinusoid, provided that $h(\theta)>0$ for $0<|\theta-\alpha|<\pi/\rho$,* (2) *the sides of the angles in which $h(\theta)$ is positive are directions of Borel of the maximum kind,* (3) *there are no other directions of Borel of the maximum kind.*

For these functions Bernstein's conjecture is true, and moreover it will appear from the proof that each direction of Borel is an effective direction of Borel for all values of $|z|$, *i.e.*

$$n(r,\alpha-\delta,\alpha+\delta,a)>\xi(\delta)V(r)$$

for all $r>r_0$, unless, in case (2), a is the exceptional value. Part 2 follows from Theorem IV; the other two parts follow immediately from the next theorem and its corollaries.

THEOREM XI. *If $f(z)$ satisfies the hypotheses of Theorem X, and if $h(\theta)$ is not a sinusoid at $\theta = \alpha$, then*

$$(5.32) \qquad \underline{\lim}_{r\to\infty} \frac{n(kr, \alpha-\delta, \alpha+\delta)-n(r, \alpha-\delta, \alpha+\delta)}{V(r)} = \xi(\delta, k) > 0$$

for every $\delta > 0$ and every $k > 1$. Also, if $h(\theta)$ is a sinusoid for $\alpha \leqslant \theta \leqslant \beta$, then

$$\lim_{r\to\infty} \frac{n(r, \alpha+\delta, \beta-\delta)}{V(r)} = 0$$

for every $\delta > 0$.

COROLLARY 1. *Suppose that $f(z)$ satisfies the hypotheses of Theorem X, and that $h(\theta)$ is not a sinusoid at $\theta = \alpha$. If $h(\theta) > 0$ for $0 < |\theta-\alpha| \leqslant \delta$, then*

$$(5.33) \qquad \underline{\lim}_{r\to\infty} \frac{n(kr, \alpha-\delta, \alpha+\delta, a)-n(r, \alpha-\delta, \alpha+\delta, a)}{V(r)} = \xi(\delta, k) > 0$$

for every a without exception.

COROLLARY 2. *Suppose that $h(\theta)$ is a sinusoid, or is negative, for $\alpha < \theta < \beta$, then*

$$\lim_{r\to\infty} \frac{n(r, \alpha+\delta, \beta-\delta, a)}{V(r)} = 0$$

for every $\delta > 0$ and every a except perhaps one. The exception only occurs if $h(\theta) \leqslant 0$ for $\alpha \leqslant \theta \leqslant \beta$.

The chief characteristics of these functions* are that they have their zeros evenly distributed as regards their distance from the origin, and that they have the largest possible number of zeros for a given $h(\theta)$. For, taking $f(0) = 1$ for simplicity, it follows from Jensen's formula that

$$\overline{\lim_{r\to\infty}} \frac{N(r)}{V(r)} = \overline{\lim_{r\to\infty}} \frac{1}{V(r)} \int_0^r \frac{n(u)}{u}\, du = \overline{\lim_{r\to\infty}} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\log|f(re^{i\theta})|}{V(r)}\, d\theta \leqslant H.$$

Hence, if $n(r)/V(r)$ tends to a limit, the limit is less than or equal to H.

By using a direct argument involving Jensen's formula, similar to that used in Theorem VI, we could probably replace $\underline{\lim}$ by lim in (5.32), and calculate $\xi(\delta, k)$ in terms of $h'(\alpha+)$, $h'(\alpha-)$, δ, and k; but I shall not attempt to do so here. It is implied by Corollary 1 that, if $h(\theta) > 0$ for $0 \leqslant \theta < 2\pi$, then $f(z)$ has no exceptional value. To be precise, we have the following result.

* Compare M. L. Cartwright, *Proc. London Math. Soc.* (2), 33 (1931), 213.

THEOREM XII. *If the hypotheses of Theorem X are satisfied and if* $h(\theta) > 0$ *for all* θ *except a set of measure zero, then*

$$\lim_{r\to\infty} \frac{n(r, a)}{\rho V(r)} = H$$

for all a *without exception.*

COROLLARY. *If* $f(z)$ *is an integral of proximate order* $\rho(r)$ *such that*

$$\overline{\lim_{r\to\infty}} \frac{\log M(r)}{V(r)} = A, \tag{5.34}$$

where $M(r) = \max|f(re^{i\theta})|$, *and if*

$$\lim_{r\to\infty} \frac{n(r)}{\rho V(r)} = A, \tag{5.35}$$

then $\lim_{r\to\infty} \frac{n(r, a)}{\rho V(r)} = A$ *for every* a, *and every line* $\arg z = \alpha$ *such that* $0 \leqslant \alpha < 2\pi$ *is a direction of Borel of the maximum kind without an exceptional value.*

It follows from Lemma 3 that the set of measure zero in which $h(\theta) = 0$ consists of a number $K \leqslant 2\rho$ of isolated points. The corollary has a special interest because it does not involve $h(\theta)$ at first sight, but it follows, as we shall see later, from (5.34) and (5.35), that $h(\theta) = A$, and so we can apply the previous theorems.

5.4. These theorems and their corollaries are all deduced from the following lemma.

LEMMA 7. *Suppose that* $f(z)$ *satisfies the hypotheses of Theorem X, then,* (1) *given any* $\epsilon > 0$, $\delta > 0$, *we can choose* $r_0(\epsilon, \delta)$ *so that*

$$\log|f(re^{i\theta})| > \{h(\theta) - \epsilon\} V(r) \tag{5.41}$$

for $r > r_0(\epsilon, \delta)$, *except in a set of* θ *of measure less than* δ, (2) *if* $h(\theta) > 0$ *for* $\alpha < \theta < \beta$, *then, given any* $\epsilon > 0$, $\delta > 0$, *and any* a, *we have*

$$\log|f(re^{i\theta}) - a| > \{h(\theta) - \epsilon\} V(r)$$

for $\alpha \leqslant \theta \leqslant \beta$, *except in a set of measure less than* δ, *provided that* $r > r_0(\epsilon, \delta, a)$.

It will appear from the proof that, if $N(r) \sim HV(r)$ in a set of r of density $\lambda > 0$, then (5.41) holds in that set. But if $\overline{\lim} \frac{N(r)}{V(r)} < H$, the method fails completely.

We observe that, if $h(\theta) > 0$, then

$$h(\theta, a) = \overline{\lim_{r\to\infty}} \frac{\log|f(re^{i\theta})-a|}{V(r)} = h(\theta),$$

and so it is easy to see that (2) follows from (1). It remains to prove (1).

We may suppose that $f(z) \sim c_0 z^m$ as $z \to 0$, and then we have

$$N(r) = \int_0^r \frac{n(u)-m}{u}\, du = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{i\theta})|\, d\theta - \log|c_0| - m\log r.$$

Since $n(r) \sim H\rho V(r)$, we have $N(r) \sim H V(r)$. Hence

$$\lim_{r\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\log|f(re^{i\theta})|}{V(r)}\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\theta)\, d\theta. \tag{5.42}$$

It follows from Lemma 4 that $\log|f(re^{i\theta})| < \{h(\theta)+\epsilon_1\} V(r)$ for every $\epsilon_1 > 0$ and every $r > r_0(\epsilon_1)$. Suppose that, for some $\epsilon_2 > 0$, $\delta > 0$, $\log|f(re^{i\theta})| < \{h(\theta)-\epsilon_2\} V(r)$ for a sequence of values of r tending to ∞, in a set of θ of measure δ. Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\log|f(re^{i\theta})|}{V(r)}\, d\theta < \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\theta)\, d\theta + \epsilon_1 - \frac{\epsilon_2\delta}{2\pi}$$

for this sequence. Choosing $\epsilon_1 < (\epsilon_2\delta_2)/(4\pi)$, we obtain a contradiction of (5.42). Hence we have the required result.

5.5. *Proof of Theorem XI.* Suppose that $h(\theta)$ is not a sinusoid at $\theta = \alpha$, and put $\alpha = 0$ for convenience. Then, by Lemma 5, the right-hand derivative $h'(0+)$ exists at $\theta = 0$, and we can put

$$h(0) = C\cos\gamma\rho, \quad h'(0+) = C\rho\sin\gamma\rho, \tag{5.51}$$

where $C \geqq 0$, $0 \leqq |\gamma| \leqq \pi/\rho$. Suppose that

$$h(\delta) = C\cos(\delta-\gamma)\rho + B\sin\delta\rho, \tag{5.52}$$

where $0 < \delta < \pi/\rho$; it then follows from Lemma 2 that

$$h(\theta) \leqq C\cos(\theta-\gamma)\rho + B\sin\theta\rho$$

for $0 \leqq \theta \leqq \delta$. Hence

$$h'(0+) = \lim_{\theta\to 0+} \frac{h(\theta)-h(0)}{\theta} \leqq C\rho\sin\gamma\rho + B\rho.$$

Comparing this with (5.51), we see that $B \geqq 0$. Also, if (5.52) holds for some δ such that $0 < \delta < \pi/\rho$ with $B = 0$, then $h(\theta) = C\cos(\theta-\gamma)\rho$ for

$0 \leqslant \theta \leqslant \delta$. Hence, either

$$h(\theta) > C \cos(\theta-\gamma)\rho \tag{5.53}$$

for $0 < \theta < \pi/\rho$, or else there is a number $\delta > 0$ such that

$$h(\theta) = C \cos(\theta-\gamma)\rho \tag{5.54}$$

for $0 \leqslant \theta \leqslant \delta$.

Suppose that (5.54) holds for $0 \leqslant \theta \leqslant \delta$, and that

$$h(\delta_1) = C \cos(\delta_1-\gamma)\rho + B \sin(\delta_1-\delta)\rho,$$

where $\delta - \pi/\rho < \delta_1 < 0$. Then $h(0) \leqslant C \cos\gamma\rho + B \sin\delta\rho$, which contradicts (5.51) unless $B \geqslant 0$. If $B = 0$, (5.54) is satisfied for $\delta_1 \leqslant \theta \leqslant \delta$, and $h(\theta)$ is a sinusoid at $\theta = 0$, which is contrary to our hypothesis. Hence $B > 0$, and (5.53) is satisfied either for $-\pi/\rho < \theta < 0$ or for $0 < \theta < \pi/\rho$.

Suppose, for convenience, that (5.53) holds for $0 < \theta < \pi/\rho$. Let $F(z) = f(z)\exp\{-Ce^{i\gamma\rho}V(z)\}$ and let

$$H(\theta) = \overline{\lim_{r\to\infty}} \frac{\log|F(re^{i\theta})|}{V(r)}.$$

Then $H(\theta) > 0$ for $0 < \theta < \pi/\rho$, $H(\theta) \geqslant 0$ for $-\pi/\rho \leqslant \theta \leqslant 0$, and

$$H(\theta) = o(\theta-\gamma)$$

as $\theta \to \gamma+0$. As usual,

$$\log|F(re^{i\theta})| < \{H(\theta)+\epsilon_1\}V(r) \tag{5.55}$$

for every $\epsilon_1 > 0$ and every $r > r_0(\epsilon_1)$, and, by Lemma 7,

$$\log|F(re^{i\theta})| > \{H(\theta)-\epsilon_2\}V(r) \geqslant -\epsilon_2 V(r) \tag{5.56}$$

for all θ except a set of measure δ_2 and all $r > r_0(\epsilon_2, \delta_2)$. Since $H(\theta)$ is continuous and $H(\theta) > 0$ for $0 < \theta < \pi/\rho$, the lower bound of $H(\theta)$ for $\delta_3 \leqslant \theta \leqslant \pi/\rho - \delta_3$ is positive. Let $\zeta(\delta_3)$ be this lower bound, and put $\epsilon_2 = \frac{1}{2}\zeta(\delta_3)$ in (5.56). Then

$$\log|F(re^{i\theta})| > \tfrac{1}{2}\zeta(\delta_3)V(r) \tag{5.57}$$

for $\delta_3 \leqslant \theta \leqslant \pi/\rho - \delta_3$, except in a set of measure δ_2 and all $r > r_0(\zeta, \delta_2)$.

Now suppose that the conclusion of the theorem is false; then, for some $k > 1$ and some $\delta > 0$, there is a sequence $R_1, R_2, \ldots, R_m \to \infty$ such that

$$\lim_{m\to\infty} \frac{n(k^2R_m, -\delta, \delta) - n(R_m, -\delta, \delta)}{V(R_m)} = 0.$$

We can apply Theorem V to $F(z)$ in the circles $|z-k R_m| \leqslant \lambda_1 R_m$, where $\lambda_1 = \min(k-1, \sin \delta)$. It is easy to see that, if

$$\lim_{m\to\infty} \frac{\log|F(kR_m)/P_m(kR_m, \lambda_1)|}{V(R_m)} = \xi > 0,$$

then (5.55) is false for $\theta = 0$. On the other hand, if $\xi < 0$, we find that (5.56) is false. If $\xi = 0$ and the points are of the first kind, then (5.57) is false. If $\xi = 0$ and the points are of the second kind, we also have a contradiction. For, if $\mu(m'') = 1$, one or other of (5.56) and (5.57) is false, and if $\mu(m'') > 1$, then both are false, owing to the negative values of $\cos\{\mu(m'')\theta + \sigma(m'')\}$ in (4.19).

It remains to prove the second part of Theorem XI. Let

$$h(\theta) = C \cos(\theta - \gamma)\rho$$

for $\alpha \leqslant \theta \leqslant \beta$, and let $F(z)$ and $H(\theta)$ be defined as above. In this case $H(\theta) = 0$ for $\alpha \leqslant \theta \leqslant \beta$; and, given $\epsilon_1 > 0$, $\delta_1 > 0$, we have

$$\log|F(re^{i\theta})| < \epsilon_1 V(r) \tag{5.58}$$

for $\alpha \leqslant \theta \leqslant \beta$, $r > r_0(\epsilon_1)$, and

$$\log|F(re^{i\theta})| > -\epsilon_1 V(r) \tag{5.59}$$

for $\alpha \leqslant \theta \leqslant \beta$, except in a set of measure δ_1 and for $r > r_0(\epsilon_1, \delta_1)$.

We may suppose that $\beta = -\alpha = \frac{1}{2}\pi$; for the other cases can be reduced to this by the transformation $z = w^{\pi/(\beta-\alpha)} e^{-i\frac{1}{2}(\alpha+\beta)}$. We can obviously choose R_0 so that (5.59) is satisfied for $r = 2^m R_0$, $m = 1, 2, \ldots$, $\theta = \theta_m$, where $|\theta_m| < \delta_1$, and also (5.58) is satisfied for $r > r_0$, $|\theta| \leqslant \frac{1}{2}\pi$. Applying Lemma 1 to $F(z)$ in the circle $|z - 2^m R_0 e^{i\theta_m}| \leqslant R_0(2^m - 1)\cos\delta_1$, we find that*, for every $\delta > 0$,

$$n(2^m R_0, \tfrac{1}{2}\pi-\delta, -\tfrac{1}{2}\pi+\delta) - n(2^{m-1} R_0, \tfrac{1}{2}\pi-\delta, -\tfrac{1}{2}\pi+\delta) < \epsilon_1 K(\delta) V(r).$$

Putting $m = 2, 3, \ldots, M$, where $2^{M-1} R_0 < r < 2^M R_0$, and summing, we have

$$\begin{aligned} n(r, \tfrac{1}{2}\pi-\delta, -\tfrac{1}{2}\pi+\delta) &< n(2^M R_0, \tfrac{1}{2}\pi-\delta, -\tfrac{1}{2}\pi+\delta) \\ &< \{\epsilon_1 K(\delta) + K\} V(2^M R_0) < \epsilon_1 K_1(\delta) V(r). \end{aligned}$$

Choosing $\epsilon_1 < \epsilon / K_1(\delta)$, we have the required result.

The corollaries follow from the obvious facts that, if $h(\theta) > 0$, then $h(\theta, a) > 0$, and, if $h(\theta) < 0$, then $h(\theta, a) = 0$, except for $a = 0$.

* This is almost a repetition of the working of Theorems I and II of H.

5.6. *Proof of Theorem XII.* Using the notation of Lemma 7, we have $h(\theta, a) = h(\theta)$, except perhaps at a number of isolated points. Since $h(\theta, a)$ and $h(\theta)$ are continuous, it follows that $h(\theta, a) \equiv h(\theta)$. We may suppose, as before, that $f(z) - a \sim c_0' z^{m'}$ as $z \to 0$, and we have

$$N(r, a) = \int_0^r \frac{n(u, a) - m'}{u}\, du < (H + \epsilon_1) V(r),$$

where $n(r, a)$ denotes the number of zeros of $f(z) - a$ for which $|z| \leqslant r$, for every $\epsilon_1 > 0$ and every $r > r_0(\epsilon_1)$. Also, given $\epsilon_2 > 0$, $\delta_2 > 0$, we have

$$\log|f(re^{i\theta}) - a| > \{h(\theta, a) - \epsilon_2\} V(r) = \{h(\theta) - \epsilon_2\} V(r)$$

for $r > r_0(\epsilon_1, \delta_2)$, except in a set of θ of measure δ_2. It follows from Lemma 1 that, given any $k > 1$ and any r, however large, we can find a circle $|z| = r'$ such that $rk^{-1} \leqslant r' \leqslant r$, and

$$\log|f(r'e^{i\theta}) - a| > -KV(r),$$

where K depends on k but not on r. Hence

$$N(r, a) \geqslant N(r', a) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log|f(r'e^{i\theta}) - a|\, d\theta$$

$$> (H - \epsilon_2) V(r') - K\delta_2 V(r)$$

for $r > r_0(\epsilon_2, \delta_2)$. Since $V(kr) \sim k^\rho V(r)$ as $r \to \infty$, given $\epsilon > 0$, we can choose $k = k(\epsilon)$ so that $V(r') > (1 - \frac{1}{3}\epsilon/H) V(r)$ for $r > r_0(\epsilon)$. Having fixed k, we choose $\delta_2 < \frac{1}{3}\epsilon/K$, and $\epsilon_2 < \frac{1}{3}\epsilon$, and we have

$$N(r, a) > (H - \epsilon) V(r)$$

for $r > r_0(\epsilon)$. We now have $N(r, a) \sim HV(r)$, from which it is easy to deduce the required result.

Proof of the corollary. We have

$$\lim_{r \to \infty} \frac{N(r)}{V(r)} = \lim_{r \to \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\log|f(re^{i\theta})|}{V(r)} = A$$

and $\log|f(re^{i\theta})| < (A + \epsilon) V(r)$ for every $\epsilon > 0$ and every $r > r_0(\epsilon)$. It follows, as in Lemma 7, that, given $\epsilon > 0$, $\delta > 0$, we have

$$\log|f(re^{i\theta})| > (A - \epsilon) V(r)$$

for $r > r_0(\epsilon, \delta)$, except in a set of θ of measure δ. Hence it is easy to see

from the continuity of $h(\theta)$ that $h(\theta) = A$, and we can apply the previous theorems which give the required results.

5.7. We next construct some examples to show that these results cannot be substantially improved. Theorems X–XII seem to be satisfactory as regards their conclusions, but the hypothesis (5.31) is very restrictive. We therefore consider functions for which

$$n(r) < (H-\delta)\rho V(r),$$

where $\delta > 0$, and we use the following lemma*.

LEMMA 8. *Suppose that $f(z)$ is a canonical product of integral order ρ and mean type such that $n(r) = o(r^\rho)$, and let*

$$S(r) = \frac{1}{\rho} \sum_{r_m \leqslant r} \frac{1}{z_n{}^\rho},$$

where $z_1, z_2, \ldots, z_n = r_n e^{i\theta_n}, \ldots$ are the zeros of $f(z)$. Then (1)

$$\log|f(re^{i\theta})| < [\Re\{S(r)\, e^{i\theta\rho}\} + \epsilon] r^\rho \tag{5.71}$$

for every $\epsilon > 0$ and every $r > r_0(\epsilon)$; (2) *given any $\epsilon > 0$, $\eta > 0$, we can choose $R_0(\epsilon, \eta)$ so that*

$$\log|f(re^{i\theta})| > [\Re\{S(r)\, e^{i\theta\rho}\} - \epsilon] r^\rho \tag{5.72}$$

for $0 < r < R$, $R > R_0(\epsilon, \eta)$, except in a set of measure ηR; (3) *if $f(z)$ has no zeros for which $\alpha < \theta_n < \beta$ then, given $\epsilon > 0$, $\delta > 0$, we can choose $r_0(\epsilon, \delta)$ so that* (5.72) *holds uniformly for $\alpha + \delta \leqslant \theta_n \leqslant \beta - \delta$, $r > r_0(\epsilon, \delta)$.*

For the sake of simplicity we shall consider only integral functions of order 1 and mean type. There is no difficulty in generalizing to integral functions of any integral order and mean type; for we can still use the lemma; and the lemma itself can be extended to functions of integral order and maximum type. We can construct similar functions of any proximate order $\rho(r)$ which are regular for $\alpha < \arg z < \beta$, $|z| \geqslant \rho$ by conformal representation $w = V(z)$ of these integral functions of integral order; but a difficulty arises if we want to construct integral functions of non-integral order. Something can probably be done by arranging the zeros so that the behaviour of $f(z)$ depends chiefly on

$$S_p(r) = \frac{1}{p} \sum_{r_n \leqslant r} \frac{1}{z_n{}^p},$$

where p is the integral part of ρ, as is the case with Valiron's† *fonctions*

* See M. L. Cartwright, *Proc. London Math. Soc.* (2), 33 (1931), 212.

† G. Valiron, *Annales de la Faculté des Sciences de Toulouse* (3), 5 (1914), 117–257.

orientées. But, since $n(r)$ is not negligible compared with $S_p(r)$, the calculations become extremely complicated when we try to vary $\arg S_p(r)$ with r, as we vary $\arg S_p(r)$ in our functions of integral order.

The general principle involved in the following examples holds good whenever $n(r) < (H-\delta)r$ and $h(\theta) > 0$. That is, if $n(r) < (H-\delta)r$, a point at which $h(\theta) > 0$ and $h(\theta)$ is not a sinusoid need not determine a direction of Borel of the maximum kind except perhaps by some accidental coincidence of circumstances depending on the form of $h(\theta)$ and the values of H and δ. But, owing to the varied forms of $h(\theta)$, it is difficult to exclude such coincidences, and prove a general result.

THEOREM XIII. *There are integral functions of order* 1 *and mean type such that*

$$h(\theta) = \max(\cos\theta, \sin\theta),$$

and the lines $\arg z = \gamma$, *where* $-\frac{1}{2}\pi \leqslant \gamma \leqslant 0$, $\frac{1}{2}\pi \leqslant \gamma \leqslant \pi$ *are directions of Borel of the maximum kind. These lines are the only lines of Julia of the function and a fortiori the only directions of Borel*; *also* $n(r) = o(r)$.

These functions have directions of Borel corresponding to the points $-\frac{1}{2}\pi < \theta \leqslant 0$, $\frac{1}{2}\pi < \gamma \leqslant \pi$ at which $h^+(\theta)$ is a sinusoid, but there is not even a line of Julia corresponding to the point $\theta = \frac{1}{4}\pi$ at which $h^+(\theta)$ is not a sinusoid. Applying the corollary of Theorem IX, we see that there must be a direction of Borel $\arg z = \gamma$, such that $0 < \gamma \leqslant \frac{1}{2}\pi$, and also one such that $0 \leqslant \gamma < \frac{1}{2}\pi$. Hence we have, in fact, the extreme case considered in that theorem.

Proof. Let $f(z)$ be a canonical product whose zeros lie only on the real and imaginary axes, and let $r_1 = 1$, $r_n = n \log n$. Since $\Sigma\{(-1)^n/n \log n\}$ is convergent and $\Sigma(1/n \log n)$ is divergent, we can choose θ_n so that $S(r)$ oscillates in any prescribed manner. That is, given any simple continuous curve of finite length, $x = x(t)$, $y = y(t)$, $0 < t < 1$, we can make

$$S(r) = x(t_r) + iy(t_r) + \epsilon(r),$$

where $\epsilon(r) \to 0$ as $r \to \infty$ and t_r oscillates between 0 and 1 as slowly as we please. For instance, suppose that the prescribed curve is the real axis from $2^{-\frac{1}{2}}$ to 1, the quarter circle of radius $2^{-\frac{1}{2}}$ with centre at the origin from $2^{-\frac{1}{2}}$ to $-i2^{-\frac{1}{2}}$, and the negative imaginary axis from $-i2^{-\frac{1}{2}}$ to $-i$. We first put $\theta_1 = 0$, then $S(1) = 1$. We make the zeros from 2 to N negative, so that $0 < S(r_{N_1}) < 2^{-\frac{1}{2}} < S(r_{N_1}-1)$. Then, by putting

$$\theta_{N_1+1} = \theta_{N_1+2} = \ldots = \theta_{N_2} = \tfrac{1}{2}\pi,$$

we diminish $\arg S(r)$ and increase $|S(r)|$. We choose N_2 so that

$$|S(r_{N_2-1})| < 2^{-\frac{1}{2}} < |S(r_{N_2})|,$$

and then put $$\theta_{N_2+1} = \theta_{N_2+2} = \ldots = \theta_{N_3} = -\pi.$$

This diminishes the real part of $S(r)$, and therefore $|S(r)|$, so that $|S(r_{N_3})| < 2^{-\frac{1}{2}}$. In this way we make $S(r)$ oscillate from one side of the curve to the other, proceeding from 1 to $-i$ and back again. For, since $\Sigma 1/r_n$ is divergent, we can increase or diminish the real and imaginary parts of $S(r)$ by any desired amount by taking a sufficient number of zeros with the same argument. Also, since the last term of $S(r)$ tends to 0 as $r \to \infty$, the distance of $S(r)$ from the nearest point on the curve tends to 0 as $r \to \infty$.

With this choice of zeros we have

$$\mathfrak{R}\{S(r)e^{i\theta}\} < \max(\cos\theta, \sin\theta) + \epsilon_1 \tag{5.73}$$

for every $\epsilon_1 > 0$ and every $r > r_0(\epsilon_1)$, and also

$$\mathfrak{R}\{S(r)e^{i\theta}\} > \max(\cos\theta, \sin\theta) - \epsilon_1 \tag{5.74}$$

for every $\epsilon_1 > 0$ and a sequence of values of r tending to infinity. Further, since $|S(r)| > 2^{-\frac{1}{2}} - \epsilon_1$, and $-\frac{1}{2}\pi - \epsilon_1 < \arg S(r) < \epsilon_1$, for every $\epsilon_1 > 0$, $r > r_0(\epsilon_1)$, given any $\epsilon_2 > 0$, $\delta > 0$, we have

$$\mathfrak{R}\{S(r)e^{i\theta}\} > 2^{-\frac{1}{2}} \sin\delta - \epsilon_2 \tag{5.75}$$

for $\delta \leqslant \theta \leqslant \frac{1}{2}\pi - \delta$, $r > r_0(\epsilon_2, \delta)$.

It follows from (5.73) and (5.74), combined with (5.71) and (5.72), that $h(\theta) = \max(\cos\theta, \sin\theta)$; and it follows from (5.75), combined with the third part of Lemma 8, that $|f(re^{i\vartheta})|$ tends to infinity uniformly in the angle $\delta \leqslant \theta \leqslant \frac{1}{2}\pi - \delta$. Hence $f(z)$ has no line of Julia in the angle $0 < \arg z < \frac{1}{2}\pi$. Since $h(\theta) < 0$ for $-\pi < 0 < -\frac{1}{2}\pi$, $|f(re^{i\theta})| \to 0$ uniformly for $-\pi + \delta \leqslant \theta \leqslant -\frac{1}{2}\pi - \delta$, and so $f(z)$ has no line of Julia in the angle $-\pi < \arg z < \frac{1}{2}\pi$. Since $S(r)$ takes values near every point $2^{-\frac{1}{2}}e^{i\alpha}$, where $-\frac{1}{2}\pi \leqslant \alpha \leqslant 0$, infinitely often, we have

$$\log|f(re^{i\vartheta})| \sim 2^{-\frac{1}{2}} \cos(\theta + \alpha) r$$

in a set of r of density 1 for $-\frac{1}{2}\pi \leqslant \alpha \leqslant 0$. It is easy to show from this that there are two sequences of points

$$Z_1, Z_2, \ldots, Z_m = R_m e^{-i(\alpha + \frac{1}{2}\pi)}, \ldots; \lim_{m=\infty} R_m = \infty$$

at which the conditions of Theorem V are satisfied. Hence every line

$\arg z = \gamma$, such that $-\frac{1}{2}\pi \leqslant \gamma \leqslant 0$, $\frac{1}{2}\pi \leqslant \gamma \leqslant \pi$, is a direction of Borel of the maximum kind.

5.8. If we apply the above method as it stands to the case $h(\theta) \equiv A$ it fails. For we have to make $S(r) \sim Ae^{i\alpha(r)}$, where $\alpha(r)$ varies from 0 to 2π, in order to get the required $h(\theta)$. But then every line $\arg z = \alpha(r) \pm \frac{1}{2}\pi$ is a direction of Borel of the maximum kind, and that is every line from $\arg z = 0$ to $\arg z = 2\pi$, so that we get the same result as if $n(r) \sim Hr$.

THEOREM XIV. *There are integral functions $f(z)$ of order 1 and mean type for which*

$$h(\theta) = 1,$$

and yet the only lines of Julia are $\arg z = 0, \pm\frac{1}{2}\pi, \pi$, *which are directions of Borel without exceptional values. For these functions,*

$$\underline{\lim}_{r\to\infty} \frac{n(r, a)}{r} = \frac{2}{\pi}, \qquad \overline{\lim}_{r\to\infty} \frac{n(r, a)}{a} = \frac{2^{\frac{3}{2}}}{\pi}$$

for all a without exception.

Proof. We consider four canonical products f_1, f_2, f_3, f_4 of the same kind as before. That is, their zeros lie on the real and imaginary axes, and if $z_n^{(s)} = r_n^{(s)} e^{i\theta_n(s)}$ denotes the n-th zero of $f_s(z)$, we have $r_n^{(s)} = n \log n$. We make $S_1(r)$ oscillate about the arc of the circle with centre at the origin and radius 1, from $-\frac{1}{2}\pi$ to 0, and back again. Since $\Sigma \frac{(-1)^n}{n \log n}$ is convergent, we can make $S(r)$ traverse the curve as slowly as we please by inserting a number of zeros which are alternately positive and negative at various stages. We can also arrange so that $\arg S(r) = \alpha(r)$ lies strictly between $-\frac{1}{2}\pi$ and 0, and approaches its upper and lower limits as slowly as we please. Hence we have

$$-\tfrac{1}{2}\pi < -\tfrac{1}{2}\pi + \zeta(r) \leqslant \alpha(r) \leqslant \zeta(r) < 0,$$

where $\zeta(r) \to 0$ as slowly as we please.

Now put

$$e^{i\theta_n(2)} = e^{-i\theta_n(1)}, \quad e^{i\theta_n(3)} = e^{-i\{\pi+\theta_n(1)\}}, \quad e^{i\theta_n(4)} = e^{i\{\pi+\theta_n(1)\}},$$

so that

$$S_2(r) = |S_1(r)| e^{-i\alpha(r)},$$

$$S_3(r) = |S_1(r)| e^{i\{\pi-\alpha(r)\}},$$

$$S_4(r) = |S_1(r)| e^{-i\{\pi-\alpha(r)\}}.$$

Then $f(z)=f_1(z)+f_2(z)+f_3(z)+f_4(z)$ has $h(\theta)=1$, and is a function of the required type, provided that $\zeta(r)\to 0$ sufficiently slowly. If $\alpha(r)$ is not near $-\frac{1}{2}\pi$ or 0, it is easy to see from Lemma 8 that, given any $\delta>0$, we have

$$\log|f(re^{i\theta})|\sim\cos\{\theta+\alpha(r)\}r\to\infty \tag{5.81}$$

for $\delta\leqslant\theta\leqslant\frac{1}{2}\pi-\delta$, as $r\to\infty$ and corresponding inequalities hold for the other quadrants. The required results follow from this by well-known methods, provided that it holds when $\alpha(r)\to-\frac{1}{2}\pi$, and when $\alpha(r)\to 0$.

Given any $\epsilon_1>0$, $\epsilon>0$, $\delta>0$, we can choose $r_0=r_0(\epsilon_1,\epsilon,\delta)$ so that

$$\left||S_3(r)|-1\right|<\epsilon_1,\quad e^{-2\epsilon r_0}<\tfrac{1}{4},$$

and also (5.71) and (5.72) hold for $r>r_0$. Let $r_0(2)$ be the value for r_0 obtained by putting $\frac{1}{2}\epsilon_1$, $\frac{1}{2}\epsilon$, $\frac{1}{2}\delta$ instead of ϵ_1, ϵ, δ, $r_0(2^2)$ the value obtained by putting $\frac{1}{4}\epsilon_1$, $\frac{1}{4}\epsilon$, $\frac{1}{4}\delta$, and so on. Since $\zeta(r)\to 0$ slowly, we may suppose that, for $r_0\leqslant r\leqslant r_0(2)$,

$$\sin\{\zeta(r)\}\sin\delta>2\epsilon+\epsilon_1.$$

Then, for $\delta\leqslant\theta\leqslant\frac{1}{2}\pi-\delta$, $r_0\leqslant r\leqslant r_0(2)$, we have

$$\begin{aligned}\Re\{S_1(r)e^{i\theta}\}-\Re\{S_2(r)e^{i\theta}\}&>\cos\{\theta+\alpha(r)\}-\cos\{\theta-\alpha(r)\}-2\epsilon_1\\&=2\sin\theta\sin\alpha(r)-2\epsilon_1>2\sin\delta\sin\{\zeta(r)\}-2\epsilon_1>4\epsilon.\end{aligned}$$

Hence
$$\left|\frac{f_2(re^{i\theta})}{f_1(re^{i\theta})}\right|<e^{-2\epsilon r}<\tfrac{1}{4}$$

for $\delta\leqslant\theta\leqslant\frac{1}{2}\pi-\delta$, $r_0\leqslant r\leqslant r_0(\epsilon)$. Similarly $|f_3/f_1|<\frac{1}{4}$, $|f_4/f_1|<\frac{1}{4}$, and we have

$$\begin{aligned}\left|\log|f(re^{i\theta})|-\log|f_1(re^{i\theta})|\right|&=\left|\log\left|1+\frac{f_2+f_3+f_4}{f_1}\right|\right|\\&<\left|\log\left|1-|(f_2+f_3+f_4)/f_1|\right|\right|<\log 4.\end{aligned}\tag{5.82}$$

Repeating the process for the interval $r_0(2)\leqslant r\leqslant r_0(2^2)$ and so on, we see that (5.82) holds for all $r>r_0$; and so

$$\log|f(re^{i\theta})|\sim\log|f_1(re^{i\theta})|\sim\cos\{\theta+\alpha(r)\}r$$

for $\delta\leqslant\theta\leqslant\frac{1}{2}\pi-\delta$ for every $\delta>0$.

Since $\log|f(re^{i\theta})|\to\infty$ uniformly in the angle $\delta\leqslant\theta\leqslant\frac{1}{2}\pi-\delta$, there is no line of Julia in the angle, and the inequality (5.81) holds with $f(re^{i\theta})-a$ in place of $f(re^{i\theta})$. Hence $f(z)$ has no exceptional value. The lower limit for

$n(r, a)$ is obtained by making $\alpha(r) \to 0$, or $-\frac{1}{2}\pi$, the upper limit by making $\alpha(r) \to \frac{1}{4}\pi$.

5.9. As we have just seen that the points at which $h^+(\theta)$ is not a sinusoid do not always determine directions of Borel of $f(z)$, it may be as well to examine what V. Bernstein* means when he says that the points at which $h^+(\theta)$ is not a sinusoid, *in general*, determine directions of Borel of the maximum kind. His actual result may be stated as follows:

Suppose that $f(z)$ is regular and of proximate order $\rho(r)$ in an angle greater than π/ρ including $\arg z = \alpha$, and that $h(\theta)$ is not a sinusoid at $\theta = \alpha$. Let q be any real number, then $\arg z = \alpha$ is a direction of Borel of the maximum kind of

$$\psi(z) = f(z) + \Pi(z)\, e^{\{h(\alpha)+iq\}V(ze^{-i\alpha})},$$

where $\Pi(z)$ is an integral function of order $\rho(r)$ and minimum type†, except perhaps for a single exceptional value of $\Pi(z)$.

It should first be observed that the theorem only gives a property peculiar to points at which $h^+(\theta)$ is not a sinusoid if

$$q = -\frac{1}{\rho}\, h'(\alpha+) \quad \text{or} \quad q = -\frac{1}{\rho}\, h'(\alpha-), \tag{5.91}$$

where $h'(\alpha+)$, $h'(\alpha-)$ are the right and left-hand derivatives of $h(\theta)$ respectively at $\theta = \alpha$. For, if (5.91) does not hold, then $\arg z = \alpha$ is a direction of Borel for every $\psi(z)$ except one, even if $h^+(\theta)$ is a sinusoid. This may easily be seen as follows: since $h^+(\theta)$ is a sinusoid at $\theta = \alpha$, $h'(\alpha+) = h'(\alpha-) = h'(\alpha)$, and we may write

$$h^+(\theta) = h(\alpha)\cos(\theta-\alpha)\rho + \frac{1}{\rho}\, h'(\alpha)\sin(\theta-\alpha)\rho$$

for $\alpha-\delta \leqslant \theta \leqslant \alpha+\delta$, where $\delta > 0$. Since $q \neq -(1/\rho)h'(\alpha)$, it is easy to see that

$$h(\theta, \psi) = \overline{\lim_{r\to\infty}} \frac{\log|\psi(re^{i\theta})|}{V(r)} = \max\{h^+(\theta),\, h(\alpha)\cos(\theta-\alpha)\rho - q\sin(\theta-\alpha)\rho\}$$

for $\alpha-\delta \leqslant \theta \leqslant \alpha+\delta$. Hence $h(\theta, \psi)$ is not a sinusoid, and we can apply Bernstein's theorem. It follows that $\arg z = \alpha$ is a direction of Borel of $\psi(z)$, except for a single exceptional function $\Pi_0(z)$, which is the required result.

* V. Bernstein, *Annali di Mat.* (4), 12 (1934), 173–196, and *Comptes rendus*, 195 (1932), 356.

† Bernstein allows the function $\Pi(z)$ to be a little more general than this.

Further, Bernstein's result has very little bearing on the directions of Borel of $f(z)$ itself. For, if $q = -(1/\rho)h'(\alpha+)$, *the exceptional function occurs in his result sometimes when* $f(z)$ *has a direction of Borel of the maximum kind without an exceptional value.* For instance, if

$$f(z) = e^z + e^{-iz},$$

then $h(\theta) = \max(\cos\theta, \sin\theta)$, $V(z) = z$; and $h^+(\theta)$ is not a sinusoid at $\theta = \frac{1}{4}\pi$. Putting $q = -2^{-\frac{1}{2}}$, $\Pi(z) = -1$, we have $h(\frac{1}{4}\pi) + iq = e^{-i\pi/4}$, and so

$$\psi(z) = e^z,$$

which has no direction of Borel such that $\arg z = \frac{1}{4}\pi$. But $f(z)$ itself has a direction of Borel without an exceptional value.

6.1. Let $$f(z) = \Sigma a_n z^n$$

be an integral function of order 1 and mean type; then

$$\phi(z) = \Sigma n! a_n z^n$$

has a finite radius of convergence, and the Borel polygon of $\phi(z)$ is defined by*

$$r \leqslant \frac{1}{h^+(\theta)} \quad (z = re^{i\theta}), \tag{6.11}$$

where $h^+(\theta) = h(\theta)$ if $h(\theta) > 0$ and 0 otherwise. Hence the theorems about $h(\theta)$ can be interpreted as results connecting the directions of Borel of $f(z)$ with the Borel polygon of $\phi(z)$. Theorems of this kind have been proved by several people†, notably V. Bernstein and Valiron. Bernstein suggested that the vertices of the Borel polygon of $\phi(z)$ determined the lines of Julia of $f(z)$. If $h(\theta) > 0$ and is of the form $C\cos(\theta - \gamma)$ for $\alpha \leqslant \theta \leqslant \beta$, then the part of the boundary of the Borel polygon lying in that angle is the straight line $rC\cos(\theta - \gamma) = 1$. Hence, if $h(\theta) > 0$, the vertices of the Borel polygon correspond to the points at which $h(\theta)$ is not a sinusoid. If $h(\gamma) = 0$, but $h^+(\theta)$ is not a sinusoid at $\theta = \gamma$, it follows that $h(\theta) > 0$ either for $-\pi + \alpha \leqslant \theta < \alpha$, or for $\alpha < \theta < \pi + \alpha$. In this case, for obvious

* See E. Borel, *Leçons sur les séries divergentes* (Paris, 1928), 152.

† The idea of analogies between the singularities of power series and the lines of Julia of integral functions is due to A. Bloch, "Les fonctions holomorphes et méromorphes dans le cercle-unite", *Mémorial des sciences mathématiques*, 20 (1926), 16. See also G. Pólya, *Math. Zeitschrift*, 29 (1929), 550–640, for theorems chiefly about gap series. The explicit formulation in terms of the Borel polygon was initiated by V. Bernstein, *Comptes rendus*, 194 (1932), 350.

reasons, we say that the Borel polygon has a vertex at infinity in the direction $\theta = \gamma$, and Theorem IV gives the following result.

THEOREM XIV. *If the Borel polygon of $\phi(z)$ has a vertex at infinity in the direction $\theta = \gamma$, then $\arg z = \gamma$ is a direction of Borel of the maximum kind.*

Theorem X gives rise to the following result, which, though not very interesting from the point of view of the power series, is useful as a means of deducing other results.

THEOREM XVI. *Suppose that*

$$\lim_{r\to\infty} \frac{n(r)}{r} = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\theta)\,d\theta. \tag{6.12}$$

Then (1) *all the vertices of the Borel polygon of $\phi(z)$ determine directions of Borel of $f(z)$ of the maximum kind, and there are no other directions of Borel of the maximum kind*; (2) *if the vertex of the Borel polygon lies in the finite part of the plane, or if there are adjacent points on both sides of it in the finite part of the plane, then the corresponding direction of Borel has no exceptional value.*

Bernstein has also applied his theorem mentioned in § 5.9 to power series.

6.2. Bernstein* and Valiron† have shown that the vertices of the Borel polygon of $\phi(z)$ determine the directions of Borel of $f(z)$ when the *nature* of the singularities of $\phi(z)$ is somewhat severely restricted. They consider only singularities of finite order, but, if the singularities are *isolated*, the order has no special significance.

THEOREM XVII. *Suppose that the Borel polygon of $\phi(z)$ lies strictly inside the circle $|z| = A$, and that the singularities of $z\phi(z)$ for which $|z| < A$ are all isolated, then the directions of Borel of $f(z)$ of the maximum kind pass through the vertices of the Borel polygon of $\phi(z)$, and have no exceptional values.*

Proof. In this case the number of singularities inside $|z| = A$ is finite, and so the Borel polygon consists of a finite number of straight lines which meet at the vertices. The vertices are finite in number, and lie in the finite part of the plane. Hence $h(\theta) > 0$ for all θ.

We can write

$$z\phi(z) = \phi_1(z) + \phi_2(z) + \ldots + \phi_m(z) + \Phi(z),$$

where $\phi_m(z)$ is regular everywhere, including the point at infinity, except

* V. Bernstein, *Comptes rendus*, 194 (1932), 350, 1629.

† G. Valiron, *Comptes rendus*, 194 (1932), 1305.

at $z = z_m$, where $|z| < A$, and $\Phi(z)$ is regular for $|z| < A$. Then, by the well-known formula,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_C e^{-zw}\phi\left(\frac{1}{w}\right)\frac{dw}{w} \\ &= \frac{1}{2\pi i}\int_C e^{-zw}\left\{\phi_1\left(\frac{1}{w}\right)+\phi_2\left(\frac{1}{w}\right)+\ldots+\phi_m\left(\frac{1}{w}\right)+\Phi\left(\frac{1}{w}\right)\right\}dw \\ &= f_1(z)+f_2(z)+\ldots+f_m(z)+F(z), \end{aligned}$$

where C is a contour including all the singularities of $\phi(1/w)/w$. It is easy to show that $f_1, f_2, \ldots, f_n, F$ are integral functions and that

$$\log|F(re^{i\theta})| < \left(\frac{1}{A}+\epsilon\right)r \tag{6.21}$$

for every $\epsilon > 0$ and every $r > r_0(\epsilon)$. Also, putting $z_s = 1/w_s$, we have

$$f_s(z) = e^{-zw_s}\psi_s(z) \quad (s = 1, 2, \ldots, m), \tag{6.22}$$

where $\psi_s(z)$ is an integral function of order 1 and minimum type. It follows from Lemma* 1 that, given $\epsilon > 0$, $\eta > 0$, we can choose $R_0(\epsilon, \eta)$ so that

$$\log|\psi_s(re^{i\theta})| > -\epsilon r \tag{6.23}$$

for $0 < r < R$, $R > R_0$, except in a set of measure ηR.

Suppose that w_m is a singularity which determines the side of the Borel polygon which stretches from the vertex on $\theta = \alpha_m$ to the vertex on $\theta = \beta_m$. Then it follows from (6.11) and the construction of the Borel polygon that

$$\mathfrak{R}(-w_s e^{i\theta}) < \mathfrak{R}(-w_m e^{i\theta}) = h(\theta),$$

where $s = 1, 2, \ldots, m-1$, for $\alpha_m < \theta < \beta_m$. Hence, given $\delta > 0$, we can find $\zeta > 0$ such that

$$\begin{cases} h(\theta)-\mathfrak{R}(-w_s e^{i\theta}) > \zeta \quad (s = 1, 2, \ldots, m-1) \\ h(\theta)-1/A > \zeta \end{cases} \tag{6.24}$$

for $\alpha_m+\delta \leqslant \theta \leqslant \beta_m-\delta$. It follows from (6.22), (6.23), (6.24) that, given $\delta > 0$, $\eta > 0$, we can choose $R_0(\delta, \eta)$ so that

$$\begin{aligned} \log|f_s(re^{i\theta})|-\log|f_m(re^{i\theta})| &< \{\mathfrak{R}(-w_s e^{i\theta})-h(\theta)+2\epsilon\}r \\ &< (-\zeta+2\epsilon)r < -\tfrac{1}{2}\zeta r < \log 1/(m+1), \end{aligned}$$

* See also M. L. Cartwright, *Proc. London Math. Soc.* (2), 33 (1931), 212.

and similarly

$$\log|F(re^{i\theta})| - \log|f_m(re^{i\theta})| < (-\zeta+2\epsilon)r < -\tfrac{1}{2}\zeta r < \log 1/(m+1)$$

for $\alpha_m+\delta \leqslant \theta \leqslant \beta_m-\delta$, $0<r<R$, $R>R_0(\delta,\eta)$, except in a set of measure ηR. Hence

$$\left|\log|f(re^{i\theta})| - \log|f_m(re^{i\theta})|\right| = \left|\log\left|1+\frac{f_1+f_2+\dots+f_{m-1}+F}{f_m}\right|\right|$$

$$< \log(m+1)$$

for $\alpha_n+\delta \leqslant \theta \leqslant \beta_m-\delta$, $0<r<R$, except in a set of measure $\eta m R$. Repeating this argument for each side of the polygon, we have

$$\left|\log|f(re^{i\theta})| - h(\theta)\,r\right| < \log(m+1) < \epsilon r$$

for $0<r<R$, except perhaps in a set of r of measure $\eta m R$, and a set of θ of measure δm, provided that $R>R_0(\epsilon,\delta,\eta)$. From this it is easy to deduce (6.12) by means of Jensen's formula and Lemma 1, and we have the required result.

6.3. Theorem IX of this paper gives rise to another kind of result.

THEOREM XVIII. *Suppose that the part of the boundary of the Borel polygon of $\phi(z)$ lying between $\theta=0$ and $\theta=\alpha$ is a straight line parallel to $\theta=\gamma$, where $0<\alpha\leqslant\gamma<\pi$. Then, either $\arg z=\gamma$ is a direction of Borel of $f(z)$ of the maximum kind, or else $\alpha<\gamma$, and there is a direction of Borel of the maximum kind without an exceptional value in the angle $\alpha\leqslant\arg z<\gamma$.*

If we make $\alpha\to 0$, the theorem seems to suggest that, *if the tangent on the right to the Borel polygon at $\theta=0$ is parallel to $\theta=\gamma$, where $0<\gamma<\pi$, then there is a direction of Borel in the angle*

$$0<\arg z\leqslant\gamma;$$

but I am unable to prove this.

6.4. The results of §5.7 and §5.8 provide examples to illustrate these results about the Borel polygon. For, corresponding to the integral functions considered in Theorems XIII and XIV, we can find power series with the appropriate Borel polygon by means of the formula

$$\phi(z)=\int_0^\infty e^{-w}f(wz)\,dw.$$

THEOREM XIX. *There are functions $\phi(z)$ whose Borel polygon consists of the straight lines $r\cos\theta = 1$, $r\sin\theta = 1$ such that $f(z)$ has no line of Julia in the angle*

$$0 < \arg z < \tfrac{1}{2}\pi.$$

THEOREM XX. *There are functions $\phi(z)$ for which the unit circle is a natural boundary, and yet $f(z)$ has only four lines of Julia* $\arg = 0, \pm\tfrac{1}{2}\pi, \pi$.

It would follow from the theorem stated at the end of § 6.3 that $f(z)$ must have at least four directions of Borel. If the series for an integral function has sufficiently large gaps, it has been shown* that every line $\arg z = \gamma$, such that $0 \leqslant \gamma < 2\pi$, is a line of Julia. But it follows from Theorem XX that any attempt to prove such theorems for integral functions of order 1 and mean type by means of the Borel polygon and gap theorems for $\phi(z)$ is doomed to failure.

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* See G. Pólya, *Math. Zeitschrift*, 29 (1929), 639 and 640, where numerous references will be found.

ON WIENER'S METHOD IN TAUBERIAN THEOREMS

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1. The application of the Cahen-Mellin integral to the Dirichlet's series

$$f(s) = \sum_1^\infty a_n e^{-\lambda_n s} = \sum_1^\infty a_n l_n^{-s} \qquad (l_n = e^{\lambda_n}) \tag{1}$$

$$(\lambda_1 < \lambda_2 < \ldots, \lambda_n \to \infty)$$

by Hardy and Littlewood has suggested an analogy between two types of Tauberian theorem†. We have, on the one hand, theorems of the "Abel-Tauber" type‡ involving hypotheses concerning the behaviour of $f(s)$ when $s \to 0$ by real positive values, and, on the other, theorems involving hypotheses concerning the behaviour of $f(s)$ in the neighbourhood of a whole line $\sigma = \alpha$ of the plane of the complex variable $s = \sigma + ti$. Theorems of the first type contain (or may contain) hypotheses relating the coefficients a_n to the sequence (λ_n) and conclusions relating the partial sums

$$A_\lambda(\omega) = \sum_{\lambda_n \leqslant \omega} a_n = \sum_{l_n \leqslant w} a_n = A_l(w) \qquad (w = e^\omega) \tag{2}$$

(for large ω) to positive or zero powers of ω. The analogy referred to consists in associating with such a theorem a theorem of the second type in which (λ_n) is replaced by (l_n) in the hypothesis concerning a_n, and a power of ω by a power of w in the conclusion.

In their original form the theorems of the second type contained additional hypotheses restricting the order of magnitude of $f(s)$ for large $|t|$§, but the general theory of Tauberian theorems developed by Wiener has

† Hardy and Littlewood, 3, Lemma 2.113, Theorems 2.12, 2.121, 2.13, 2.14, 2.15. (The references in heavy type are to the list at the end of this paper.)

‡ For an explanation of the terms "Abel-Tauber", "special", "general", etc., see Hardy and Littlewood, 1, § 2.

§ Landau, 1, (ii), 874–877; Hardy and Littlewood, 3, Theorems 2.121–2.15.

shown that such restrictions are entirely unnecessary†. Wiener's theory in its most general form is difficult, but, owing to the special nature of the "kernels" involved, his method may be applied to the particular case of Dirichlet's series without explicit appeal to the general theory, or to the theory of Fourier transforms on which it is founded. This has been done by Bochner, and further simplifications have been effected by Heilbronn and Landau, and by Landau‡. Heilbronn and Landau have shown, moreover, that Tauberian theorems of the second type are capable of a remarkable kind of refinement. This consists in confining the hypothesis concerning $f(s)$ to a finite portion $-T \leqslant t \leqslant T$ of the line $\sigma = \alpha$, and replacing the conclusion that a certain function tends to a limit by the assertion that the limits of indetermination of this function lie between certain bounds; these bounds may be expressed in terms of functions not depending on the particular series under consideration, and they vary with T in such a way as to give a non-trivial result for every $T > 0$, and to yield the original theorem on making $T \to \infty$, if the conditions are satisfied for every T.

These researches have added a new interest and importance to Tauberian theorems of the second type, and it now seems appropriate to follow out more systematically the analogy between these theorems and those of the Abel-Tauber type.

As fundamental theorems of the Abel-Tauber theory ("λ-theorems") we select the following. We use the notations (1) and (2) throughout the paper. It is to be understood always that $f(s)$ is defined in the first instance only in the half-plane explicitly mentioned, say $\sigma > \alpha$. We reserve the right to define $f(s)$, and associated functions, as we please at points on the line $\sigma = \alpha$; and, whenever we make an assertion involving a reference to such a function at points on this line, we mean that values may be assigned in such a way as to make the assertion true. When we speak of the existence of a limit, we always mean a finite limit, unless the contrary is stated.

Theorem 1 (λ). *Suppose that, for some* θ,

$$\lim_{\omega \to \infty} \omega^{-\theta} \sum_{0 < \lambda_n \leqslant \omega} \lambda_n^{\theta} a_n = 0,$$

so that (1) *is certainly convergent for* $\sigma > 0$. *Suppose further that, when* $s \to 0$ *by real positive values,* $f(s)$ *tends to a limit* $f(0)$. *Then* (1) *converges for*

† For details and references see Wiener, 1, ch. II, III.

‡ Bochner, 1; Heilbronn and Landau, 1, 2; Landau, 2.

$s=0$ and its sum is $f(0)$, that is to say

$$\lim_{\omega\to\infty}\{A_\lambda(\omega)-f(0)\}=0.$$

Theorem 2 (λ). *Suppose that $a_n\geqslant 0$ for $n\geqslant n_0$, that* (1) *is convergent for $\sigma>0$, and that, for an $\alpha>0$, $s^\alpha f(s)$ tends to a limit a as $s\to 0$ by real positive values. Then*

$$\lim_{\omega\to\infty}\omega^{-\alpha}A_\lambda(\omega)=\frac{a}{\Gamma(\alpha+1)}.$$

Theorem 3 (λ). *Suppose that one of the following sets of conditions,* (i) *or* (ii), *is satisfied (K being a number independent of n):*

(i) $$|a_n|\leqslant K\frac{\lambda_n-\lambda_{n-1}}{\lambda_n}\quad(n\geqslant n_0),$$

so that (1) *is certainly convergent for $\sigma>0$;*

(ii) *a_n is real,*

$$a_n\geqslant -K\frac{\lambda_n-\lambda_{n-1}}{\lambda_n}\quad(n\geqslant n_0),\qquad \varliminf_{n\to\infty}a_n\geqslant 0,$$

and (1) *is convergent for $\sigma>0$.*

Suppose further that, when $s\to 0$ by real positive values, $f(s)$ tends to a limit $f(0)$. Then (1) *is convergent for $s=0$, that is to say*

$$\lim_{\omega\to\infty}\{A_\lambda(\omega)-f(0)\}=0.$$

The special case $\theta=1$ of Theorem 1 (λ) is the familiar extension, due to Schnee, of Tauber's "converse of Abel's theorem on power series"†. It is convenient for our purpose to state the theorem for general (complex) θ; the extension presents no difficulty. Theorem 3 (λ) (i) is Littlewood's famous "converse of Abel's theorem"‡, and Theorems 2 (λ) and 3 (λ) (ii) are due to Hardy and Littlewood§. Theorem 3 (λ) is usually deduced from Theorem 2 (λ) with the help of the special case $\theta=1$ of Theorem 1 (λ).

The "l-analogues" of these theorems, with the Heilbronn-Landau refinement, are as follows. In all these theorems T must be strictly positive, so that the interval $(-T, T)$ does not reduce to a point.

† See Hardy and Riesz, 1, ch. VI, § 7.

‡ Littlewood, 1. Theorem 3 (λ) is a "general", Theorem 1 (λ) a "special", Tauberian theorem.

§ Hardy and Littlewood, 2, Theorems D, F. With regard to the condition $\lambda_{n+1}/\lambda_n\to 1$, compare the remarks in § 7 below.

Theorem 1 (l). *Suppose that, for some* θ,

$$\overline{\lim_{w\to\infty}}\left| w^{-\theta}\sum_{l_n\leqslant w} l_n{}^{\theta} a_n\right| = \Theta \tag{3}$$

is finite, so that (1) *is certainly convergent for* $\sigma > 0$. *Suppose further that*

$$\frac{f(s)-f(0)}{s}$$

is continuous† *for* $\sigma \geqslant 0$, $-T \leqslant t \leqslant T$. *Then*

$$\overline{\lim_{w\to\infty}} |A_l(w)-f(0)| \leqslant 2\left(1+3\frac{|\theta|}{T}\right)\Theta.$$

Theorem 2 (l). *Suppose that* $a_n \geqslant 0$ *for* $n \geqslant n_0$, *that* (1) *is convergent for* $\sigma > \alpha > 0$, *and that, for an appropriate* α,

$$f(s)-\frac{\alpha}{s-\alpha}$$

is continuous for $\sigma \geqslant \alpha$, $-T \leqslant t \leqslant T$. *Then*

$$\mathfrak{p}\left(\frac{T}{\alpha}\right)\frac{a}{\alpha} \leqslant \overline{\lim_{w\to\infty}}\, w^{-\alpha} A_l(w) \leqslant \mathfrak{P}\left(\frac{T}{\alpha}\right)\frac{a}{\alpha}, \tag{4}$$

where $\mathfrak{p}(\tau)$ *and* $\mathfrak{P}(\tau)$ *are functions of* τ *only, which are defined for every* $\tau > 0$, *and are such that*

$$0 < \mathfrak{p}(\tau) < 1 < \mathfrak{P}(\tau) \quad (\tau > 0),$$

$$\mathfrak{p}(\tau) \to 1, \quad \mathfrak{P}(\tau) \to 1 \quad as \quad \tau \to \infty.$$

Theorem 3 (l). *Suppose that one of the following sets of conditions,* (i) *or* (ii), *is satisfied* (K *being a positive number independent of* n):

(i) $$|a_n| \leqslant K\frac{l_n-l_{n-1}}{l_n} \quad (n \geqslant n_0),$$

so that (1) *is certainly convergent for* $\sigma > 0$‡;

(ii) a_n *is real,*

$$a_n \geqslant -K\frac{l_n-l_{n-1}}{l_n} \quad (n \geqslant n_0), \qquad \underline{\lim_{n\to\infty}}\, a_n \geqslant 0,$$

and (1) *is convergent for* $\sigma > 0$.

† Compare the remarks immediately preceding the enunciation of Theorem 1 (λ). The conditions obviously allow no ambiguity in the determination of $f(0)$, and $f(0)$ is the value of $f(s)$ at the point $s = 0$ in the ordinary sense whenever $f(s)$ is regular there.

‡ Because (3) then holds with $\theta = 1$, $\Theta = K$. That (3) implies the convergence of (1) for $\sigma > 0$ follows, of course, by partial summation.

Suppose further that

$$\frac{f(s)-f(0)}{s}$$

is continuous for $\sigma \geqslant 0$, $-T \leqslant t \leqslant T$. *Then*

$$\overline{\lim_{w\to\infty}} \, |A_l(w)-f(0)| \leqslant \mathfrak{q}(T)\, K,$$

where $\mathfrak{q}(T)$ *is a function of* T *only, which is defined for every* $T > 0$ *and tends to* 0 *when* $T \to \infty$.

Along with Theorem 3 (l) it is convenient to consider the following theorem, which may be regarded as a parallel theorem in that the two are easily derived from a common source†.

Theorem $3_1(l)$. *Suppose that* $a_n \geqslant 0$ *for* $n \geqslant n_0$, *that* (1) *is convergent for* $\sigma > 0$, *and that, with appropriate* a *and* a',

$$\frac{1}{s}\left(f(s)-\frac{a}{s}-a'\right)$$

is continuous for $\sigma \geqslant 0$, $-T \leqslant t \leqslant T$. *Then*

$$\overline{\lim_{w\to\infty}} \, |A_l(w)-a\log w-a'| \leqslant \mathfrak{q}_1(T)\, a,$$

where $\mathfrak{q}_1(T)$ *is a function of* T *only, which is defined for every* $T > 0$ *and tends to* 0 *when* $T \to \infty$.

Theorem 2 (l) is a known theorem due to Heilbronn and Landau‡; it is a generalization of Ikehara's theorem§, to which it reduces when $T \to \infty$. Theorem $3_1(l)$ is a sharpened form of two further theorems of Heilbronn and Landau, corresponding respectively to "$T \to \infty$" and "T fixed" in Theorem $3_1(l)$‖. Of these two theorems the former was deduced from Theorem 2 (l) with the aid of M. Riesz's well-known generalization of a theorem of Fatou, and the latter from a part of Theorem 2 (l) with the aid of a similar (but simpler) theorem of Riesz. But the theorem for fixed T, as presented by Heilbronn and Landau, lacked the precision of their Theorem 2 (l), or of our Theorem $3_1(l)$, because Riesz's theorem was not immediately available in a suitable form. The present paper is the outcome of an attempt to introduce greater precision and unity into

† For a regular and not too rapidly increasing sequence l_n, such as $l_n = n$, either may be deduced directly from the other.

‡ Heilbronn and Landau, 1; Landau, 2.

§ Wiener, 1, 127–130.

‖ Heilbronn and Landau, 2.

the theory at this point by formulating an appropriate version of Riesz's theorem and proving it by Wiener's method. The result is Theorem 1 (l)†. The use of this theorem is analogous to the use of Theorem 1 (λ) in deducing Theorem 3 (λ) from Theorem 2 (λ), and the analogy suggests a proof on the desired lines. Theorem 3 (l) emerges naturally as the analogue of Theorem 3 (λ); it is a generalization of a theorem proved by Hardy and Littlewood subject to an "order" condition on $f(s)$‡.

The theorem of Fatou to which we have referred is a theorem on power series, but its immediate extension to Dirichlet's series is as follows: *If* $a_n = o(\lambda_n - \lambda_{n-1})$ *and* $a_n = o(1)$, *so that* (1) *is certainly convergent for* $\sigma > 0$, *and if* $f(s)$ *is regular for* $s = 0$, *then* (1) *is convergent for* $s = 0$ *and its sum is* $f(0)$. This is included in Theorem 1 (l)§. It is also included in Theorem 3 (l), for the conditions imply those of Theorem 3 (l) (ii) with an arbitrarily small positive K and some positive T.

By way of comparing λ-theorems and l-theorems we may note that, when $\lambda_n - \lambda_{n-1}$ is bounded (as in the most important cases $\lambda_n = n$ and $\lambda_n = \log n$), $(l_n - l_{n-1})/l_n$ is of the same order as $\lambda_n - \lambda_{n-1}$, so that the restrictions on a_n are lighter in Theorem 3(l) than in Theorem 3 (λ); to compensate for this a heavier restriction is imposed on $f(s)$.

The fundamental theorems, from which the theorems (l) may be deduced, are best stated, for the sake of generality and formal convenience, in terms of Stieltjes integrals of the type

$$f(s) = \int_0^\infty e^{-us}\, dA(u), \quad A(0) = 0. \tag{5}$$

We suppose always that $A(u)$ is of bounded variation in any finite interval $(0, \omega)$. With regard to the definition of $f(s)$, we adopt the same conventions as in the case of the series (1).

Theorem I. *Suppose that, for some* θ,

$$\overline{\lim_{\omega \to \infty}} \left| e^{-\omega\theta} \int_0^\omega e^{u\theta}\, dA(u) \right| = \Theta$$

† It yields the most familiar form of Riesz's theorem on taking $\Theta = 0$, making a change of origin, and noting that the hypothesis concerning $f(s)$ is certainly satisfied for some $T > 0$ if $f(s)$ is regular at $s = 0$. See Hardy and Riesz, 1, Theorem 37; Riesz, 1, 2 (Satz I and I'), 3. Riesz considers uniform convergence along a segment of a line, and (in 3) summation by typical means; he also relaxes the condition on $f(s)$. We shall not be concerned with these extensions here.

‡ Hardy and Littlewood, 3, Theorems 2.13, 2.15. With regard to Theorems 2.121, 2.14, see § 4 below.

§ With $\theta = 1$, $\Theta = 0$. *Cf.* Riesz, 2, 359–360.

is finite, so that (5) *is certainly convergent for* $\sigma > 0$. *Suppose further that* $f(s)$ *satisfies the same conditions as in Theorem* 1 (*l*). *Then*

$$\overline{\lim_{\omega\to\infty}}\,|A(\omega)-f(0)| \leqslant 2\left(1+3\,\frac{|\theta|}{T}\right)\Theta.$$

Theorem II. *Suppose that* $A(u)$ *increases (in the wide sense) with* u *for* $u \geqslant u_0$, *that* (5) *is convergent for* $\sigma > \alpha > 0$, *and that* $f(s)$ *satisfies the same conditions as in Theorem* 2 (*l*). *Then*

$$\mathfrak{p}\left(\frac{T}{\alpha}\right)\frac{a}{\alpha} \leqslant \overline{\lim_{\omega\to\infty}}\, e^{-\omega\alpha}A(\omega) \leqslant \mathfrak{P}\left(\frac{T}{\alpha}\right)\frac{a}{\alpha}, \tag{6}$$

where $\mathfrak{p}(\tau)$ *and* $\mathfrak{P}(\tau)$ *are as in Theorem* 2 (*l*).

Theorem III. *Suppose that* $A(u)$ *is real, that there is a function* $\mathrm{H}(\delta)$ *such that, for every* $\delta > 0$,

$$A(\omega')-A(\omega) \geqslant -\mathrm{H}(\delta) \quad [\omega \leqslant \omega' \leqslant \omega+\delta,\ \omega \geqslant \omega_0(\delta)], \tag{7}$$

that (5) *is convergent for* $\sigma > 0$, *and that* $f(s)$ *satisfies the same conditions as in Theorem* 3 (*l*). *Then*

$$\overline{\lim_{\omega\to\infty}}\,|A(\omega)-f(0)| \leqslant \mathfrak{C}\,\mathrm{H}(T^{-1}), \tag{8}$$

where $\mathfrak{C}$ *is an absolute constant.*

If, further, there exist a sequence (ω_n) *tending to infinity, and a function* $\eta(\delta)$, *such that, for every* $\delta > 0$,

$$|A(\omega')-A(\omega_n)| \leqslant \eta(\delta) \quad [\omega_n \leqslant \omega' \leqslant \omega_n+\delta,\ n \geqslant n_0(\delta)],$$

then

$$\overline{\lim_{n\to\infty}}\,|A(\omega_n)-f(0)| \leqslant \mathfrak{C}T^{-\frac{1}{2}}\mathrm{H}(T^{-\frac{1}{2}})+\eta(T^{-\frac{1}{2}}). \tag{9}$$

It will be clear from the proofs of these theorems that the continuity hypotheses concerning the behaviour of $f(s)$ in the neighbourhood of the line $\sigma = 0$ or $\sigma = \alpha$ are needlessly restrictive, and may be replaced by wider conditions of the type familiar in the theory of functions of a real variable†.

The analogy between λ-theorems and l-theorems suggests a further theorem. It is well known that the rate of increase of λ_n (though not alluded to in our enunciations) has an important bearing on λ-theorems. In particular, if λ_n increases with such rapidity and regularity that $\lim \lambda_n/\lambda_{n-1} > 1$, then Theorem 3 ($\lambda$) is true without any explicit restriction on a_n; this is the "high-indices theorem" of Hardy and Littlewood‡.

† See Riesz, **3**, Satz II, for conditions appropriate to Theorems I and 1 (*l*).

‡ Hardy and Littlewood, **4**.

The natural l-analogue of this is: *If* $\underline{\lim} l_n/l_{n-1} > 1$, *and if the hypothesis concerning* $f(s)$ *in Theorem* 3 (l) *is satisfied for every* T *(however large), then the conclusion of the theorem, namely that* Σa_n *converges to the sum* $f(0)$, *is valid without any explicit restriction on* a_n. This is in fact true, but it does not tell the whole truth, and is not in itself a particularly interesting result. We reserve the discussion of this theorem, and its relation to other theorems in the theory of Dirichlet's series, for another occasion.

We shall set out the proofs of Theorems I, II, III in § 2. A proof of II is included for the sake of completeness, though it is a direct reproduction in our notation of the final and very simple proof of Theorem 2 (l) obtained by Landau. We deduce Theorems 1 (l), 2 (l), 3 (l), 3_1 (l) in § 3. In § 4 we discuss a generalization of Theorem 2 (l). The concluding sections, §§ 5, 6, 7, are devoted to miscellaneous observations†.

2. *Proof of Theorems* I, II, III. We begin with a lemma, in which we set out the relevant properties of the kernels which will be employed. Throughout the paper we use German letters to represent functions of the variables explicitly indicated (in brackets or as suffixes) *and of no others*; a German letter with no variables attached denotes an *absolute constant*.

Lemma. (I) *There exists a* $\mathfrak{k}(x)$, *even and continuous for* $-1 \leqslant x \leqslant 1$, *such that, if*

$$\mathfrak{K}(y) = \int_{-1}^{1} \mathfrak{k}(x)\, e^{-yxi}\, dx = 2\int_{0}^{1} \mathfrak{k}(x) \cos yx\, dx,$$

then $\mathfrak{K}(y) \geqslant 0$ *for all real* y *(with equality only at isolated points), and*

$$\mathfrak{K} = \int_{-\infty}^{\infty} \mathfrak{K}(y)\, dy \quad \text{is convergent.} \tag{10}$$

(II) *There exists a* $\mathfrak{k}(x)$ *with the properties enumerated under* (I), *and such that, in addition,*

$$\mathfrak{K}' = \int_{-\infty}^{\infty} |y|\, \mathfrak{K}(y)\, dy \quad \text{is convergent,} \tag{11}$$

and

$$\mathfrak{K}' < 6\mathfrak{K}. \tag{12}$$

† [*Added in proof* (*December*, 1934).]—Our main results for integrals have been anticipated by Karamata in a paper published during the printing of this, but completed before it (Karamata, 1). Thus, apart from the conditions on $f(s)$, his Satz B' is substantially equivalent to the case $\theta = 1$ of our Theorem I, and the theorem stated at the end of his paper is practically our Theorem III. Karamata does not, however, discuss what is the main theme of this paper—the analogy between λ- and l-theorems.

For (I) it is sufficient to take $\mathfrak{f}(x) = 1-|x|$; this gives the familiar Fejér kernel

$$\mathfrak{K}(y) = \frac{2(1-\cos y)}{y^2} = \left(\frac{\sin \frac{1}{2}y}{\frac{1}{2}y}\right)^2$$

(to be defined as 1 when $y = 0$).

For (II) we start from the formulae

$$\sin^4 y = \tfrac{3}{8} - \tfrac{1}{2}\cos 2y + \tfrac{1}{8}\cos 4y, \quad \left(\frac{d}{dy}\right)^4 \sin^4 y = -8\cos 2y + 32\cos 4y.$$

Integrating the last equation four times from 0 to y (noting that $\sin^4 y$ and its first three derivatives vanish for $y = 0$), and expressing the right-hand side as a single integral, we obtain

$$\sin^4 y = \frac{1}{3!}\int_0^y (y-u)^3 (-8\cos 2u + 32\cos 4u)\, du,$$

whence, splitting up the last integral and changing the variables,

$$\frac{\sin^4 y}{y^4} = -\tfrac{8}{3}\int_0^{\frac{1}{2}} (1-2x)^3 \cos 4yx\, dx + \tfrac{16}{3}\int_0^1 (1-x)^3 \cos 4yx\, dx.$$

Thus, if we take

$$\mathfrak{f}(x) = \begin{cases} -\frac{4}{3}(1-2|x|)^3 + \frac{8}{3}(1-|x|)^3 & (|x| \leqslant \frac{1}{2}) \\ \frac{8}{3}(1-|x|)^3 & (\frac{1}{2} \leqslant |x| \leqslant 1), \end{cases}$$

we have

$$\mathfrak{K}(y) = \left(\frac{\sin \frac{1}{4}y}{\frac{1}{4}y}\right)^4,$$

and it remains only to verify (12). Now

$$\tfrac{1}{2}\mathfrak{K}' = 16\int_0^\infty y\left(\frac{\sin y}{y}\right)^4 dy < 16\int_0^\infty \frac{\sin^2 y}{y^2}\, dy = 16\mathfrak{I},$$

say, and

$$\tfrac{1}{2}\mathfrak{K} = 4\int_0^\infty \frac{\sin^4 y}{y^4}\, dy = \tfrac{2}{3}\int_0^\infty \frac{1}{y^2}\frac{d^2}{dy^2}(\sin^4 y)\, dy$$

$$= \tfrac{4}{3}\int_0^\infty \frac{\cos 2y - \cos 4y}{y^2}\, dy = \tfrac{8}{3}\int_0^\infty \frac{\sin^2 2y}{y^2}\, dy - \tfrac{8}{3}\int_0^\infty \frac{\sin^2 y}{y^2}\, dy = \tfrac{8}{3}\mathfrak{I},$$

whence the result.

We next obtain a general formula. Suppose that (5) is convergent for $\sigma > \alpha \geqslant 0$. Then, by partial integration, $A(u) = O(e^{u\alpha'})$ for every $\alpha' > \alpha$,

and [since $A(0)=0$]

$$\frac{f(s)}{s}=\int_0^\infty A(u)\,e^{-us}\,du=\int_0^\infty A(u)\,e^{-u\sigma}\,e^{-uti}\,du \quad (\sigma>\alpha). \tag{13}$$

For any fixed $\sigma>\alpha$ the last integral is absolutely and uniformly convergent for all real t. Now let $\mathfrak{f}(x)$ be a function satisfying the conditions of either part of the lemma, and write

$$\mathfrak{f}_T(t)=\mathfrak{f}\left(\frac{t}{T}\right), \quad \mathfrak{K}_T(y)=\int_{-T}^{T}\mathfrak{f}_T(t)\,e^{-yti}\,dt=T\mathfrak{K}(yT).$$

Multiply (13) by $\mathfrak{f}_T(t)\,e^{\omega ti}$ (ω real) and integrate from $t=-T$ to $t=T$; the integration on the right may be performed under the integral sign, and we obtain

$$\int_{-T}^{T}\frac{f(\sigma+ti)}{\sigma+ti}\,\mathfrak{f}_T(t)\,e^{\omega ti}\,dt=\int_0^\infty A(u)\,e^{-u\sigma}\,\mathfrak{K}_T(u-\omega)\,du \quad (\sigma>\alpha).$$

Subtracting from this the corresponding formula for the special case

$$A(0)=0, \quad A(u)=Ae^{u\alpha} \ (u>0), \quad \frac{f(s)}{s}=\frac{A}{s-\alpha},$$

where A is a constant, we obtain

$$\int_{-T}^{T}\phi(\sigma+ti)\,\mathfrak{f}_T(t)\,e^{\omega ti}\,dt=\int_0^\infty R(u)\,\mathfrak{K}_T(u-\omega)\,e^{-u(\sigma-\alpha)}\,du \quad (\sigma>\alpha), \tag{14}$$

where

$$\phi(s)=\frac{f(s)}{s}-\frac{A}{s-\alpha}, \quad R(u)=A(u)\,e^{-u\alpha}-A.$$

This is the desired formula†. From this point we consider the three theorems separately.

Theorem I. We take the second form (II) of $\mathfrak{f}(x)$, and use (14) with $\alpha=0$, $A=f(0)$. Then $\phi(s)$ is continuous for $\sigma\geqslant 0$, $-T\leqslant t\leqslant T$. Now, writing

$$B(\omega)=\int_0^\omega e^{u\vartheta}\,dA(u),$$

† It may be regarded as an instance of "Parseval's theorem" in the trivial form

$$\int_{-\infty}^{\infty}f(u)\,g^*(u)\,du=\int_{-\infty}^{\infty}f^*(v)\,g(v)\,dv\left(=C\int\int_{-\infty}^{\infty}f(u)\,g(v)e^{-uvi}\,du\,dv\right),$$

where f, g belong to the Lebesgue class $L(-\infty,\infty)$ and f^*, g^* are their Fourier transforms; for $\phi(\sigma+ti)$ is (substantially) the transform of $R(u)e^{-u(\sigma-\alpha)}$ (defined to be 0 when $u<0$), and $\mathfrak{K}_T(u-\omega)$ that of $\mathfrak{f}_T(t)e^{\omega ti}$ (defined to be 0 when $|t|>T$), the variables being t and u. The classical methods of estimating partial sums of Dirichlet's series by means of Perron's formulae (Hardy and Riesz, 1, Theorems 13, 39, 40) correspond to the use of formulae similar to (14), but with $R(u)$, instead of $\psi(\sigma+ti)$, multiplied by a function which picks out a finite range of values of the variable.

we have (since $\alpha = 0$)

$$R(\omega')-R(\omega) = \int_{\omega}^{\omega'} dA(u) = \int_{\omega}^{\omega'} e^{-u\theta}\, dB(u)$$

$$= \left[e^{-u\theta} B(u) \right]_{\omega}^{\omega'} + \int_{\omega}^{\omega'} \theta e^{-u\theta} B(u)\, du.$$

Hence, if Θ_1 is any number greater than Θ,

$$|R(\omega')-R(\omega)| < 2\Theta_1 + |\theta(\omega'-\omega)|\,\Theta_1 \quad [\omega,\ \omega' > \omega_0 = \omega_0(\Theta_1)]. \tag{15}$$

We deduce in particular that

$$R(u) = O(u) \quad \text{as} \quad u \to \infty. \tag{16}$$

Now let $\sigma \to +0$ (*i.e.* $\alpha + 0$) in (14). The limits may be taken under the integral signs, and we obtain

$$\int_{-T}^{T} \phi(ti)\, \mathfrak{k}_T(t)\, e^{\omega t i}\, dt = \int_0^{\infty} R(u)\, \mathfrak{K}_T(u-\omega)\, du = \int_{-T\omega}^{\infty} R\left(\omega + \frac{y}{T}\right) \mathfrak{K}(y)\, dy. \tag{17}$$

In fact, the process is legitimate for the integral on the left by the continuity of $\phi(s)$, and for the integral on the right because the third, and therefore the second, integral in (17) is (absolutely) convergent by (16) and (11).

Now let $\omega \to \infty$ in (17). The first integral tends to 0 by the Riemann-Lebesgue theorem. Hence the third also tends to 0. Defining $R(u)$ to be 0 when $u < 0$, so that the lower limit of integration may be replaced by $-\infty$, we deduce that

$$R(\omega) \int_{-\infty}^{\infty} \mathfrak{K}(y)\, dy = \int_{-\infty}^{\infty} \left\{ R(\omega) - R\left(\omega + \frac{y}{T}\right) \right\} \mathfrak{K}(y)\, dy + o(1),$$

when $\omega \to \infty$. We take $\omega > 0$ and split the range of integration on the right into the two parts $(-\infty, -\frac{1}{2}T\omega)$ and $(-\frac{1}{2}T\omega, \infty)$. In the first range, $\omega \leqslant 2|y|/T$ and $\omega + y/T \leqslant \frac{1}{2}\omega \leqslant |y|/T$; hence, by (16), the contribution to the integral is

$$O\left(\int_{-\infty}^{-\frac{1}{2}T\omega} |y|\, \mathfrak{K}(y)\, dy \right) = o(1),$$

by (11). In the second range, $\omega + y/T \geqslant \frac{1}{2}\omega$; hence we may apply (15) to the expression $\{\ldots\}$ if $\omega > 2\omega_0$. It follows that

$$|R(\omega)|\,\mathfrak{K} \leqslant \int_{-\frac{1}{2}T\omega}^{\infty} \left(2 + \left|\frac{\theta y}{T}\right|\right) \Theta_1\, \mathfrak{K}(y)\, dy + o(1) \leqslant \left(2\mathfrak{K} + \frac{|\theta|}{T}\, \mathfrak{K}'\right) \Theta_1 + o(1).$$

Since $\mathfrak{K}' < 6\mathfrak{K}$ (and $\mathfrak{K} > 0$) it follows that

$$\overline{\lim_{\omega \to \infty}}\, |R(\omega)| \leqslant \left(2 + \frac{6|\theta|}{T}\right) \Theta_1,$$

whence the theorem, since $R(\omega) = A(\omega) - f(0)$ for $\omega > 0$, and Θ_1 may be taken arbitrarily near to Θ.

Theorem II†. We may suppose that $A(u)$ increases for all $u \geqslant 0$ [otherwise modify $A(u)$ over a finite range and add a constant]. We may now use the simpler form (I) of $\mathfrak{k}(x)$. In (14) let α (> 0) be the "α" of the theorem, and take $A = a/\alpha$. Writing $B(u) = A(u)\, e^{-u\alpha}$, we obtain, for $\sigma > \alpha$,

$$\int_{-T}^{T} \phi(\sigma+ti)\, \mathfrak{k}_T(t)\, e^{\omega ti}\, dt$$

$$= \int_0^\infty B(u)\, \mathfrak{K}_T(u-\omega)\, e^{-u(\sigma-\alpha)}\, du - \int_0^\infty \frac{a}{\alpha}\, \mathfrak{K}_T(u-\omega)\, e^{-u(\sigma-\alpha)}\, du.$$

Here $$\phi(s) = \frac{f(s)}{s} - \frac{a}{\alpha(s-\alpha)} = \frac{1}{s}\left(f(s) - \frac{a}{s-\alpha} - \frac{a}{\alpha}\right),$$

so that $\phi(s)$ is continuous for $\sigma \geqslant \alpha$, $-T \leqslant t \leqslant T$. Also $B(u) \geqslant 0$ for $u \geqslant 0$, since $A(u)$ increases with u and is 0 for $u = 0$.

Now let $\sigma \to \alpha + 0$ in the relation just obtained. By the continuity of $\phi(s)$ the limit may be taken under the integral sign in the integral on the left, and, since the integrands are of fixed sign, the same is consequently true for the integrals on the right, unless the resulting integrals are both divergent. But this is not the case for the last integral, by (10). Thus the process is justified, and, making the substitution $u = \omega + y/T$ in the resulting equation, we obtain

$$\int_{-T}^{T} \phi(\alpha+ti)\, \mathfrak{k}_T(t)\, e^{\omega ti}\, dt = \int_{-T\omega}^{\infty} B\left(\omega + \frac{y}{T}\right) \mathfrak{K}(y)\, dy - \frac{a}{\alpha} \int_{-T\omega}^{\infty} \mathfrak{K}(y)\, dy.$$

Now let $\omega \to \infty$. The first integral tends to 0 by the Riemann-Lebesgue theorem, and the third tends to $\mathfrak{K}$. Hence

$$(18) \qquad \int_{-T\omega}^{\infty} B\left(\omega + \frac{y}{T}\right) \mathfrak{K}(y)\, dy \to \frac{a}{\alpha}\, \mathfrak{K} \quad \text{as} \quad \omega \to \infty.$$

Now take $Y > 0$, and suppose $\omega > Y/T$. In the range $-Y \leqslant y \leqslant Y$ we have, since $B(u)\, e^{u\alpha} = A(u)$ is a non-negative increasing function of u,

$$B\left(\omega + \frac{y}{T}\right) \geqslant B\left(\omega - \frac{Y}{T}\right) e^{-2(Y/T)\alpha}.$$

† The argument which follows is that of Landau; Landau, 2. The reason for the trifling modification at the end is that we do not wish to use irrelevant properties of the Fejér kernel.

Since $B(u)$ and $\mathfrak{K}(y)$ are never negative, we deduce from (18), writing $\tau = T/a$, that

$$\overline{\lim_{\omega\to\infty}}\, B\Big(\omega-\frac{Y}{T}\Big)e^{-2Y/\tau}\int_{-Y}^{Y}\mathfrak{K}(y)\,dy \leqslant \frac{a}{\alpha}\,\mathfrak{K}.$$

Hence, since $\mathfrak{K}(y)$ is obviously an even function,

$$\overline{\lim_{\omega\to\infty}}\, B(\omega) = \overline{\lim_{\omega\to\infty}}\, B\Big(\omega-\frac{Y}{T}\Big) \leqslant \frac{a}{\alpha}\,\frac{e^{2Y/\tau}}{1-\mathfrak{r}(Y)},$$

where $$\mathfrak{r}(Y) = \frac{2}{\mathfrak{K}}\int_{Y}^{\infty}\mathfrak{K}(y)\,dy.$$

Now $$0<\mathfrak{r}(Y)<1 \quad (Y>0), \qquad \mathfrak{r}(Y)\to 0 \text{ as } Y\to\infty.$$

Hence, taking $Y=\tau^{\frac{1}{2}}$, we obtain

$$\overline{\lim_{\omega\to\infty}}\, B(\omega) \leqslant \frac{a}{\alpha}\,\frac{e^{2/\tau^{\frac{1}{2}}}}{1-\mathfrak{r}(\tau^{\frac{1}{2}})} = \frac{a}{\alpha}\,\mathfrak{P}(\tau), \tag{19}$$

where $\mathfrak{P}(\tau)>1$ for all $\tau>0$, and $\mathfrak{P}(\tau)\to 1$ as $\tau\to\infty$; and the second of the inequalities (6) is proved. It follows from (19) that $a\geqslant 0$.

To obtain the other part of (6) take $Y>0$, suppose $\omega > 2Y/T$, and dissect the integral I on the left of (18) thus:

$$I=\int_{-T\omega}^{\infty} = \int_{-T\omega}^{-\frac{1}{2}T\omega}+\int_{-\frac{1}{2}T\omega}^{-Y}+\int_{-Y}^{Y}+\int_{Y}^{\infty} = I_1+I_2+I_3+I_4,$$

say. By (19), $B(u)$ is bounded; hence (when $\omega\to\infty$)

$$I_1 = O\Big(\int_{-T\omega}^{-\frac{1}{2}T\omega}\mathfrak{K}(y)\,dy\Big) = o(1).$$

In I_2 and I_4, $\omega+y/T\geqslant\frac{1}{2}\omega$; hence, by (19),

$$\overline{\lim_{\omega\to\infty}}\,(I_2+I_4) \leqslant \frac{a}{\alpha}\,\mathfrak{P}(\tau)\Big(\int_{-\infty}^{-Y}+\int_{Y}^{\infty}\Big)\,\mathfrak{K}(y)\,dy = \frac{a}{\alpha}\,\mathfrak{P}(\tau)\,\mathfrak{r}(Y)\,\mathfrak{K}.$$

Also

$$I_3 \leqslant B\Big(\omega+\frac{Y}{T}\Big)\,e^{2Y/\tau}\int_{-Y}^{Y}\mathfrak{K}(y)\,dy \leqslant B\Big(\omega+\frac{Y}{T}\Big)\,e^{2Y/\tau}\,\mathfrak{K}.$$

Hence, by (18),

$$\begin{aligned}\frac{a}{\alpha}\,\mathfrak{K} = \underline{\lim_{\omega\to\infty}}\, I &\leqslant \overline{\lim_{\omega\to\infty}}\,(I_1+I_2+I_4)+\underline{\lim_{\omega\to\infty}}\, I_3\\ &\leqslant \frac{a}{\alpha}\,\mathfrak{P}(\tau)\,\mathfrak{r}(Y)\,\mathfrak{K}+\underline{\lim_{\omega\to\infty}}\, B\Big(\omega+\frac{Y}{T}\Big)\,e^{2Y/\tau}\,\mathfrak{K}.\end{aligned}$$

Thus

$$\lim_{\omega\to\infty} B(\omega) = \lim_{\omega\to\infty} B\left(\omega+\frac{Y}{T}\right) \geqslant \frac{a}{\alpha}\{1-\mathfrak{P}(\tau)\mathfrak{r}(Y)\}e^{-2Y/\tau}.$$

Now take $Y=\max(\mathfrak{v}_\tau, \tau^{\frac{1}{2}}) = \mathfrak{y}_\tau$, say, where $\mathfrak{v}_\tau$ is defined by $\mathfrak{r}(\mathfrak{v}_\tau)=\{2\mathfrak{P}(\tau)\}^{-1}$. [This equation defines a unique positive $\mathfrak{v}_\tau$, since $\mathfrak{r}(v)$ is a strictly decreasing continuous function of v, $\mathfrak{r}(0)=1$, $\mathfrak{r}(v)\to 0$ as $v\to\infty$, and $0<\{2\mathfrak{P}(\tau)\}^{-1}<\frac{1}{2}$.] Then

$$\lim_{\omega\to\infty} B(\omega) \geqslant \frac{a}{\alpha}\{1-\mathfrak{P}(\tau)\mathfrak{r}(\mathfrak{y}_\tau)\}\exp\left(-\frac{2\mathfrak{y}_\tau}{\tau}\right) = \frac{a}{\alpha}\mathfrak{p}(\tau),$$

say. Now $\mathfrak{P}(\tau)\mathfrak{r}(\mathfrak{y}_\tau) \leqslant \mathfrak{P}(\tau)\mathfrak{r}(\mathfrak{v}_\tau) = \frac{1}{2}$, so that $\mathfrak{p}(\tau)>0$ for all $\tau>0$. Also, since $\{2\mathfrak{P}(\tau)\}^{-1}\to\frac{1}{2}$, $\mathfrak{v}_\tau$ is bounded, as $\tau\to\infty$; hence, for all sufficiently large τ, we have $\mathfrak{y}_\tau=\tau^{\frac{1}{2}}$ and so

$$\mathfrak{p}(\tau) = \{1-\mathfrak{P}(\tau)\mathfrak{r}(\tau^{\frac{1}{2}})\}e^{-2/\tau^{\frac{1}{2}}}\to 1 \quad \text{as} \quad \tau\to\infty.$$

This completes the proof.

Theorem III. We deduce this from Theorems I and II. With a view to applications we have adopted a somewhat elaborate form of statement having the appearance of great generality. We may remark, however, that the generality embodied in the arbitrary function $\mathrm{H}(\delta)$ is more apparent than real, for we shall deduce the general theorem from what is on the face of it a highly specialized form of itself. The deduction involves, indeed, a "Tauberian" argument, but one of a very simple kind.

Suppose first that there is a number K such that $A(u)+Ku$ increases with u for $u\geqslant u_0$. Take $\theta>0$, and let

$$B(\omega) = \int_0^\omega e^{u\theta}d\{A(u)+Ku\} = \int_0^\omega e^{u\theta}dA(u)+\frac{K}{\theta}(e^{\omega\theta}-1). \tag{20}$$

Then $B(\omega)$ increases with ω for $\omega\geqslant u_0$; and $B(0)=0$. Also, for $\sigma>\theta$,

$$g(s) = \int_0^\infty e^{-us}dB(u) = \int_0^\infty e^{-us}e^{u\theta}d\{A(u)+Ku\} = f(s-\theta)+\frac{K}{s-\theta},$$

so that $g(s)-K/(s-\theta)=f(s-\theta)$ is continuous for $\sigma\geqslant\theta$, $-T\leqslant t\leqslant T$. It follows from Theorem II that

$$\mathfrak{p}\left(\frac{T}{\theta}\right)\frac{K}{\theta} \leqslant \overline{\lim_{\omega\to\infty}}\, e^{-\omega\theta}B(\omega) \leqslant \mathfrak{P}\left(\frac{T}{\theta}\right)\frac{K}{\theta}.$$

We deduce that $K\geqslant 0$, and, by (20), that

$$-\mathfrak{Q}\left(\frac{T}{\theta}\right)\frac{K}{\theta} \leqslant \overline{\lim_{\omega\to\infty}}\, e^{-\omega\theta}\int_0^\omega e^{u\vartheta}dA(u) \leqslant \mathfrak{Q}\left(\frac{T}{\theta}\right)\frac{K}{\theta},$$

where $$\mathfrak{Q}(\tau) = \max\{1-\mathfrak{p}(\tau),\ \mathfrak{P}(\tau)-1\}.$$

It now follows from Theorem I that

$$\overline{\lim_{\omega\to\infty}}\ |A(\omega)-f(0)| \leqslant 2\left(1+3\frac{\theta}{T}\right)\mathfrak{Q}\left(\frac{T}{\theta}\right)\frac{K}{\theta}.$$

Taking $\theta = T$, we obtain

$$\overline{\lim_{\omega\to\infty}}\ |A(\omega)-f(0)| \leqslant \mathfrak{C}_0 K T^{-1},$$

where $\mathfrak{C}_0 = 8\mathfrak{Q}(1)$, a positive absolute constant. This is a special form of Theorem III, for (since $K \geqslant 0$) the assumption which we have made about $A(u)$ implies (7) with $\mathrm{H}(\delta) = K\delta$ (and indeed more).

Now suppose only that $A(u)$ satisfies the condition (7) of Theorem III. Take $\delta > 0$, define $A(u)$ to be 0 when $u < 0$, and let

$$A^*(\omega) = \frac{1}{\delta}\int_{\omega-\delta}^{\omega} A(u)\,du = \frac{1}{\delta}\int_0^{\omega}\{A(u)-A(u-\delta)\}\,du.$$

Then $A^*(0) = 0$, and, by (7),

$$A^*(\omega)+\frac{\mathrm{H}(\delta)}{\delta}\,\omega = \frac{1}{\delta}\int_0^{\omega}\{A(u)-A(u-\delta)+\mathrm{H}(\delta)\}\,du$$

is an increasing function of ω for $\omega \geqslant \omega_0(\delta)+\delta$. Also, for $\sigma > 0$,

$$f^*(s) \equiv \int_0^{\infty} e^{-us}\,dA^*(u) = \frac{1}{\delta}\int_0^{\infty} e^{-us}\{A(u)-A(u-\delta)\}\,du$$

$$= \frac{1}{\delta}\int_0^{\infty} e^{-us}A(u)\,du - \frac{1}{\delta}\int_0^{\infty} e^{-(u+\delta)s}A(u)\,du = \frac{1-e^{-\delta s}}{\delta}\,\frac{f(s)}{s}.$$

Thus

$$\frac{f^*(s)-f(0)}{s} = \frac{1-e^{-\delta s}}{\delta s}\,\frac{f(s)-f(0)}{s} - \frac{e^{-\delta s}-1+\delta s}{\delta s^2}\,f(0)$$

is continuous for $\sigma \geqslant 0$, $-T \leqslant t \leqslant T$. Hence, by the special result already obtained, $\mathrm{H}(\delta) \geqslant 0$ and

$$\overline{\lim_{\omega\to\infty}}\ |A^*(\omega)-f(0)| \leqslant \mathfrak{C}_0\,\mathrm{H}(\delta)\,\delta^{-1}\,T^{-1}. \tag{21}$$

Now it follows from (7) that, if $\omega \geqslant \omega_0(\delta)+\delta$,

$$A(\omega-\delta)-\mathrm{H}(\delta) \leqslant A^*(\omega) \leqslant A(\omega)+\mathrm{H}(\delta).$$

This combined with (21) gives

$$\overline{\lim_{\omega\to\infty}}\ |A(\omega)-f(0)| = \overline{\lim_{\omega\to\infty}}\ |A(\omega-\delta)-f(0)| \leqslant (\mathfrak{C}_0\delta^{-1}T^{-1}+1)\,\mathrm{H}(\delta).$$

Taking $\delta = T^{-1}$ we obtain the first part of Theorem III. To obtain the second part we note that

$$|A^*(\omega_n+\delta)-A(\omega_n)| = \left| \frac{1}{\delta} \int_{\omega_n}^{\omega_n+\delta} \{A(u)-A(\omega_n)\}\, du \right| \leqslant \eta(\delta)$$

if $n \geqslant n_0(\delta)$, whence, by (21),

$$\overline{\lim_{n\to\infty}}\, |A(\omega_n)-f(0)| \leqslant \mathfrak{C}_0 \delta^{-1} T^{-1} \mathrm{H}(\delta)+\eta(\delta).$$

Taking $\delta = T^{-\frac{1}{2}}$, we obtain the desired result.

The precise form of the inequalities (8) and (9) of Theorem III is of no great importance. What is important is that the right-hand sides tend to 0 when $T\to\infty$, if (in the first case) $\mathrm{H}(\delta)$ tends to 0 with δ, or (in the second) $\mathrm{H}(\delta)$ is bounded, and $\eta(\delta)\to 0$, when $\delta\to 0$. Other forms of inequality having these properties may be obtained by different choices of θ and δ. We may take, for example, $\theta = 1$ and $\delta = \{\mathfrak{Q}(T)\}^{\frac{1}{2}}$. It will be noticed, however, that our original choice has the advantage that it does not demand the full force of Theorem II, but only the finiteness of $\mathfrak{P}(\tau)$ for $\tau = 1$.

3. *Proof of Theorems* 1 (*l*), 2 (*l*), 3 (*l*), 3_1 (*l*). We may suppose that $\lambda_1 > 0$ (otherwise change λ_n to $\lambda_n+\gamma$, where $\gamma > -\lambda_1$). Theorems 1 (*l*), 2 (*l*), 3 (*l*) follow from Theorems I, II, III, respectively, on taking

$$A(\omega) = A_\lambda(\omega) = \sum_{\lambda_n \leqslant \omega} a_n, \qquad w = e^{\omega}. \tag{22}$$

The deduction is immediate in the case of the first two theorems, but a little manipulation of the hypotheses is necessary for Theorem 3 (*l*).

We take the case (ii) first. Let $\omega \leqslant \omega' \leqslant \omega+\delta$ ($\delta > 0$), and let λ_N, $\lambda_{N'}$ be the greatest λ_n not exceeding ω, ω' respectively. Then, since $(l_n-l_{n-1})/l_n = 1-e^{-(\lambda_n-\lambda_{n-1})} < \lambda_n-\lambda_{n-1}$, we have, for sufficiently large ω,

$$A(\omega')-A(\omega) = \sum_{N+1}^{N'} a_n \geqslant a_{N+1} - \sum_{N+2}^{N'} K(\lambda_n-\lambda_{n-1}) \geqslant a_{N+1}-K\delta \tag{23}$$

if $N' > N$, and $A(\omega')-A(\omega) = 0$ otherwise. Since $\underline{\lim}\, a_n \geqslant 0$ and $K > 0$, the condition (7) of Theorem III is therefore satisfied with $\mathrm{H}(\delta) = 2K\delta$, and Theorem 3 (*l*) (ii) follows.

In case (i) we note first that we may suppose a_n real; for, if $a_n = \alpha_n+\beta_n i$, then $\Sigma \alpha_n l_n^{-s} = \{f(s)+\overline{f}(s)\}/2$ and $\Sigma \beta_n l_n^{-s} = \{f(s)-\overline{f}(s)\}/2i$ satisfy the conditions of the theorem. We then have (23) as before (if $N' > N$), and, since $(l_n-l_{n-1})/l_n < 1$, $a_{N+1} > -K$ if $N+1 \geqslant n_0$; hence (7) is satisfied with

$\mathrm{H}(\delta) = K(1+\delta)$. Further, if $\lambda_n \leqslant \omega' \leqslant \lambda_n + \delta$ and $\lambda_{N'}$ is as before, we have

$$|A(\omega') - A(\lambda_n)| \leqslant \sum_{n+1}^{N'} |a_\nu| \leqslant \sum_{n+1}^{N'} K(\lambda_\nu - \lambda_{\nu-1}) \leqslant K\delta \quad (n \geqslant n_0),$$

so that the conditions of the last part of Theorem III are satisfied with $\omega_n = \lambda_n$, $\eta(\delta) = K\delta$; and Theorem 3 (l) (i) follows, since

$$\overline{\lim}\, |A(\omega) - f(0)| = \overline{\lim}\, |A(\lambda_n) - f(0)|.$$

To deduce Theorem 3_1 (l) from Theorem III we note first of all that $a \geqslant 0$, since $sf(s) \to a$ as $s \to +0$ and $a_n \geqslant 0$ for $n \geqslant n_0$. Let

$$B(\omega) = A(\omega) - a\omega,$$

where $A(\omega)$ is defined by (22). Then, since $B(\omega) + a\omega = A(\omega)$ increases with ω for $\omega \geqslant \omega_0$ (say), we have

$$B(\omega') - B(\omega) \geqslant -a(\omega' - \omega) \geqslant -a\delta \quad (\omega \leqslant \omega' \leqslant \omega + \delta,\ \omega \geqslant \omega_0);$$

also $B(0) = 0$. Further, for $\sigma > 0$,

$$g(s) \equiv \int_0^\infty e^{-us}\, dB(u) = f(s) - \frac{a}{s},$$

so that

$$\frac{g(s) - a'}{s} = \frac{1}{s}\left(f(s) - \frac{a}{s} - a'\right)$$

is continuous for $\sigma \geqslant 0$, $-T \leqslant t \leqslant T$. Hence, by the first part of Theorem III with $\mathrm{H}(\delta) = a\delta$, we have

$$\overline{\lim_{\omega \to \infty}}\, |B(\omega) - a'| \leqslant \mathfrak{C} a T^{-1},$$

which implies the result stated. It will be noticed that we do not require the complete Theorem III, but only the simpler form contained in the first part of the proof in § 2.

4. It is well known that there is a generalized form of Theorem 2 (λ) in which the condition $a_n \geqslant 0$ is replaced by conditions similar to those of Theorem 3 (λ) (ii), and that the latter theorem may be included as the case $\alpha = 0$†. It is possible to formulate an l-analogue of this theorem, but it is somewhat complicated owing to the number of parameters involved in the expressions for the limits of oscillation, and the significance of the assertion $\mathfrak{p}(\tau) > 0$ in Theorem 2 (l) tends to become obscured in the general statement. Moreover, although the cases $\alpha > 0$ and $\alpha = 0$ can be included formally in a single statement, they differ essentially (just as in the λ-theory) in that the

† Hardy and Littlewood, 2, Theorems E, F.

case $\alpha = 0$ demands greater subtlety of treatment. For these reasons we prefer to regard the simpler theorems as the fundamental ones, but it may be worth while to indicate how the general theorem, in the case $\alpha > 0$, may be derived from Theorem II. For simplicity we confine ourselves to the form to which the theorem reduces when T may be taken arbitrarily large†.

Theorem 2_1 (l). *Suppose that $\alpha > 0$, and that one of the following sets of conditions,* (i) *or* (ii), *is satisfied*:

(i) $$|a_n| \leqslant K(l_n - l_{n-1}) l_n^{\alpha-1} \quad (n \geqslant n_0),$$

so that (1) *is certainly convergent for $\sigma > \alpha$;*

(ii) *a_n is real,*

$$a_n \geqslant -K(l_n - l_{n-1}) l_n^{\alpha-1} \quad (n \geqslant n_0), \qquad \varliminf_{n\to\infty} l_n^{-\alpha} a_n \geqslant 0,$$

and (1) *is convergent for $\sigma > \alpha$.*

Suppose further that, for an appropriate a,

$$f(s) - \frac{a}{s-\alpha}$$

is continuous for $\sigma \geqslant \alpha$. Then

$$w^{-\alpha} A_l(w) \to \frac{a}{\alpha}, \tag{24}$$

when $w \to \infty$.

We might deduce this from an extended form of Theorem II just as Theorem 3 (l) was deduced from Theorem III. The extended theorem would contain a hypothesis of the form

$$A(\omega') - A(\omega) \geqslant -\mathrm{H}(\delta) e^{\omega\alpha} \quad \{\omega \leqslant \omega' \leqslant \omega + \delta,\ \omega \geqslant \omega_0(\delta)\} \tag{25}$$

† See Hardy and Littlewood, **3**, Theorems 2.121, 2.14, 2.15, for the general theorem subject to an "order" condition on $f(s)$, and on the hypothesis that $l_n/l_{n-1} \to 1$; and Wiener, **1**, 126–127, 130–135, for the theorem without order condition, but with a heavier restriction on l_n. In place of the convergence of (1) for $\sigma > \alpha$ these authors assume absolute convergence for $\sigma > \sigma_0 \geqslant \alpha$ together with regularity of $f(s)$ for $\sigma > \alpha$; but the two sets of conditions are equivalent in virtue of the other hypotheses, as may be seen by supposing a_n real and applying to the series $\Sigma \{a_n + K(l_n - l_{n-1}) l_n^{\alpha-1}\} l_n^{-s}$ Landau's theorem on Dirichlet's series with non-negative coefficients (Hardy and Riesz, **1**, Theorem 10). A similar remark applies (but for other reasons) to theorems of the "high-indices" type, in which $\varliminf l_n/l_{n-1} > 1$ but a_n is subject to no explicit restriction; for when $\varliminf l_n/l_{n-1} > 1$, $f(s)$ necessarily has singular points on the line of convergence of the series (Pólya, **1**; Bernstein, **1**, 140), and the half-planes of convergence, absolute convergence, and regularity coincide.

[corresponding to (7)] and a suitably modified conclusion; and there would be a second part similar to the second part of Theorem III. For the proof we should first consider the special case in which, for some K, $A(\omega)+Ke^{\omega\alpha}$ increases with ω for $\omega \geqslant \omega_0$. In this case we should simply apply Theorem II to $A(\omega)+Ke^{\omega\alpha}-K$; there would be no need to introduce a factor $e^{u\theta}$ or to apply Theorem I, but we should have to use the full force of Theorem II. The general theorem could then be deduced by considering the function $A^*(\omega)$ of § 2 with an appropriate choice of δ.

For the sake of variety, however, we indicate another line of argument†. We may suppose that $\lambda_1 > 0$, that a_n is real, and that $K > 0$. From the relations

$$l_n^{\alpha}-l_{n-1}^{\alpha} = (l_n-l_{n-1})\,\alpha\xi_n^{\alpha-1} \quad (l_{n-1} < \xi_n < l_n),$$

$$l_n^{\alpha}-l_{n-1}^{\alpha} = (l_n-l_{n-1})\,l_n^{\alpha-1}+l_{n-1}(l_n^{\alpha-1}-l_{n-1}^{\alpha-1}),$$

we infer that

$$l_n^{\alpha}-l_{n-1}^{\alpha} \geqslant (l_n-l_{n-1})\,l_n^{\alpha-1}\min(\alpha, 1).$$

Hence, writing $A = K\max(\alpha, 1)$, we have in all cases

$$a_n \geqslant -(l_n^{\alpha}-l_{n-1}^{\alpha})\,A/\alpha$$

for $n \geqslant n_0$, and, defining l_0 to be 1 and replacing a finite number of a_n by 0 if necessary, we may suppose the inequality to hold for all n. Then, by summation,

$$A(\omega) \equiv \sum_{\lambda_n \leqslant \omega} a_n \geqslant -\frac{A}{\alpha}\,e^{\omega\alpha} \quad (\omega \geqslant 0). \tag{26}$$

Hence $A(\omega)+(A/\alpha)\,e^{\omega\alpha} \geqslant 0$, and so

$$B(\omega) \equiv \int_0^{\omega}\left(A(u)+\frac{A}{\alpha}\,e^{u\alpha}\right)du = \int_0^{\omega}A(u)\,du+\frac{A}{\alpha^2}\,(e^{\omega\alpha}-1) \tag{27}$$

is an increasing function; and $B(0) = 0$. Also, for $\sigma > \alpha$,

$$g(s) \equiv \int_0^{\infty}e^{-us}\,dB(u) = \int_0^{\infty}e^{-us}\left(A(u)+\frac{A}{\alpha}\,e^{u\alpha}\right)du = \frac{f(s)}{s}+\frac{A}{\alpha(s-\alpha)},$$

so that

$$g(s)-\frac{\alpha+A}{\alpha(s-\alpha)} = \frac{1}{s}\left(f(s)-\frac{\alpha}{s-\alpha}-\frac{a}{\alpha}\right)$$

is continuous for $\sigma \geqslant \alpha$. Hence, by Theorem II,

$$\lim_{\omega\to\infty} e^{-\omega\alpha}\,B(\omega) = (\alpha+A)/\alpha^2,$$

† Theorem 3 (*l*) can be proved by a somewhat similar method, with, of course, an added subtlety involving the use of Theorem I.

and so, by (27),

$$\int_0^\omega A(u)\,du = \frac{a}{\alpha^2}e^{\omega\alpha}+o(e^{\omega\alpha}) \quad \text{as } \omega\to\infty. \tag{28}$$

At this point it is convenient to remark that, when $a\neq 0$, it follows from (28) that $l_n/l_{n-1}\to 1$ as $n\to\infty$†. For, writing $\delta_n=\frac{1}{2}(\lambda_n-\lambda_{n-1})$, taking $\omega=\lambda_n$, $\lambda_n-\delta_n$, $\lambda_n-2\delta_n$ in (28), combining the three results suitably, and noting that $A(u)$ is constant for $\lambda_{n-1}<u<\lambda_n$, we obtain

$$0=\frac{a}{\alpha^2}(e^{\lambda_n\alpha}-2e^{(\lambda_n-\delta_n)\alpha}+e^{(\lambda_n-2\delta_n)\alpha})+o(e^{\lambda_n\alpha}) \quad \text{as } n\to\infty,$$

whence (if $a\neq 0$) $$(1-e^{-\delta_n\alpha})^2=o(1),$$

and this is equivalent to $l_n/l_{n-1}\to 1$.

We can now complete the proof of the theorem. Writing

$$S(u)=A(u)-(a/\alpha)e^{u\alpha},$$

we infer from (28) that

$$S^*(\omega)\equiv\frac{1}{\delta}\int_{\omega-\delta}^{\omega}S(u)\,du=o(e^{\omega\alpha}) \quad \text{as} \quad \omega\to\infty. \tag{29}$$

Now from the hypotheses (ii) we may deduce that $A(u)$, and therefore $S(u)$, satisfies a condition of the form (25), where $\mathrm{H}(\delta)$ tends to 0 with δ. Arguing substantially as at the bottom of p. 472, we infer from (29) that

$$-\mathrm{H}(\delta)\leqslant\varlimsup_{\omega\to\infty}e^{-\omega\alpha}S(\omega)\leqslant\mathrm{H}(\delta),$$

and the case (ii) of the theorem follows on making $\delta\to 0$. In the case (i) we can show (as on pp. 473, 474) that (24) holds at any rate when $w\to\infty$ through the sequence l_n. But we have seen that either $a=0$ or $l_n/l_{n-1}\to 1$; and in either case the restriction to the sequence l_n in (24) is plainly irrelevant.

5. It will be observed that the logical relationship of the theorems of the λ-theory is imitated in the l-theory in that Theorem $2_1(l)$ depends on Theorem II only, while Theorems $3(l)$ and $3_1(l)$ involve (through Theorem III) Theorem I also. On the other hand, the proof of Theorems $3(l)$ and $3_1(l)$ depends (in virtue of the remark at the end of § 2) only on the easiest part of Theorem II, and it is natural to suppose that there must be an analogous proof of Theorem $3(\lambda)$ based on a simpler theorem in place of Theorem $2(\lambda)$. Such a proof may, indeed, be constructed, but it does not

† This remark and its applications have been added in proof (*December*, 1934).

represent any advance on known methods. For, whereas the introduction of a factor $e^{u\theta}$ or l_n^{θ} in (5) or (1) affects $f(s)$ only to the extent of a trivial change of variable, a factor u^{θ} or λ_n^{θ} involves a much more elaborate type of transformation, of the nature of repeated differentiation. Thus the procedure suggested (involving, as it would, the use of a large value of θ, corresponding to $\theta = T$ in §2) would amount effectively to applying a repeated differentiation argument to Theorem 3(λ) directly; and this is precisely what Littlewood does in his original proof†. The more usual practice of deducing Theorem 3(λ) from the full Theorem 2(λ) by differentiating once and applying Theorem 1(λ) with $\theta = 1$ corresponds to the alternative choice of θ suggested at the end of § 2‡.

6. The restrictions imposed on a_n in Theorem 3(λ) are the most familiar ones, but others of a different kind are known, and these have analogues in the l-theory. For example, the analogue of Schmidt's condition§ is that

$$\underline{\lim}\{A_l(w') - A_l(w)\} \geqslant 0 \tag{30}$$

when w and w' tend to infinity in any way so that $w' > w$ and $w'/w \to 1$. With this condition, which is wider than (ii), Theorem 3(l) remains true, provided that we replace the right-hand side of the final inequality by a function $\mathfrak{q}(T, f)$, which, for a given f, is defined for every $T > 0$ and tends to 0 when $T \to \infty$. To see this we need only observe that (30) implies that $A(\omega) = A_\lambda(\omega)$ satisfies a condition of the type (7), where $\mathrm{H}(\delta)$ tends to 0 with δ, and apply the first part of Theorem III.

By choosing the $A(\omega)$ of Theorems I, II, III suitably, we may also obtain theorems concerning Riesz's typical means.

7. We conclude with some observations on the relevance of the rate of increase of l_n in the l-theory.

1) If $l_n/l_{n-1} \to 1$ as $n \to \infty$, the second clause ($\underline{\lim} a_n \geqslant 0$) of conditions (ii) of Theorem 3(l) is superfluous since it is implied by the first. In the special case when $l_n = n$ it is not difficult to find examples to show that the restrictions on a_n in this theorem are substantially the widest possible of their kind. For example, if

$$a_n = \frac{\log_k n}{n} \cos(\log n \log_{k+1} n)$$

† Littlewood, 1.

‡ Hardy and Littlewood, 2, Theorem F. In carrying out the details of the analogous technique we should naturally state the basic theorems in Stieltjes integral form, just as we have done for the l-theory.

§ Schmidt, 1, 136, 151; Vijayaraghavan, 1.

for sufficiently large n (where $\log_2 n = \log\log n$, etc.), the series $\Sigma a_n n^{-s}$ converges for $\sigma > 0$ and represents an integral function, but is not convergent for $s = 0$. It could, no doubt, be shown that, whenever $l_n/l_{n-1} \to 1$, the theorem is a best possible one in the complete sense in which this has been done for Theorem 3 (λ)†.

2) If $\underline{\lim}\, l_n/l_{n-1} > 1$, the conditions of Theorem 3 (l) are in no sense best possible ones, as is obvious from the "high-indices theorem" enunciated in § 1. We shall not discuss this theorem in detail here, but we may remark that, in the special case when $l_n = e^n$, the theorem is trivial‡; for, if we write $e^{-s} = z$, $f(s) = \phi(z)$, $a_1 + a_2 + \ldots + a_n = A_n$, the function

$$\frac{\phi(z) - f(0)}{1-z} = \sum_0^\infty \{A_n - f(0)\} z^n \quad (A_0 = 0)$$

(suitably defined for $|z| = 1$) is regular for $|z| < 1$ and continuous for $|z| \leqslant 1$, whence $A_n - f(0) \to 0$ (by the Riemann-Lebesgue theorem or by Parseval's theorem).

3) When $\underline{\lim}\, l_n/l_{n-1} = 1 < \overline{\lim}\, l_n/l_{n-1}$ (the only case to which neither of the preceding remarks is applicable), the hypotheses of Theorem 3 (l) are all required. If we simply omit the condition $\underline{\lim}\, a_n \geqslant 0$ in (ii), we can no longer assert that $\mathfrak{q}(T) \to 0$ as $T \to \infty$. This may be seen from the example§

$$\sum_1^\infty (-1)^n l_n^{-s},$$

where $$l_{2m} \leqslant l_{2m-1} + 1, \quad l_{2m+1} \geqslant 2 l_{2m} \quad (m = 1, 2, \ldots).$$

This series converges for $\sigma > 0$, the function represented is regular for $\sigma > -1$ (as may be verified by taking the terms $2m-1$, $2m$ together), and all conditions, except $\underline{\lim}\, a_n \geqslant 0$, are satisfied with $K = 2$ and arbitrarily large T; but the series is not convergent for $s = 0$. Without the condition $\underline{\lim}\, a_n \geqslant 0$ we can, however, prove a similar theorem for the Rieszian mean of type l and any *positive* order κ, with a $\mathfrak{q}_\kappa(T)$ which tends to 0 when $T \to \infty$. This is an easy deduction from Theorem III.

4) In Theorem 2_1 (l) (ii) the condition $\underline{\lim}\, l_n^{-\alpha} a_n \geqslant 0$ may be omitted when $\alpha \neq 0$; for we have shown (without, of course, using this condition)

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† Littlewood, 1; Ingham, 1.

‡ *Cf.* Hardy and Littlewood, 3, 2.15 (ii).

§ For the corresponding phenomenon in the λ-theory see Ananda Rau, 1. Ananda Rau discusses the "Riesz-Tauber" theorem, but his example applies also to the "Abel-Tauber" Theorem 3 (λ).

that in this case $l_n/l_{n-1} \to 1$, so that the condition in question is implied by the first clause of (ii). The condition $\underline{\lim}\, l_n^{-\alpha} a_n \geqslant 0$ must, however, be retained when $\alpha = 0$; an example to show this may be constructed by taking l_n as in 3) above and $a_n = (-1)^n l_n^{\alpha}$.

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THE DENSITY DIRECTIONS OF IRREGULAR LINEARLY MEASURABLE PLANE SETS

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1. The abstract foundation of the Theory of Linear Measure was given by Carathéodory†. In the paper "On the fundamental geometrical properties of linearly measurable plane sets of points", Besicovitch‡ bases his discussion on the density properties of such sets.

Let A be a measurable§ plane set of points, a any point of the plane, $c(a, r)$ the circle with centre a and radius r, and let μ denote linear measure. We consider the upper and lower limits of $\mu[A \,.\, c(a, r)]/(2r)$, as r tends to zero, called respectively the upper and lower density of A at a. When they are equal we speak of the density $D(a, A)$. A point of A is "regular" if $D(a, A) = 1$; "irregular" otherwise. We call a set regular if almost all of its points are regular; irregular if almost all of its points are irregular.

A "tangent" of a plane set is defined thus: Let a be any point of A at which the upper density is positive, and let TT' be a straight line drawn through a. Let (α) denote the part of the plane lying within the double angle α, with vertex a and bisector TT'. If, for arbitrarily small values of α,

$$D[a,\ A - A(\alpha)] = 0,$$

the line TT' is said to be the tangent to A at a. With these definitions, Besicovitch proves the following theorems.

A (THEOREM ON THE PERMANENCE OF DENSITY). *If A and B are measurable sets, and $A \subset B$, then at almost all points of A the upper and lower densities of A are equal respectively to those of B.*

† C. Carathéodory, *Göttinger Nachrichten* (1914), 404–426.

‡ A. S. Besicovitch, *Math. Annalen*, 98 (1928), 422–464.

§ "Measurable" will always mean "linearly measurable".

B (THEOREM OF DECOMPOSITION). *The set of all regular points of a measurable set is regular: the set of all its irregular points is irregular.*

C (MAIN THEOREM). *A regular set has a tangent at almost all points: at almost all points of an irregular set there is no tangent.*

In this paper I investigate certain properties of irregular measurable sets, with respect to directions at a point.

2. *Definitions and notations.*

By (a, ϑ) we denote the half-line with origin at the point a and direction ϑ (referred to some fixed axis). By (a, ϑ, ϵ) we denote the angle of magnitude ϵ whose bisector is (a, ϑ). By $(a, \vartheta, \epsilon, r)$ we denote the sector cut off from (a, ϑ, ϵ) by the circle $c(a, r)$. By $(A, a, \vartheta, \epsilon, r)$ we denote the part of the measurable plane set A contained in $(a, \vartheta, \epsilon, r)$.

We write

$$\delta(A, a, \vartheta) = \lim_{\epsilon \to 0} \left(\overline{\lim_{r \to 0}} \frac{\mu(A, a, \vartheta, \epsilon, r)}{r} \right),$$

calling this number the "density coefficient" at the point a, in the direction ϑ. We shall speak of an (a, ϑ) for which $\delta(A, a, \vartheta) > 0$ as a "density direction" of the set A.

Write

$$\begin{aligned} f(a, \vartheta, \epsilon, r) &= \mu(A, a, \vartheta, \epsilon, r)/r, \\ f_1(a, \vartheta, \epsilon, h) &= \text{u.b.}_{0 < r \leqslant h} f(a, \vartheta, \epsilon, r), \\ f_2(a, \vartheta, \epsilon) &= \lim_{h \to 0} f_1(a, \vartheta, \epsilon, h), \\ f_3(a, \vartheta) &= \lim_{\epsilon \to 0} f_2(a, \vartheta, \epsilon). \end{aligned}$$

Then $$\delta(A, a, \vartheta) = f_3(a, \vartheta).$$

3. THEOREM 1. *A plane set A of finite exterior linear measure, which is the projection of a set, in three or more dimensions, of the form*

$$\prod_m \left(\sum_n F_{mn} \right),$$

where F_{mn} is closed (or, more shortly, of the form $\Pi\Sigma F$), is linearly measurable.

We write $$\mathfrak{A} = \Pi\Sigma F, \quad A = P\mathfrak{A}$$

(denoting the projection of $\mathfrak{A}$). It is sufficient to prove the theorem

under the conditions

(i) Two sets F_{mi}, F_{mj} have no point in common;

(ii) Two sets F_{mi}, F_{nj} $(m > n)$ either have no point in common, or $F_{mi} \subset F_{nj}$.

For consider the sets F_{11}, F_{12}, ..., and write

$$G_{11} = F_{11},$$

$$G_{12} = F_{12} - F_{12} . F_{11},$$

$$G_{13} = F_{13} - F_{13}(F_{11} + F_{12}),$$

and so on. Then each set G_{12}, G_{13}, ... is the product of an open set and a closed set, and can be expressed as the sum of an enumerable infinity of distinct closed sets. Moreover, the sets G_{11}, G_{12}, ... are distinct. We have thus replaced F_{11}, F_{12}, ... by a new set H_{11}, H_{12}, ... of distinct, closed sets, such that

$$\sum_n F_{1n} = \sum_n H_{1n}.$$

We now call this new set F_{11}, F_{12}, ...

Writing now

$$F_{2i} = \sum_n (F_{2i} . F_{1n}) + G_{2i},$$

we see that

$$G_{2i} . \prod_m \left(\sum_n F_{mn}\right) = 0.$$

Thus the sets F_{2i} $(i = 1, 2, \ldots)$ can be replaced by the doubly infinite array of closed sets $F_{2i} . F_{1n}$, enumerated again as F_{2i}, without altering $\mathfrak{A}$. Proceeding in this way we replace a general set of closed F_{mn} by a set satisfying conditions (i) and (ii).

We now prove the theorem. Given $\epsilon > 0$, we can find n_1 such that

$$\mu^* P\left(\mathfrak{A} . \sum_{n=1}^{n_1} F_{1n}\right) > \mu^* A - \tfrac{1}{2}\epsilon.$$

For $P\left(\mathfrak{A} . \sum_1^{n_1} F_{1n}\right)$ is contained in A, and tends to A as n_1 tends to infinity; and we use the theorem "If $A_n \subset A_{n+1}$, then $\lim_{n\to\infty} \mu^* A_n = \mu^* \lim_{n\to\infty} A_n$". (See, for example, Carathéodory, *loc. cit.*, Theorem 13.)

Among the F_{2n} contained in F_{11}, F_{12}, ..., we can select a finite number n_2 such that

$$\mu^* P\left(\mathfrak{A} \,.\, \sum_{n=1}^{n_2} F_{2n}\right) > \mu^* A - \tfrac{1}{2}\epsilon - \tfrac{1}{4}\epsilon;$$

and so on. We thus form sums S_1, S_2, ... of closed sets F_{mn} such that

$$S_m = \sum_{n=1}^{n_m} F_{mn}, \quad S_m \supset S_{m+1},$$

and $$\mu^* P(\mathfrak{A} \,.\, S_m) > \mu^* A - 2^{-1}\epsilon - \ldots - 2^{-m}\epsilon > \mu^* A - \epsilon.$$

Now PS_m is closed, and contains PS_{m+1}; thus $E = \Pi PS_m$ is a closed set and

$$\mu E \geqslant \mu^* A - \epsilon.$$

But $\mathfrak{E} = \Pi S_m$ is a closed set, contained in $\mathfrak{A}$. It is easily seen that

$$E = P\mathfrak{E} \subset P\mathfrak{A} = A,$$

wherefore $$\mu_* A \geqslant \mu E \geqslant \mu^* A - \epsilon,$$

and, since ϵ is arbitrary, A is measurable.

4. Let us now suppose that A is irregular, measurable, and of finite measure, and let us consider the functions defined in § 2.

(i) $f(a, \vartheta, \epsilon, r)$ is continuous in $(a, \vartheta, \epsilon, r)$ for $r > 0$. It is sufficient to show that the same is true of $\mu(A, a, \vartheta, \epsilon, r)$. But this is evident, since A is irregular. and has only a nul set in common with the contour of $(a, \vartheta, \epsilon, r)$. From the theorem "If $E_m \supset E_{m+1}$ and μE_1 is finite, then $\mu(\lim E_m) = \lim (\mu E_m)$" (Carathéodory, *loc. cit.*, Theorem 9), we see that the part of A lying within a sufficiently small distance of the contour of $(a, \vartheta, \epsilon, r)$ has arbitrarily small measure; the result follows at once.

(ii) $\underset{1/n \leqslant r \leqslant h}{\text{u.b.}} f(a, \vartheta, \epsilon, r)$ is continuous in (a, ϑ, ϵ), increasing as n tends to infinity, and decreasing as $h \to 0$. Thus $\underset{0 < r \leqslant h}{\text{u.b.}} f(a, \vartheta, \epsilon, r)$ is a lower-semi-continuous function of (a, ϑ, ϵ), decreasing as h or $\epsilon \to 0$.

(iii) $f_2(a, \vartheta, \epsilon)$ is an upper-lower semi-continuous function of (a, ϑ), decreasing as $\epsilon \to 0$. Thus $f_3(a, \vartheta)$ is an upper-lower semi-continuous function of (a, ϑ).

If $\{\phi_n\}$ is a decreasing sequence of functions having limit function ϕ, and $E(\phi_n < \alpha)$ denotes the set of points at which $\phi_n < \alpha$, then

$$E(\phi < \alpha) = \Sigma E(\phi_n < \alpha).$$

We apply this to the functions f_1, f_2, f_3 defined above. Since $f_1(a, \vartheta, \epsilon, h)$ is a lower semi-continuous function of (a, ϑ), we have, in space of (a, ϑ),

$$\begin{aligned} E\{f_1(a, \vartheta, \epsilon, h) < \alpha\} &= \Sigma F, \\ E\{f_2(a, \vartheta, \epsilon) < \alpha\} &= \Sigma\Sigma F = \Sigma F, \\ E\{f_3(a, \vartheta) < \alpha\} &= \Sigma\Sigma F = \Sigma F \\ E\{f_3(a, \vartheta) \geqslant \alpha\} &= \Pi O = \Pi\Sigma F. \end{aligned}$$

Here, as in Theorem 1, ΣF denotes the sum of an enumerable set of closed sets; and ΠO denotes the product of an enumerable set of open sets.

Since almost everywhere outside A the upper density of A is zero, it follows *a fortiori* that, outside A, $\delta(A, a, \vartheta)$ is positive on a nul set at most. Hence the set of points at which $\delta(A, a, \vartheta) \geqslant \alpha$ has finite exterior measure, and is of the form $P\Pi\Sigma F$. It is measurable, by Theorem 1. We have, therefore,

THEOREM 2. *The set $A(\alpha)$ of points of A at which, for some ϑ, $f_3(a, \vartheta) \geqslant \alpha$, is a $P\Pi\Sigma F$, and is linearly measurable. The set of points of A at which there is a density direction is linearly measurable.*

THEOREM 3. (DECOMPOSITION THEOREM.) *Let A_1 be the set of points of A at which A has a density direction, and let $A_2 = A - A_1$. Then A_2 has zero density at almost every point of A_1, and at such points A_1 has the same density directions and coefficients of density as A. Further, the set A_2 has no density direction at any of its points.*

The last part of the theorem is a consequence of A having no density direction at any point of A_2.

5. THEOREM 4. *At almost every point of A (measurable, irregular) the density directions are symmetrical: that is, for all ϑ, and almost all a,*

$$\delta(A, a, \vartheta + \pi) = \delta(A, a, \vartheta).$$

Suppose the theorem to be false: then at a set $A_1 \subset A$, such that $\mu^* A_1 > 0$, we have the inequality

$$f_3(a, \vartheta) > f_3(a, \vartheta + \pi)$$

for some $\vartheta = \vartheta(a)$. Then there is a $\delta > 0$, and $A_2 \subset A_1$, such that $\mu^* A_2 > 0$, and in A_2,

$$f_3(a, \vartheta) > f_3(a, \vartheta+\pi)+\delta.$$

It follows from the bisection argument that there is a K_0 such that the subset $A_2(\eta)$ of A_2, at which $K_0-\eta \leqslant f_3(a, \vartheta) < K_0+\eta$, has positive exterior measure for all $\eta > 0$.

Take $\eta < \frac{1}{2}\delta$: there is an A_3 which is contained in A_2, such that $\mu^* A_3 > 0$, and in which

$$K_0-\eta \leqslant f_3(a, \vartheta) < K_0+\eta,$$

$$f_3(a, \vartheta+\pi) < K_0+\eta-\delta < K_0-\eta.$$

That is, there exist k, K $(k < K)$, and $A_3 \subset A_2$ such that $\mu^* A_3 > 0$ and

$$\begin{cases} f_3(a, \vartheta) \geqslant K, \\ f_3(a, \vartheta+\pi) < k, \end{cases} \tag{1}$$

at every point of A_3 for $\vartheta = \vartheta(a)$.

Next we can find a subset A_4 of A_3 $(\mu^* A_4 > 0)$, and positive numbers r_0, ϵ_0 such that $\mu^* A_4 > 0$ and

$$f(a, \vartheta+\pi, r, \epsilon) < k \quad \text{if} \quad r < r_0, \ \epsilon < \epsilon_0 \tag{2}$$

at every point of A_4. This presents no difficulty. Now divide the angle $(0, 2\pi)$ into equal angles of magnitude less than $\phi = \lambda\epsilon_0$ (λ is a function of K and k which we shall calculate below). There is a subset A_5 of A_4 $(\mu^* A_5 > 0)$ at all points of which $\vartheta = \vartheta(a)$ lies within an angle $\beta \leqslant \vartheta < \gamma$ of magnitude $\gamma-\beta = \psi < \phi$.

At almost every point of A_5, $D^* A \leqslant 1$; hence, at all points of $A_6 \subset A_5$ $(\mu^* A_6 > 0)$,

$$\mu\{A \,.\, c(a, r)\} < 2r(1+\alpha) \tag{3}$$

for $r < r_1(a)$.

Now A_6 is measurable (for convenience this is proved below); it follows from Theorem 3 that at almost all points of A_6 there is a density line of A_6, coinciding with the density line of A, and having the same coefficient.

Let B be the set of points of A_6 at which A_6 (and therefore B, since $\mu B = \mu A_6$) has, for all ϑ,

$$\delta(A_6, a, \vartheta) = \delta(A, a\ \vartheta).$$

At any point P of B we have

(i) A density direction $\vartheta(P)$ lying in the angle ψ (which we include in an angle χ, bisector ϑ_0, of magnitude less than $\lambda\epsilon_0$), such that

$$\delta\{A,\ P,\ \vartheta(P)\} \geqslant K > K' = \tfrac{1}{2}(K+k),$$

(ii) $\quad f\{P,\ \vartheta(P)+\pi,\ \epsilon,\ r\} < k \quad \text{if} \quad \epsilon < \epsilon_0,\ \ r < r_0,$

(iii) $\quad \mu\{A \,.\, c(P,\ r)\} < 2(1+\alpha)r, \quad r < r_1.$

Take any point P of B, and $r < \min(r_0, r_1)$, so that

$$f(P,\ \vartheta_0,\ \chi,\ r) > K',$$

and so that there is in (P, ϑ_0, χ) a point Q of B with $PQ = r$. It is evident that this can always be done. Let R, R' $(QR \geqslant QR')$ be the points common to the rays enclosing $(Q, \vartheta_0+\pi,\ \frac{1}{2}\epsilon_0)$ and the rays enclosing (P, ϑ_0, χ). (See Fig. 1.) If $\vartheta(Q)$, lying in χ, is the ϑ (and there is such a ϑ) for which

$$\delta\{A,\ Q,\ \vartheta(Q)\} \geqslant K,$$

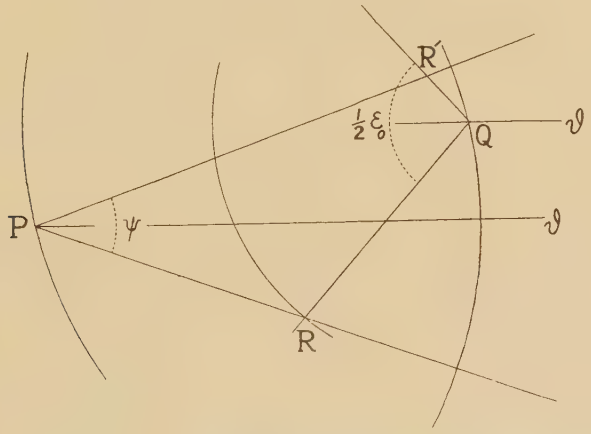

Fig. 1.

then $\qquad \{Q,\ \vartheta(Q)+\pi,\ \epsilon_0) \supset (Q,\ \vartheta_0+\pi,\ \tfrac{1}{2}\epsilon_0)$

and $\qquad QR \geqslant r \sin\chi / \sin(\tfrac{1}{4}\epsilon_0 + \tfrac{1}{2}\chi).$

Thus

$$\begin{aligned} f\{Q,\ \vartheta(Q)+\pi,\ \epsilon_0,\ r\} &\geqslant f(P,\ \vartheta_0,\ \chi,\ r) - r^{-1}\mu\{A \,.\, c(Q,\ QR)\} \\ &> K' - 2(1+\alpha)\sin\chi\ \operatorname{cosec}(\tfrac{1}{4}\epsilon_0 + \tfrac{1}{2}\chi) \\ &> k, \end{aligned}$$

by suitable choice of λ, as a function of K, k, α. This contradicts (ii) among the properties of B, and the theorem is therefore proved.

6. *Measurability of the sets considered in Theorem* 4.

The set of points in space of (a, ϑ), at which equation (1) is true, is of the form $\Pi\Sigma F$; it is, as we have seen,

$$E\{f_3(a, \vartheta) \geqslant K\} . E\{f_3(a, \vartheta+\pi) < k\} = (\Pi\Sigma F)(\Sigma F);$$

and, since $A_3 = P\Pi\Sigma F$ has finite exterior measure, it follows from Theorem 1 that A_3 is measurable.

Next,

$$A_4 = A_3 . CP\Big\{E\{a, \vartheta;\ f_3(a, \vartheta) \geqslant K\} . \prod_n E\Big(f(a, \vartheta+\pi, \epsilon, r) > k - \frac{1}{n}\Big) . E(\epsilon < \epsilon_0,\ r < r_0)\Big\}$$

$$= A_3 . CP(\Pi\Sigma F . \Pi O . O) = A_3 . CP\Pi\Sigma F,$$

and is measurable in virtue of Theorem 1.

A_5 gives no difficulty.

Lastly,

$$A_6 = A_5 . E\,[\mu\{A . c(a, r)\} < 2r(1+\alpha),\ r < r_1]$$

and is measurable, being the product of A_5 and an open set.

7. We show now by examples that there are sets which have density directions, considering first an irregular set described by Besicovitch†, and then a class of sets investigated by Walker‡.

8. The first set is constructed thus: Let c be a circle of radius r: by the "operation of class n with respect to the circle c" we mean the construction of n circles as follows. Draw the concentric circle of radius $(1-1/n)r$, divide its circumference into n equal parts, and describe circles of radius r/n about the points of division.

We begin with c_1 of radius 1 and perform on it the operation of class 4, obtaining a set c_4 of four circles each of radius $\frac{1}{4}$ and each denoted by γ_4. We perform the operation of class 5 on each γ_4 and obtain the set c_5 of

† *Math. Annalen*, 98 (1928), 431–434. I should like to thank Mr. Besicovitch for directing my attention to this set.

‡ *Proc. London Math. Soc.* (2), 30 (1929), 481–499.

circles γ_5. And so on. We perform the operations in such a way that every γ_n lies entirely inside some γ_{n-2} (that is, γ_n and γ_{n-2} do not touch). Then A denotes the set $c_1 \times c_4 \times c_5 \times \ldots$, and it is shown, in the paper quoted, that $\mu A = 2$.

We shall prove here that every direction is a density direction at almost every point of A; and that the set A has a projection of zero measure in every direction. This shows how radically different is a density direction of an irregular set from a tangent of a regular set.

9. *At almost all points of A every direction is a density direction.*

We number the γ_n belonging to any γ_{n-1} as $\gamma_n(1), \gamma_n(2), \ldots, \gamma_n(n)$, and take any point a of A in $\gamma_n(1)$. We take for $\vartheta = 0$ the common tangent of γ_{n-1} and $\gamma_n(1)$. The common tangent of $\gamma_n(1)$ and $\gamma_n(\nu)$ (see Fig. 2)

Fig. 2.

makes an angle ϵ_n with $\vartheta = 0$. For large n, ν, and small ν/n we have $\epsilon_n \sim \nu\pi/n$. It is readily seen, therefore, that the sector $(a, 0, 2\epsilon_n, 2\nu\pi r_n)$ contains the circles $\gamma_n(i)$ for $(\sqrt{n})/\pi \leqslant i \leqslant \nu$; and that, for large n,

$$\frac{\mu(A, a, 0, 2\epsilon_n, 2\nu\pi r_n)}{2\nu\pi r_n} > \left(\nu - \frac{\sqrt{n}}{\pi}\right) \frac{2r_n}{2\nu\pi r_n} > \frac{1}{\pi} - \eta,$$

where η is any positive number, and ν is taken to be, say, $[n^{\frac{3}{4}}]$.

Any point a of A is strictly interior to a sequence of circles $\gamma_1 \supset \gamma_4 \supset \gamma_5 \supset \ldots$; moreover, a is not at the centre of any circle γ_n. Through a we draw the line t_n parallel to the common tangent of γ_n and γ_{n-1}. We show that any limit line of the sequence $(t_1, t_4, t_5, \ldots)$ is a density direction of A, having $\delta(A, a, \vartheta) \geqslant 1/\pi$.

Let ϑ_n denote the direction of t_n and ϑ that of t, any limit line of the sequence. Given any positive ϵ and n_0, we can find $n \geqslant n_0$ so that

$$|\vartheta - \vartheta_n| < \tfrac{1}{4}\epsilon, \quad \epsilon_n < \tfrac{1}{4}\epsilon.$$

It follows that $(a, \vartheta, \epsilon, r) \supset (a, \vartheta_n, 2\epsilon_n, r)$, and therefore

$$f(a, \vartheta, \epsilon, r) \geqslant f(a, \vartheta_n, 2\epsilon_n, r) \geqslant \pi^{-1} - \eta$$

for $r = 2\nu\pi r_n$. This proves our subsidiary assertion.

The main result is proved if we show that at almost every point of A the directions $\{t_n\}$ are dense in $(0, \pi)$.

If this is not so, the subset A_1 of A, of points at which the directions $\{t_n\}$ are not everywhere dense in $(0, \pi)$, has positive exterior measure. At every point of A_1 we define an interval of directions containing no t_n, and call such an interval "blank". A_2 is the subset of A_1 at which the blank interval is greater than $\delta > 0$. For small enough δ we have $\mu^* A_2 > 0$. Divide $(0, \pi)$ into N equal intervals less than $\frac{1}{2}\delta$, and divide A_2 into subsets $A_2^{(1)}, \ldots, A_2^{(N)}$, putting in $A_2^{(i)}$ those points of A_2 at which the bisector of the blank interval $(> \delta)$ lies in

$$\frac{(i-1)\pi}{N} \leqslant \vartheta < \frac{i\pi}{N}.$$

For some i, $\mu^* A_2^{(i)} > 0$; write $A_2^{(i)} = A_3$. Then the interval $\left(\frac{(i-1)\pi}{N}, \frac{i\pi}{N}\right)$ is blank for every point of A_3. Now this is impossible; in fact, if a set B of points of A is such that at no point of B is there a t_n lying in a certain angle of magnitude $\delta > 0$, then $\mu B = 0$. For there are (asymptotically) $(\delta n/\pi)$ γ_n in each γ_{n-1} such that the corresponding t_n lies within the angle δ. Thus within each γ_n, at most $(1-\delta/\pi)^p$ circles γ_{n+p} contain points of B; it follows that $\mu B = 0$.

10. *The set A has, in every direction, a projection of measure zero.*

In what follows, the statements are "approximately" true, in the sense that they are asymptotically true as n tends to infinity. Take any direction, any $\epsilon > 0$, and any γ_{n-1}. With vertex at centre of γ_{n-1}, and the given direction as bisector, we draw an angle $4\nu\pi/n$; it contains 4ν circles γ_n. (We shall suppose n and ν large, and ν/n small.) The measure of the projection of these circles on the given direction is less than $2r_{n-1}\{1-\cos(2\nu\pi n^{-1})\}$, that is, $4r_{n-1}\sin^2(\nu\pi n^{-1})$. We shall suppose that $4\sin^2(\nu\pi/n) < 4\nu\epsilon/n$. Then a part of A in γ_{n-1}, of measure $4\nu r_{n-1}/n$, has a projection of measure less than $4\nu\epsilon/n$. From each remaining γ_n in γ_{n-1} we remove a similar and similarly situated sector of angle $2\alpha = 4\nu\pi/n$, obtaining in all a further part of A of measure $4\nu n\ ^1(1-4\nu n^{-1})r_{n-1}$ with projection of measure less than $4\nu\epsilon n^{-1}(1-4\nu n^{-1})$; and so on. At any stage in this process we have

projected a part of A in γ_{n-1} of measure

$$r_{n-1}\frac{2\alpha}{\pi}\left\{1+\left(1-\frac{2\alpha}{\pi}\right)+\ldots+\left(1-\frac{2\alpha}{\pi}\right)^{m}\right\}$$

on a linear set of measure less than ϵr_{n-1}; and have left a remainder of A in γ_{n-1} of measure

$$r_{n-1}\left(1-\frac{2\alpha}{\pi}\right)^{m+1}.$$

By taking m large enough we see that the projection of $A\,.\,\gamma_{n-1}$ has measure not greater than $2\epsilon r_{n-1}$, and A therefore has a projection of measure not greater than 4ϵ, for $\sum_{\gamma_{n-1}} r_{n-1}=2$. This proves the second statement.

11. We consider the sets investigated by Walker for the reason that some of them have a unique density line at almost every point, and $\delta(A, a, \vartheta)=1$, which is the greatest value that it can assume; and that others have every direction a density direction at almost every point, with $\delta=1$ in every direction.

Construction.

If $\alpha>\frac{1}{3}$, let $m_0=1$. If $\alpha\leqslant\frac{1}{3}$, let m_0 be defined by

$$\tfrac{1}{2}\left(\frac{1}{\alpha}-1\right)<m_0\leqslant\tfrac{1}{2}\left(\frac{1}{\alpha}+1\right).$$

$$A=\frac{1}{2\alpha}\sqrt{\{1-4\alpha^2(m_0-\tfrac{1}{2})^2\}}\quad[0\leqslant A<\sqrt{(2m_0)}].$$

The operation $\Omega(n)$, performed on a rectangle R, with sides l and d $(d<l)$, is defined as follows. Divide d into m_0+1 equal parts, and denote by L_1' the set of m_0 lines parallel to the long side of R, drawn through the points of division. Construct parallel lines L_2' at a distance $x=Al/(m_0n^2)$ from each line of L_1', and on the same side of those lines. Divide these parallel lines into n^2 equal segments. On the odd segments of a line of L_1', and on the even segments of a line of L_2' (counting from one end of R), construct rectangles R_n of sides

$$\frac{l}{m_0n^2},\quad\frac{l}{m_0n^3},$$

concentric with the segments and having their long sides parallel to the segments. In addition, when $m_0=1$, we rotate the whole system of n^2 rectangles about the centre of R through an angle $\sin^{-1}(2n)^{-1}$. Then the

operation $\Omega(n)$ consists in replacing R by the $m_0 n^2$ rectangles R_n, all strictly interior to it. Take now

$$n_1 > 4(A+1),$$

$$n_i = n_1 + i - 1,$$

and perform the operation $\Omega(n_1)$ on the rectangle P_0 $(l = 1, d = \frac{1}{2})$: perform the operation $\Omega(n_2)$ on each of the $m_0 n_1^2$ rectangles so obtained; let P_2 denote the set of $m_0 n_1^2 . m_0 n_2^2$ rectangles R_2 so obtained, and also the set of points forming these rectangles; and so on. Of the set

$$P = P_0 \times P_1 \times P_2 \times \ldots$$

it is shown (*loc. cit.*) that

(i) P is perfect and nowhere dense,

(ii) $\mu P = 1$,

(iii) $D_*(a, P) = a$, p.p. in P $(a \leqslant \frac{1}{2})$,

(iv) $D^*(a, P) = 1$, p.p. in P.

We assert that, if $a \leqslant \frac{1}{3}$, the line parallel to the length of rectangles is a density direction, and is the only one, and that $\delta(P, a, \vartheta) = 1$, p.p. in P. And, if $\frac{1}{3} < a \leqslant \frac{1}{2}$, then every direction is a density direction with coefficient equal to 1 at almost every point of P. Since the proof is similar to that given above, but much simpler, we do not give it here.

12. Not every irregular set has density directions, as is shown by an example. Divide a square S into sixteen equal squares by parallels to its sides; let the operation W performed on S consist of replacing S by the four squares (taken closed) at the corners of S. Take a square c_0 of unit dimensions; c_1 is the set of points of the four squares WS; c_2 is the set of points of the sixteen squares obtained by applying W to each square of c_1; and so on. Thus c_n consists of the points of 4^n squares, each of side 4^{-n}, and each denoted by γ_n. The set†

$$A = c_0 \times c_1 \times c_2 \times \ldots$$

is perfect, nowhere dense, is irregular, and has measure $\sqrt{2}$. We shall refer to the part of A lying in a square γ_n as a "component" of A, of rank n.

† See, for example, W. Gross, *Monatshefte für Math. und Phys.*, 29 (1918), 168.

A has nowhere a density direction. Assign $\epsilon > 0$. It is enough to calculate an upper bound for $f(a, \vartheta, \epsilon, r)$ for values of r in the range $2(4^{-k-1}) \leqslant r \leqslant \sqrt{2}(4^{-k})$. [For this supposes the sector $(a, \vartheta, \epsilon, r)$ to contain points of more than one component of rank $k+1$, but of not more than one component of rank k.] We calculate, therefore, the maximum number of components of rank n (n much larger than k) lying in a component of rank $k+1$, and in a rectangle of shorter side $\epsilon . 4^{-k}\sqrt{2}$, which is the greatest width of the sector. It is evident that the sector contains less than twice this number of components of rank n, and the rectangle is most advantageously placed with its length parallel to the side of the square. Since a rectangle of breadth $4^{-k-1} . 4^{-m}$ contains at most $2^{n-m-k-1}$ rows of components of rank n, a rectangle of breadth $b = \epsilon . 4^{-k}\sqrt{2}$ contains at most

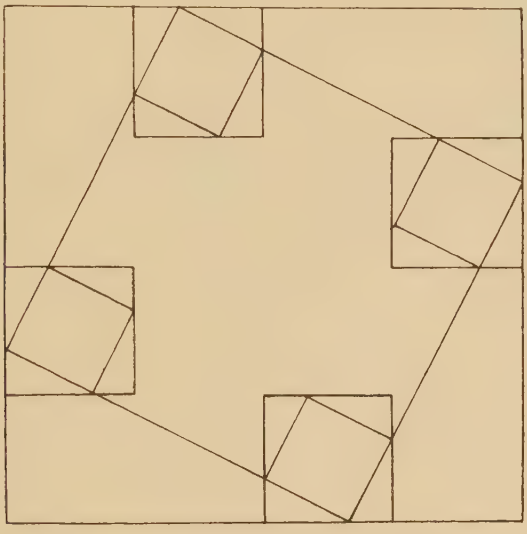

Fig. 3.

$2^n \sqrt{b} = 2^{n-k}\sqrt{(\epsilon\sqrt{2})}$ components of rank n, and there are 2^{n-k-1} components in each row. Therefore

$$f(a, \vartheta, \epsilon, r) \leqslant \frac{2 . 2^{n-k}\sqrt{(\epsilon\sqrt{2})} . 2^{n-k-1} . 4^{-n}\sqrt{2}}{2 . 4^{-k}} = \sqrt{\left(\frac{\epsilon}{\sqrt{2}}\right)}.$$

This proves our statement.

Referred to axes at angles $\tan^{-1}(\frac{1}{2})$, $\tan^{-1}(-2)$ to the base of the square c_0, the set A provides a biuniform correspondence between the intervals $\{0, \frac{1}{2}(1+\sqrt{5})\}$ on the two axes; there is, however, an ambiguity for an enumerable set of values in each interval. This is most readily verified by considering Fig. 3.

The functional relation is, in fact,

$$x = \sum_1^\infty a_n 2^{-n} \qquad (a_n = 0 \text{ or } 1),$$

$$y = f(x) = \sum_1^\infty \beta_n 2^{-n} \quad (\beta_{2n-1} = a_{2n},\ \beta_{2n} = 1 - a_{2n-1}).$$

This displays the source of the ambiguity in the double representation of terminating binary decimals. We see, therefore, that the curve of a uniform function, even of Baire class 1 as above, may have a density direction at no point.

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APPLICATIONS OF MELLIN'S TRANSFORMATION TO INTEGRAL EQUATIONS

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1. *Introduction.*

1.1. The object of this paper is to solve certain types of Fredholm's* equation by means of Mellin's transformation. The methods are similar to those used by Hardy and Titchmarsh† in their paper on Fourier kernels and also to applications of Laplace's transformation made by Doetsch‡ and Fock§.

The equation which we shall discuss is

$$F(x) = G(x) + \int_0^\infty H(ux)\, F(u)\, du, \tag{1.1.1}$$

where $G(x)$ and $H(x)$ are known functions and $F(x)$ is to be determined. The method also applies to simultaneous equations of the type

$$F_r(x) = G_r(x) + \int_0^\infty H_r(ux)\, F_{r+1}(u)\, du, \tag{1.1.2}$$

where $r = 1, 2, \ldots, n$; $F_{n+1}(x) = F_1(x)$; the functions $G_r(x)$ and $H_r(x)$ are known and $F_r(x)$ are to be determined. The case $n = 2$ will be briefly considered at the end of the paper.

1.2. Let $s = \sigma + it$, where σ and t are real. Then, as an illustration of the method, multiply equation (1.1.1) throughout by x^{s-1} and integrate

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* Solutions of Fredholm's equation and references to the literature of the subject are given by Whittaker and Watson, 1, Ch. 11, and Bôcher, 2.

† Hardy and Titchmarsh, 3.

‡ Doetsch, 4 and 5.

§ Fock, 6.

with respect to x. Adopting the notation of Hardy and Titchmarsh, we write

$$f(s) = \int_0^\infty F(x)\, x^{s-1}\, dx,$$

and similarly for other letters. We then have

$$f(s) = g(s) + \int_0^\infty dx \int_0^\infty H(ux)\, F(u)\, x^{s-1}\, du.$$

If we assume that we may change the order of integration, this becomes

$$f(s) = g(s) + h(s) f(1-s).$$

Changing s to $1-s$ we also have

$$f(1-s) = g(1-s) + h(1-s) f(s).$$

The elimination of $f(1-s)$ leads to

$$f(s)[1 - h(s)\, h(1-s)] = g(s) + h(s)\, g(1-s),$$

and so, when $h(s)\, h(1-s) \neq 1$, we may expect from Mellin's formula that a solution of (1.1.1) is given by

$$F(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{g(s) + h(s)\, g(1-s)}{1 - h(s)\, h(1-s)}\, x^{-s}\, ds. \tag{1.2.1}$$

Hardy and Titchmarsh have shown that, when $h(s)\, h(1-s) = 1$, $H(x)$ is a Fourier kernel and it is evident that in this case there can be no solution unless

$$g(s) + h(s)\, g(1-s) = 0,$$

a result which, as we shall show later, is equivalent to

$$G(x) + \int_0^\infty H(ux)\, G(u)\, du = 0, \tag{1.2.2}$$

that is to say, if $H(x)$ is a Fourier kernel, there is no solution of (1.1.1) unless $G(x)$ is a self-reciprocal function of the kernel $H(x)$ with characteristic number -1. This can also be proved very easily as follows:

$$\begin{aligned} F(x) &= G(x) + \int_0^\infty H(ux)\, F(u)\, du \\ &= G(x) + \int_0^\infty H(ux)\, G(u)\, du + \int_0^\infty \int_0^\infty H(ux)\, H(vu)\, F(v)\, du\, dv, \end{aligned}$$

which, if $H(x)$ is a Fourier kernel, reduces to (1.2.2).

In § 2 conditions for Mellin's formula are stated, and in §§ 3 and 4 we consider equations (1.1.1) and (1.1.2) respectively. In § 4.2 we illustrate the methods by solving the equation

$$F(x) = e^{-x} + \lambda \int_0^\infty e^{-ux} F(u)\, du.$$

2. *Mellin's formula.*

The well-known formula of Mellin, which is a variant of Fourier's transformation, is as follows:

$$f(s) = \int_0^\infty F(x)\, x^{s-1}\, dx, \tag{2.1}$$

$$F(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s)\, x^{-s}\, ds, \tag{2.2}$$

and corresponding to Parseval's* theorem we have

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s)\, g(1-s)\, x^{-s}\, ds = \int_0^\infty F(ux)\, G(u)\, du. \tag{2.3}$$

Conditions for the validity of these formulae are given by Bochner† and Pollard‡. Those given by Bochner are

B. *If*

(i) $\int_0^\infty F(x)\, x^{s-1}\, dx$ *is absolutely convergent in the strip*

$$\alpha < R(s) = \sigma < \beta,$$

(ii) $F(x)$ *is of bounded variation*§ *at the point* x,

then (2.1) *implies* (2.2).

If, in addition,

(iii) $\int_0^\infty G(x)\, x^{s-1}\, dx$ *converges absolutely in the same strip* $\alpha < \sigma < \beta$,

then (2.3) *is also true.*

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* I am indebted to the Referee for pointing out that (2.3) is Parseval's formula and also for the references to Doetsch.

† Bochner, **7**, 35, p. 148 (10); 36, p. 151, (5).

‡ Pollard, **8**, Theorem A., 462.

§ If $F(x)$ in (2.2) is not continuous it must be replaced by $\frac{1}{2}[F(x+0)+F(x-0)]$.

The conditions given by Pollard are

P. *If*

(i) *for a particular value of* c *and all positive values of* x, $|f(s)|$ *is bounded,*

(ii) *for some positive* a,

$$\int_0^\infty \frac{|f(c+it)+f(c-it)|}{t^2}\,dt \quad \text{and} \quad \int_{-a}^{a} |f(c+it)|\,dt$$

exist,

(iii) $F(x)$ *has a finite integral in every finite interval consisting of positive values of* x,

then, for almost all values of s *whose real part is* c, (2.2) *implies* (2.1).

3. *Theorem* 1.

3.1. *Suppose that* $h(s)h(1-s) \neq 1$ *and let*

$$k(s) = \frac{g(s)+g(1-s)h(s)}{1-h(s)h(1-s)}.$$

Then a solution of equation (1.1.1) *is given by* (1.2.1) *if there is a common strip* $\alpha < \sigma = R(s) < \beta$ *for which the functions* $g(s)$, $h(s)$, $k(s)$, $G(x)$, $H(x)$, $K(x)$ *all satisfy conditions* B *and* P *above.*

If $h(s)h(1-s) = 1$, *then there can be a solution of* (1.1.1) *only if* $G(x)$ *is s self-reciprocal function of the kernel* $H(x)$ *with characteristic number* -1.

Proof. The conditions enable us to apply Mellin's formula to (1.2.1) when we have

$$f(s) = \frac{g(s)+g(1-s)h(s)}{1-h(s)h(1-s)}. \tag{3.1.1}$$

Writing $1-s$ for s, we have

$$f(1-s) = \frac{g(1-s)+g(s)h(1-s)}{1-h(s)h(1-s)}.$$

On multiplying this by $h(s)$ and subtracting from (3.1.1), we get

$$f(s) = g(s)+h(s)f(1-s).$$

Now multiply this equation by x^{-s} and integrate along the line $\sigma = c$, where $\alpha < c < \beta$. Then, on using (2.3), we finally obtain

$$F(x) = G(x)+\int_0^\infty H(ux)\,F(u)\,du.$$

If $h(s)h(1-s)=1$ there can evidently be no solution of (1.1.1) unless

$$g(s)+g(1-s)h(s)=0.$$

On multiplying this equation by x^{-s}, integrating along the line $\sigma=c$, and using (2.3) again, we find that this condition becomes equivalent to

$$G(x)=-\int_0^\infty H(ux)\,G(u)\,du,$$

a result already proved in §1.2 by a different method.

3.2. As an illustration consider the equation

$$F(x)=e^{-x}+\lambda\int_0^\infty e^{-ux}F(u)\,du. \tag{3.2.1}$$

We have $$g(s)=\Gamma(s),\quad h(s)=\lambda\Gamma(s).$$

Hence, subject to the conditions of §2 being satisfied, we have

$$\begin{aligned}F(x)&=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s)+\lambda\Gamma(s)\,\Gamma(1-s)}{1-\lambda^2\,\Gamma(s)\,\Gamma(1-s)}\,x^{-s}\,ds\\&=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s)\sin s\pi+\lambda\pi}{\sin s\pi-\lambda^2\pi}\,x^{-s}\,ds.\end{aligned}$$

The integrand has poles at $\sin s\pi=\lambda^2\pi$. If $0\leqslant\lambda^2\pi\leqslant 1$ and m is any integer, then there are two poles on the real axis lying between $2m$ and $2m+2$, at $s=2m+p$ and $s=2m+1-p$ say, where $0<p<1-p<1$. All the poles are then given by

$$s=2m+p+2n,\quad s=2m+1-p+2n,$$

where n has any integral value, positive, negative, or zero. For $0<x<1$, $F(x)$ can be evaluated by considering the residues of $f(s)$ on the left of the line $\sigma=c$. For $x>1$,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s)\sin s\pi}{\sin s\pi-\lambda^2\pi}\,x^{-s}\,ds$$

can be evaluated as before by considering the residues on the left of the line $\sigma=c$ and

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\lambda\pi}{\sin s\pi-\lambda^2\pi}\,x^{-s}\,ds$$

by considering the residues on the right of the line $\sigma=c$. The result is the same whether $0<x<1$ or $x>1$, but depends upon the value of c. There are two cases according as

$$2m+p<c<2m+1-p\quad\text{or}\quad 2m+1-p<c<2m+2+p.$$

Dealing with the second case first, we have

$$(3.2.2)\quad F(x)=\frac{\lambda}{\cos p\pi}x^{-2m}\left(\frac{x^{-p}-x^{p-1}}{|1-x^2|}\right)+\frac{x^{-2m}}{\cos p\pi}\sum_{n=0}^{\infty}\left\{\frac{x^{2n-p}}{\Gamma(1-p+2n-2m)}-\frac{x^{2n+p-1}}{\Gamma(p+2n-2m)}\right\}.$$

In order to determine the interval in which c lies we observe that, in virtue of the conditions of § 2, the integral

$$(3.2.3)\qquad \int_0^{\infty}F(x)x^{s-1}dx$$

must converge absolutely. For convergence at the lower limit we must have $m\leqslant 0$. For convergence at the upper limit we can easily find the asymptotic expansion of the series on the right-hand side of (3.2.2) by a method such as Barnes's* method and it is then found that, for sufficiently large values of x,

$$|F(x)|<Ax^{-2m-2-p},$$

where A is a constant independent of x. Hence $m>-1+\frac{1}{2}(1-p)$ and so $m=0$. The solution then becomes

$$(3.2.4)\quad F(x)=\frac{\lambda}{\cos p\pi}\left(\frac{x^{-p}-x^{p-1}}{|1-x^2|}\right)+\frac{1}{\cos p\pi}\sum_{n=0}^{\infty}\left\{\frac{x^{2n-p}}{\Gamma(1-p+2n)}-\frac{x^{2n+p-1}}{\Gamma(p+2n)}\right\}.$$

If we deal with the case when $2m+p<c<2m+1-p$ by the same methods, we find that it is impossible to make (3.2.3) converge absolutely for any value of m. Hence (3.2.4) is the only solution of (3.2.1).

We must still deal with the case $\lambda^2\pi\geqslant 1$. When $\lambda^2\pi=1$ the equation $\sin s\pi=1$ has double zeros at $s=2n+\frac{1}{2}$ (n any positive or negative integer or zero). The corresponding residues are

$$A_n x^{2n-\frac{1}{2}}+B_n x^{2n-\frac{1}{2}}\log x,$$

where A_n and B_n are constants. These constants are not difficult to evaluate, but it is laborious to do so and consequently we shall not calculate them.

When $\lambda^2\pi>1$, the equation $\sin s\pi=\lambda^2\pi$ has only complex roots. In the interval $0<\sigma<1$ it has two roots, μ and ν say, both of which lie on the line $\sigma=\frac{1}{2}$. The general roots are then $2n+\mu$, $2n+\nu$ (n any positive or negative integer or zero) and the solution of (3.2.1) is substantially the same as that given by (3.2.4).

* Barnes, **9**.

It is worth noting that the solution of the equation

$$F(x) = \lambda \int_0^\infty e^{-ux} F(u)\, du \tag{3.2.5}$$

also involves the consideration of the roots of the equation $\sin s\pi = \lambda^2 \pi$. When $\lambda^2 \pi \neq 1$, a solution of (3.2.5) is

$$A\left\{\lambda x^{-p} - \frac{x^{p-1}}{\Gamma(p)}\right\}, \tag{3.2.6}$$

where (i) in the case $\lambda^2 \pi < 1$, p is the smallest positive root of $\sin \pi = s\lambda^2 \pi$ and (ii) in the case $\lambda^2 \pi > 1$, p is a root on the line $\sigma = \frac{1}{2}$. The expression (3.2.6) with $A = \sec p\pi$ forms part of the right-hand side of (3.2.4)*.

4. *Simultaneous equations.*

The simultaneous equations

$$\left\{\begin{aligned} F_1(x) &= G_1(x) + \int_0^\infty H_1(ux)\, F_2(u)\, du, \\ F_2(x) &= G_2(x) + \int_0^\infty H_2(ux)\, F_1(u)\, du, \end{aligned}\right. \tag{4.1}$$

can be dealt with in the same way as (1.1.1). I state the results without proof.

If $h_1(s)\, h_2(1-s) \neq 1$, then the solutions of equations (4.1) are

$$\left\{\begin{aligned} F_1(x) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{g_1(s) + h_1(s)\, g_2(1-s)}{1 - h_1(s)\, h_2(1-s)}\, x^{-s}\, ds, \\ F_2(x) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{g_2(s) + h_2(s)\, g_1(1-s)}{1 - h_2(s)\, h_1(1-s)}\, x^{-s}\, ds. \end{aligned}\right. \tag{4.2}$$

The conditions for the validity of these solutions are easily determined with the help of the statements of § 2.

If $h_1(s)\, h_2(1-s) = 1$, then there can be solutions of (4.1) only if

$$G_1(x) + \int_0^\infty H_1(ux)\, G_2(u)\, du = 0,$$

$$G_2(x) + \int_0^\infty H_2(ux)\, G_1(u)\, du = 0,$$

that is to say, if $G_1(x)$ and $-G_2(x)$ are transforms of each other for the pair of Fourier kernels $H_1(x)$, $H_2(x)$.

* See Hardy and Titchmarsh, **10**, where equation (3.2.5) is discussed and references to other literature on the subject are given.

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ON THE DIRECTIONS OF BOREL OF FUNCTIONS WHICH ARE REGULAR AND OF FINITE ORDER IN AN ANGLE

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1.1. Let $f(z)$ be regular and of finite proximate order $\rho(r)$ for $|z| \geqslant l$, $\alpha \leqslant \arg z \leqslant \beta$. The function* $\rho(r)$ is a real continuous function, differentiable on the right and on the left for $r > r_0$; and, as $r \to \infty$, $\rho(r) \to \rho$, where $0 < \rho < \infty$. Let

$$h(\theta) = \overline{\lim_{r \to \infty}} \frac{\log |f(re^{i\theta})|}{r^{\rho(r)}}$$

for $\alpha \leqslant \theta \leqslant \beta$; then $h(\theta) < \infty$ for $\alpha \leqslant \theta \leqslant \beta$.

This paper is a continuation of my two previous papers†, and the principal object of it is to prove two theorems, which are limiting cases of theorems in my former papers‡. Let $n(r, \alpha', \beta')$ denote the number of zeros of $f(z)$ for which $l \leqslant |z| \leqslant r$, $\alpha < \alpha' \leqslant \arg z \leqslant \beta' < \beta$, and let A and B denote positive or negative real numbers which remain the same throughout a theorem and its proof. I showed that, *if*

$$h(\theta) = A \cos \theta\rho + B \sin \theta\rho, \tag{1.11}$$

for $\alpha \leqslant \theta \leqslant \beta$, *where* $\beta - \alpha > \pi/\rho$, *then, as* $r \to \infty$

$$n(r, \alpha+\delta, \beta-\delta) = o(r^{\rho(r)})$$

for every $\delta > 0$. If $A = B = 0$, then $f(z) - a$ also satisfies (1.11) for $\alpha \leqslant \theta \leqslant \beta$ for every a, except perhaps one; and it follows that $f(z)$ has no

* See G. Valiron, *Lectures on the general theory of integral functions* (1933), 64. For functions of order ρ and mean type $\rho(r) \equiv \rho$.

† M. L. Cartwright, *Proc. London Math. Soc.* (2), 38 (1935), 158–179, 417–457.

‡ *Loc. cit.*, 166, Theorem II, and *Comptes rendus*, 194 (1932) 1889–1891.

directions of Borel* of proximate order $\rho(r)$ for which $\alpha < \arg z < \beta$. I also showed that, *if* $h(\theta_0) > 0$ *for some* θ_0 *such that* $\alpha < \theta_0 < \beta$, *and if* $\beta - \alpha > \pi/\rho$, *then* $f(z)$ *has a direction of Borel of proximate order* $\rho(r)$ *in the angle* $\alpha < \arg z < \beta$.

These two theorems are concerned with an angle greater than π/ρ; the two main theorems which I shall prove here are the corresponding theorems for an angle equal to π/ρ. In order to avoid complications we suppose that $f(z)$ *is of proximate order* $\rho(r)$ *in an angle*, $\alpha \leqslant \arg z \leqslant \beta$, *containing the angle of the theorem*†, which we take to be $|\arg z| \leqslant \frac{1}{2}\pi/\rho$.

THEOREM A. *If* (1.11) *is satisfied for* $\alpha < -\frac{1}{2}\pi/\rho \leqslant \theta \leqslant \frac{1}{2}\pi/\rho < \beta$, *then, as* $r \to \infty$,

$$n(r, -\tfrac{1}{2}\pi/\rho + \delta, \tfrac{1}{2}\pi/\rho - \delta) = o(r^{\rho(r)})$$

for every $\delta > 0$. *In particular, if* $B = 0$ *so that*

$$h(\theta) = h(0)\cos\theta\rho \tag{1.12}$$

for $|\theta| \leqslant \frac{1}{2}\pi/\rho$, *then* $f(z)$ *has no direction of Borel of proximate order* $\rho(r)$ *for which* $|\arg z| < \frac{1}{2}\pi/\rho$.

THEOREM B. *If* $f(z)$ *has no direction of Borel of proximate order* $\rho(r)$ *for which* $\alpha \leqslant -\frac{1}{2}\pi/\rho < \arg z < \frac{1}{2}\pi/\rho \leqslant \beta$, *then* (1.12) *holds for*

$$|\theta| \leqslant \tfrac{1}{2}\pi/\rho.$$

Theorem A yields some interesting results when it is applied to integral functions of order $\rho > 1$ whose directions of Borel of proximate order $\rho(r)$ lie in an angle not greater than π/ρ. For if we suppose that all the directions of Borel lie in the angle $|\arg z| \leqslant \frac{1}{2}\pi/\rho$, there are none in the angle $\frac{1}{2}\pi/\rho < \arg z < 2\pi - \frac{1}{2}\pi/\rho$. Since $\rho > 1$, this angle is greater than π/ρ, and so it follows from Theorem B‡ that $h(\theta) = 0$ for

$$\tfrac{1}{2}\pi\rho < \theta < 2\pi - \tfrac{1}{2}\pi\rho.$$

It then follows from the properties of $h(\theta)$ that

$$h(\theta) = h(0)\cos\theta\rho \quad \text{for} \quad |\theta| \leqslant \tfrac{1}{2}\pi\rho.$$

* For a definition see § 2.2.

† In the applications to integral functions of proximate order $\rho(r)$ this hypothesis is, of course, always satisfied. See § 5.1.

‡ This also follows from the cruder theorem quoted above. See M. L. Cartwright, *Comptes rendus*, 194 (1932), 1889–1891.

Hence these functions have no direction of Borel of proximate order $\rho(r)$ inside the angle $|\arg z| < \frac{1}{2}\pi/\rho$; and so $\arg z = \pm\frac{1}{2}\pi/\rho$ are the only directions of Borel of proximate order $\rho(r)$. A similar result for functions of order 1 can be obtained by using Theorems A and B. In proving Theorem A we actually prove considerably more about the general behaviour of $f(z)$, and by using the complete result we can determine very precisely the number of zeros of the functions whose directions of Borel lie in an angle π/ρ.

My thanks are due to Dr. E. F. Collingwood, whose advice has been a great help to me in writing this and the preceding paper on the same subject.

2.1. We have already stated some of the properties of $\rho(r)$. It is also differentiable in adjacent intervals for $r > r_0$, and

$$\lim_{r\to\infty} r \log r\, \rho'(r) = 0,$$

where $\rho'(r)$ denotes either the right- or the left-hand derivative. Further, there is a function* $V(z)$, regular for $|z| \geqslant r_0$, $\alpha \leqslant \arg \leqslant \beta$, such that, as $r \to \infty$,

$$\frac{V(kr)}{V(r)} \sim k^\rho, \text{ and } V(re^{i\theta}) \sim r^{\rho(r)} \cos\theta\rho, \tag{2.11}$$

for $\alpha \leqslant \theta \leqslant \beta$. The function $f(z)$ is said to be of proximate order $\rho(r)$ for $|z| \geqslant l$, $\alpha \leqslant \arg z \leqslant \beta$, if

$$\overline{\lim_{r\to\infty}} \frac{\log M(r, \alpha, \beta)}{r^{\rho(r)}} = K,$$

where

$$M(r, \alpha, \beta) = \max_{\alpha \leqslant \theta \leqslant \beta} \log|f(re^{i\theta})| \quad (0 < K < \infty).$$

2.2. Directions of Borel of proximate order $\rho(r)$ may be defined in various ways. Let $n(r, \alpha', \beta', a)$ denote the number of zeros of $f(z)-a$ for which $l \leqslant |z| \leqslant r$, $\alpha < \alpha' \leqslant \arg z \leqslant \beta' < \beta$, and suppose that

$$\overline{\lim_{r\to\infty}} n(r, \gamma-\delta, \gamma+\delta, a)/r^{\rho(r)} = \zeta(\delta) > 0$$

for every $\delta > 0$. Then $\arg z = \gamma$ is said to be *a direction of Borel of proximate*

* In order to avoid complicated indices, we shall sometimes write $V(r)$ in place of $r^{\rho(r)}$. This makes no real difference to the working.

order $\rho(r)$ *of* $f(z)$. A more precise definition was given in my former paper* and shown to be equivalent to this one. If $f(z)$ is of proximate order $\rho(r)$ in the angle $\alpha \leqslant \arg z \leqslant \beta$, then a direction of Borel of proximate order $\rho(r)$ is said to be *a direction of Borel of the maximum kind.*

2.3. A summary of the results about $h(\theta)$, with references, will be found in my former paper†. Since $f(z)$ is of proximate order $\rho(r)$ in the angle $\alpha \leqslant \arg z \leqslant \beta$, given any $\epsilon > 0$, we can find $r_0 = r_0(\epsilon)$ such that

$$\log|f(re^{i\theta})| < \{h(\theta)+\epsilon\} r^{\rho(r)} \tag{2.31}$$

for $\alpha \leqslant \theta \leqslant \beta$, $r \geqslant r_0$. Also there is a number K $(0 < K < \infty)$ such that $h(\theta) \leqslant K$ for $\alpha \leqslant \theta \leqslant \beta$, and $h(\theta_0) = K$ for some θ_0 such that $\alpha \leqslant \theta_0 \leqslant \beta$.

Most of the other properties of $h(\theta)$ can be deduced from the following inequality‡:

$$h(\theta_1)\sin\rho(\theta_3-\theta_2)+h(\theta_2)\sin\rho(\theta_1-\theta_3)+h(\theta_3)\sin\rho(\theta_2-\theta_1) \geqslant 0, \tag{2.32}$$

where $$\alpha \leqslant \theta_1 < \theta_2 < \theta_3 \leqslant \beta, \quad \theta_2-\theta_1 < \pi/\rho, \quad \theta_3-\theta_2 < \pi/\rho.$$

In particular, if $\beta-\alpha > \pi/\rho$, $h(\theta)$ is finite, and continuous and differentiable on the right and on the left for $\alpha < \theta < \beta$; we denote by $h'(\theta+)$, $h'(\theta-)$ the right- and left-hand derivatives respectively.

2.4. The following lemma will be frequently used. The first part is trivial, the second is due in its present form to V. Bernstein§, and the third is taken from one of my former papers||.

LEMMA 1. *Suppose that* $F(z)$ *is regular for* $|z| \leqslant R$,

$$\log|F(0)| \geqslant -K,$$

$$\log|F(re^{i\theta})| \leqslant K \quad \textit{for} \quad r \leqslant R.$$

Then (1) $n(r) < 2K/\log(R/r)$, *where* $n(r)$ *denotes the number of zeros of* $F(z)$ *for which* $|z| \leqslant r < R$.

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* M. L. Cartwright, *Proc. London Math. Soc.* (2), 38 (1935), 421.

† *Loc. cit.*, 420.

‡ See G. Pólya, *Math. Zeitschrift*, 29 (1929), 573. Another form is as follows: If $h(\theta) \leqslant A\cos\theta\rho + B\sin\theta\rho$, for $\theta = \theta_1$, $\theta = \theta_2$, where $|\theta_2-\theta_1| \leqslant \pi/\rho$, then $h(\theta) \leqslant A\cos\theta\rho + B\sin\theta\rho$ for $\theta_1 \leqslant \theta \leqslant \theta_2$.

§ V. Bernstein, *Annali di Mat.* (4), 12 (1934), 179.

|| M. L. Cartwright, *Quart. J. of Math.* (Oxford Series), 1 (1930), 44–46.

(2) *Given any* $k>1$, $\zeta>0$, *we can choose* $H^*=H(\zeta, k)$ *independently of* z *and* K *so that*

$$\log|F(z)|>-HK$$

for $|z|\leqslant R/k$, *except in circles the sum of whose radii is less than* ζR.

(3) *Given any* $k>1$, *we can choose* $H=H(k)$ *so that*

$$\left|\int d\{\arg F(z)\}\right|<HK, \tag{2.41}$$

where the integral is taken along any arc of $|z|=r<R/k$ *less than or equal to a whole circle, or along any curve inside* $|z|=r$ *without double points whose ends lie on* $|z|=r$.

It follows from (1) that, if (2.41) holds with the integral taken along the arc of $|z|=r$, it will also hold along any curve inside $|z|=r$ without double points whose ends lie on $|z|=r$. For in deforming one curve into the other we meet not more than $2K/\log k$ zeros, so that (2.41) is not increased by more than $4\pi K/\log k$.

3.1. The complete form of Theorem A is as follows.

THEOREM I. *Suppose that* $f(z)$ *is regular and of proximate order* $\rho(r)$ *for* $|z|\geqslant l$, $\alpha\leqslant\arg z\leqslant\beta$, *and that*

$$h(\theta)=A\cos\theta\rho+B\sin\theta\rho, \tag{3.11}$$

where

$$\alpha<-\tfrac{1}{2}\pi/\rho\leqslant\theta\leqslant\tfrac{1}{2}\pi/\rho<\beta.$$

Then (1) *as* $r\to\infty$,

$$n(r,\ -\tfrac{1}{2}\pi/\rho+\delta,\ \tfrac{1}{2}\pi/\rho-\delta)=o(r^{f(r)}) \tag{3.12}$$

for every $\delta>0$.

(2) *Given* $\epsilon, \delta, \eta, \zeta>0$, *we can choose* $R_0=R_0(\epsilon, \delta, \eta, \zeta)$ *so that*

$$\left|\log|f(re^{i\theta})|-\{A(R)\cos\theta\rho+B\sin\theta\rho\}r^{f(r)}\right|<\epsilon r^{f(r)}, \tag{3.13}$$

where

$$\overline{\lim_{R\to\infty}}\,A(R)=A,\quad \underline{\lim_{R\to\infty}}\,\rho A(R)\geqslant\max\{-h'(-\tfrac{1}{2}\pi/\rho-);\ -h'(\tfrac{1}{2}\pi/\rho+)\}, \tag{3.14}$$

* I use H throughout to denote a constant which may depend on parameters, ζ, k, say. H need not be the same in each place, but H_1, H_2, K preserve their identity throughout a theorem and its proof. K usually refers to the proximate order of $f(z)$ as defined in § 2.1.

for $|\theta| \leqslant \frac{1}{2}\pi/\rho - \delta$, $\eta R \leqslant r \leqslant R$, $R > R_0$, *except in circles the sum of whose radii is less than* ζR.

(3) *If* $B = 0$, *then* $f(z)$ *has no direction of Borel of proximate order* $\rho(r)$ *for which* $|\arg z| < \frac{1}{2}\pi/\rho$.

The last part follows immediately from (1). For $h(\theta)$ is the same for $f(z) - a$ for every value of a if $A > 0$, and $h(\theta) \leqslant 0$ otherwise.

It should be observed that, putting $\theta_1 = 0$, $\theta_2 = \frac{1}{2}\pi/\rho$, $\theta_3 = \frac{1}{2}\pi/\rho + \delta$ in (2.32), we have

$$h(0)\sin\delta\rho - h(\tfrac{1}{2}\pi/\rho)\cos\delta\rho + h(\tfrac{1}{2}\pi/\rho+\delta) \geqslant 0,$$

and so

$$A = h(0) \geqslant -\{h(\tfrac{1}{2}\pi/\rho+\delta) - h(\tfrac{1}{2}\pi/\rho)\}\operatorname{cosec}\delta\rho - h(\tfrac{1}{2}\pi/\rho)(1-\cos\delta\rho)\operatorname{cosec}\delta\rho$$

$$(3.15) \qquad \geqslant -\frac{h(\frac{1}{2}\pi/\rho+\delta) - h(\frac{1}{2}\pi/\rho)}{\delta}\,\frac{\delta}{\sin\delta\rho} - O(\delta) \to -\frac{h'(\frac{1}{2}\pi/\rho+)}{\rho}$$

as $\delta \to 0$. Similarly $\rho A \geqslant -h'(-\frac{1}{2}\pi/\rho-)$. Hence, if

$$\max\{-h'(-\tfrac{1}{2}\pi/\rho-);\ -h'(\tfrac{1}{2}\pi/\rho+)\} = \rho A,$$

we have

$$\overline{\lim}\, A(R) = \underline{\lim}\, A(R) = \lim A(R) = A;$$

and so

$$\log|f(re^{i\theta})| \sim (A\cos\theta\rho + B\sin\theta\rho)\, r^{\rho(r)}$$

in a set of linear* density 1.

3.2. The following result is considerably easier to obtain than Theorem I, and will be used as an intermediate step.

THEOREM II. *Suppose that* $f(z)$ *is regular and of proximate order* $\rho(r)$ *for* $|z| \geqslant l$, $\alpha \leqslant \arg z \leqslant \beta$, *where* $\beta - \alpha > \pi/\rho$. *Then* (1)

$$n(r,\ \alpha+\delta,\ \beta-\delta) < KH(\delta)\, r^{\rho(r)}$$

for every $\delta > 0$, $r > l$, (2) *given* $\delta > 0$, $\eta > 0$, $\zeta > 0$, *we can choose* $H = H(\delta, \eta, \zeta)$ *and* $R_0(\delta, \eta, \zeta)$ *so that*

$$(3.21) \qquad \log|f(re^{i\theta})| > -KH\, r^{\rho(r)}$$

* See M. L. Cartwright, *Proc. London Math. Soc.* (2), 38 (1935), 166.

for $\alpha+\delta \leqslant \theta \leqslant \beta-\delta$, $\eta R \leqslant r \leqslant R$, $R > R_0$, *except in circles the sum of whose radii is less than* ζR.

This may easily be proved by following the method of Theorem II of my former paper*. In that theorem $h(\theta)$ was of the form $A\cos\theta\rho + B\sin\theta\rho$, but we reduced the proof to a consideration of the function

$$F(z) = f(z)\exp\{-(A-iB)V(z)\}$$

for which $h(\theta) \equiv 0$. Hence we see that this theorem is the O result corresponding to the o result in my former paper.

Proof of Theorem II. As we have already mentioned, since $\beta-\alpha > \pi/\rho$, $h(\theta)$ is finite for $\alpha \leqslant \theta \leqslant \beta$. Suppose for convenience that $\alpha = -\beta$, so that $\beta > \frac{1}{2}\pi/\rho$. Then it follows from (2.32) that

$$h(0) \geqslant \tfrac{1}{2}\{h(\beta)+h(-\beta)\}\sec\beta\rho.$$

Consider the function

$$F(z) = f(z)\exp\{(C+iD+\zeta)V(z) - \xi V(r_0)\},$$

where $\zeta > 0$, $\xi > 0$,

$$C = -\tfrac{1}{2}\{h(\beta_1)+h(-\beta_1)\}\sec\beta_1\rho,\quad D = \tfrac{1}{2}\{h(\beta_1)-h(-\beta_1)\}\operatorname{cosec}\beta_1\rho,$$

and $\frac{1}{2}\pi/\rho < \beta_1 < \beta$. It follows from (2.31) that, given any $\epsilon_1 > 0$, we have

$$\log|f(re^{i\theta})| < \{h(\theta)+\epsilon_1\}r^{\rho(r)}$$

for $|\theta| \leqslant \beta_1$, $r > r_0(\epsilon_1)$; and, since $\zeta\cos\beta_1\rho > 0$, it follows from (2.11) that

$$\log|F(re^{i\theta})| < 0$$

for $\theta = \pm\beta_1$, $r > r_0$, provided that $r_0 = r_0(\zeta)$ is chosen sufficiently large. We choose ξ so that

$$\log|F(r_0 e^{i\theta})| < 0$$

for $|\theta| \leqslant \beta_1$.

* See M. L. Cartwright, *Proc. London Math. Soc., ibid.* See also M. L. Cartwright, *Quart. J. of Math.* (Oxford Series), 4 (1933), 246–257; the working of Lemma 2 is the same as that for the above theorem simplified by the use of Carathéodory's theorem in place of Lemma 1 of this paper. In this theorem the constants $H(\delta)$ and H depend on the function f.

It follows from the definition of $h(0)$ that, given any $\epsilon_2 > 0$, we can find a sequence $R_1, R_2, \ldots, R_m \to \infty$ such that

$$\log|f(R_m)| > \{h(0)-\epsilon_2\} V(R_m) \geqslant [\tfrac{1}{2}\{h(\beta_1)+h(-\beta_1)\}\sec\beta_1\rho-\epsilon_2] V(R_m).$$

Hence

$$\log|F(R_m)| > (\zeta-\epsilon_2) V(R_m)-\xi V(r_0) > 0,$$

provided that $\epsilon_2 < \zeta$, $m \geqslant m_0(\epsilon_1, \epsilon_2, \xi)$. Applying the maximum modulus principle to $F(z)$ in the region $r_0 \leqslant r \leqslant R$, $R > R_{m_0}$, $|\theta| \leqslant \beta_1$, we see that there must be a point $Re^{i\Theta}$, on $|z| = R$, $|\theta| \leqslant \beta_1$, at which

$$\log|F(Re^{i\Theta})| > 0.$$

Taking this point as centre, we apply Lemma 1 to $F(z)$ in the circle

$$|z-Re^{i\Theta}| \leqslant R\sin(\beta-\beta_1),$$

and we have

$$\log|F(z)| > -H(\eta) K R^{\rho(R)}$$

for $|z-Re^{i\Theta}| \leqslant \frac{1}{2}R\sin(\beta-\beta_1)$, except in circles the sum of whose radii is less than $\eta R\sin(\beta-\beta_1)$; and so

$$\log|F(Re^{i\theta})| > -H(\eta) K R^{\rho(R)} \tag{3.22}$$

for $|\theta-\Theta| \leqslant \frac{1}{2}(\beta-\beta_1)$, except perhaps in a set of θ of measure less than $\eta(\beta-\beta_1)$. Let $Re^{i\Theta_1}$ be one of the points at which (3.22) holds, and apply the lemma to $F(z)$ again in the circle $|z-Re^{i\Theta_1}| \leqslant R\sin(\min|\beta\mp\Theta_1|)$. After not more than $2\beta/(\beta-\beta_1)$ applications of Lemma 1, we obtain a point R' on the real axis such that $|R'-R| \leqslant \eta R$ and

$$\log|F(R')| > -H(\eta) K R^{\rho(R)}.$$

If $\beta = \frac{1}{2}\pi$, we can obtain the required results by applying Lemma 1 in the circle $|z-R'| \leqslant R'-l$; and if $\beta \neq \frac{1}{2}\pi$, we put $z = w^{\frac{1}{2}\pi/\beta}$, apply the lemma, and then transform back again.

3.3. It appears from the form of the result (3.13) [which is, fundamentally, a necessary preliminary to obtaining (3.12)] that we have to consider regions where $|f(z)|$ is comparatively small, as well as those where $|f(z)|$ is large and therefore obviously controlled by $h(\theta)$. In order to do this we require a lemma which is similar to the fundamental theorem*

* See (2.32) and foot-note. Compare H. Milloux, *Acta Math.*, 52 (1928), 225.

about $h(\theta)$ for a certain sequence of intervals of $|z|$. We prove it in a general form for a sequence of functions corresponding to the intervals of $|z|$, because the general form will be needed in the proof of Theorem III.

LEMMA 2. *Let $R_1, R_2, \ldots, R_m \to \infty$, be a sequence of positive numbers. Suppose that, for every $\eta > 0$, we have a sequence of functions $F_m(z, \eta)$ which are regular and of the form*

$$\log|F_m(re^{i\theta}, \eta)| < Kr^{\rho(r)}$$

for $|\theta| \leqslant \gamma < \frac{1}{2}\pi/\rho$, $\eta^2 R_m \leqslant r \leqslant R_m/\eta^2$, $m > m_0(\eta)$. Suppose also that

$$\log|F_m(re^{i\theta}, \eta)| \leqslant \{A(R_m, \eta)\cos\theta\rho + B(R_m, \eta)\sin\theta\rho\} r^{\rho(r)},$$

where $|A(R_m, \eta)| + |B(R_m, \eta)| < K$, for $\theta = \pm\gamma$, $\eta^2 R_m \leqslant r \leqslant R_m/\eta^2$, $m > m_0(\eta)$. Then, given any $\epsilon > 0$, we can choose $\eta_0(\epsilon)$ so that

$$\log|F_m(re^{i\theta}, \eta)| < \{A(R_m, \eta)\cos\theta\rho + B(R_m, \eta)\sin\theta\rho + \epsilon\} r^{\rho(r)}$$

for $|\theta| \leqslant \gamma$, $\eta R_m \leqslant r \leqslant R_m/\eta$, provided that $\eta < \eta_0(\epsilon)$, $m > m_0(\epsilon, \eta)$.

Proof. Let

$$\Phi_m(z, \eta) = F_m(z, \eta)\exp[-\{A(R_m, \eta) - iB(R_m, \eta)\} V(z) - \zeta z^{\rho_1} - \xi(\eta^2 R_m)^{\rho(\eta^2 R_m)}],$$

where $\zeta > 0$, $\xi > 0$, $\rho < \rho_1 < \frac{1}{2}\pi/\gamma$, and $V(z)$ is the function referred to in §2.1. Then

$$|\Phi_m(z, \eta)| < 1 \tag{3.31}$$

on the boundary of the region $|\arg z| \leqslant \gamma$, $\eta^2 R_m \leqslant |z| \leqslant R_m/\eta^2$, provided that ζ and ξ are suitably chosen, and m is sufficiently large. For (3.31) is obviously satisfied on $\arg z = \pm\gamma$, $\eta^2 R_m \leqslant |z| \leqslant R_m/\eta^2$, and

$$\log|\Phi_m(\eta^2 R_m e^{i\theta}, \eta)| < (2K - \xi)(\eta^2 R_m)^{\rho(\eta^2 R_m)} < 0, \tag{3.32}$$

provided that $\xi > K$. Also for $|\theta| \leqslant \gamma$, we have

$$\log\left|\Phi_m\left(\frac{R_m e^{i\theta}}{\eta^2}, \eta\right)\right| < 2K\left(\frac{R_m}{\eta^2}\right)^{\rho(R_m/\eta^2)} - \zeta\cos\theta\rho_1\left(\frac{R_m}{\eta^2}\right)^{\rho_1} < 0,$$

provided that

$$\zeta\cos\gamma\rho_1\left(\frac{R_m}{\eta^2}\right)^{\rho_1} > 2K\left(\frac{R_m}{\eta^2}\right)^{\rho(R_m/\eta_2)}. \tag{3.33}$$

Let $\xi > 2K$, and $\zeta = \zeta'(R_m/\eta^2)^{\rho(R_m/\eta^2)-\rho_1}$, where $\zeta' > 2K\sec\gamma\rho_1$. Then (3.32) and (3.33) are satisfied.

It follows from the maximum modulus principle that (3.31) holds for $|\arg z| \leqslant \gamma$, $\eta^2 R_m \leqslant |z| \leqslant R_m \eta^2$. Hence

$$\log|F_m(re^{i\vartheta}, \eta)| - \{A(R_m, \eta)\cos\theta\rho + B(R_m, \eta)\sin\theta\rho + \tfrac{1}{2}\epsilon\} r^{\rho(r)}$$
$$< \zeta'(R_m/\eta^2)^{\rho(R_m/\eta^2)-\rho_1}\cos\theta\rho_1 r^{\rho_1} + \xi(\eta^2 R_m)^{\rho(\eta^2 R_m)}$$

for every $\epsilon > 0$ and every $m > m_0(\epsilon, \eta)$. It remains to show that the right-hand side of this equality is less than $\frac{1}{2}\epsilon r^{\rho(r)}$ for $\eta R_m \leqslant r \leqslant R_m/\eta$, provided that $\eta < \eta_0(\epsilon)$, $m > m_0(\epsilon, \eta)$. This will certainly be the case if

$$\zeta' r^{\rho_1-\rho(r)}(R_m/\eta^2)^{\rho(R_m/\eta^2)-\rho_1} + \xi(\eta^2 R_m)^{\rho(\eta^2 R_m)} r^{-\rho(r)} < \tfrac{1}{2}\epsilon.$$

But, since $\eta R_m \leqslant r \leqslant R_m/\eta$, the left-hand side is less than

$$\zeta' \eta^{\rho_1-\rho-\epsilon(m)} + \xi\eta^{\rho \epsilon(m)},$$

where $\epsilon(m) \to 0$ as $m \to \infty$. Since $\rho_1 > \rho$, this is less than $\frac{1}{2}\epsilon$, provided that $\eta < \eta_0(\epsilon)$, $m > m_0(\epsilon, \eta)$, and so we have the required result.

3.4. We also require the following lemma.

LEMMA 3. *Let $\phi(x)$ be a real function, defined for $0 < l \leqslant x < \infty$, let $k > 1$, and let*

$$A(X, k) = \operatorname*{upper\ bound}_{X/k \leq x < kX} \phi(x).$$

Suppose that $\phi(x) \leqslant K_1$ for $l \leqslant x < \infty$, and that

$$A(X, 2) \geqslant -K_2 = -K_2(2)$$

for $0 < X < \infty$. Then, given any sequence $X_1, X_2, \ldots, X_m \to \infty$, we have

$$\lim_{k\to\infty}\lim_{m\to\infty}\{A(X_m, k^2) - A(X_m, k)\} = 0. \tag{3.41}$$

Proof. We have to show that, given any $\epsilon > 0$, we can choose $k_0 = k_0(\epsilon)$ so that

$$A(X_m, k^2) - A(X_m, k) < \epsilon$$

for $k > k_0$, $m > m_0(\epsilon, k)$. For, since $A(X_m, k)$ is a non-decreasing function of k, the left-hand side cannot be negative. For the same reason it is sufficient to show that

$$A\{X_m, \exp(2^{q+1})\} - A\{X_m, \exp(2^q)\} < \epsilon \tag{3.42}$$

for every integer $q > q_0(\epsilon)$, $m > m_0(\epsilon, q)$.

Suppose that this is not true. Then, given any integer Q, however large, we can find a subsequence $X_1', X_2', \ldots, X_m' \to \infty$, and Q integers $q_1, q_2, \ldots, q_Q$, such that

$$A\{X_m', \exp(2^{q_s+1})\} - A\{X_m', \exp(2^{q_s})\} \geqslant \epsilon.$$

Since
$$A\{X_m', \exp(2^{q+1})\} - A\{X_m', \exp(2^{q})\} \geqslant 0$$

for all q, putting $q = 1, 2, \ldots, q_Q$, and adding, we have

$$A\{X_m', \exp(2^{q_Q+1})\} - A\{X_m', \exp(2)\} \geqslant Q\epsilon,$$

and so
$$A\{X_m', \exp(2^{q_Q+1})\} \geqslant Q\epsilon - K_2(2).$$

Choosing $Q > \{K_1 - K_2(2)\}/\epsilon$, we have

$$A\{X_m', \exp(2^{q_Q+1})\} > K_1$$

for $m = 1, 2, \ldots$, and so

$$\phi(x) > K_1$$

for a sequence of values of x tending to infinity. But this contradicts our hypothesis. Hence (3.42) holds, and therefore (3.41).

3.5. We shall also require*

CARLEMAN'S FORMULA. *Suppose that $F(z)$ is regular for $l \leqslant |z| \leqslant R$, $|\arg z| \leqslant \frac{1}{2}\pi$. Let $z_1, z_2, \ldots, z_n = r_n e^{i\theta_n}$, be the zeros of $F(z)$ for which $l < r_n < R$, $|\theta_n| < \frac{1}{2}\pi$. Then*

$$\sum_{l<r_n<R} \frac{\cos\theta_n}{r_n}\left(1-\frac{r_n^2}{R^2}\right)$$

$$= \frac{1}{\pi R}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \log|F(Re^{i\theta})| \cos\theta\, d\theta$$

$$+ \frac{1}{2\pi}\int_l^R \{\log|F(ir)| + \log|F(-ir)|\}\left(\frac{1}{r^2}-\frac{1}{R^2}\right) dr$$

$$- \frac{l}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \left\{\log|F(le^{i\theta})| \cos\theta\left(\frac{1}{l^2}+\frac{1}{R^2}\right) - \arg F(le^{i\theta}) \sin\theta\left(\frac{1}{l^2}-\frac{1}{R^2}\right)\right\} d\theta,$$

where any determination of arg $F(le^{i\theta})$ *may be chosen, provided that it is the same for all values of θ, and if any zeros lie on the arc $|z| = l$, $|\arg z| \leqslant \frac{1}{2}\pi$, the variation of* arg $F(z)$ *is found by indenting the contour so that the zeros lie on the side nearer the origin.*

* See G. Pólya and G. Szegö, *Aufgaben und Lehrsätze*, 1 (1925), 310, ex. 240.

3.6. *Proof of Theorem I.* We first prove that, *given any* $\epsilon > 0$, *we can find* $k_0 = k_0(\epsilon) > 1$ *such that*

$$n(r, -\tfrac{1}{2}\pi/\rho+\delta, \tfrac{1}{2}\pi/\rho-\delta) < \epsilon r^{\rho(r)}$$

for $r = R_1, R_2, \ldots, R_m < k_0 R_{m-1}, R_m \to \infty$. *Also part* (2)* *of Theorem I holds for a sequence of values of* $R = R_1, R_2, \ldots, R_m \to \infty$, *such that* $R_m < k R_{m-1}$, *where* k *depends on* ϵ, δ, η.

We suppose for convenience that $\rho = 1$. The general case can be reduced to this by the transformation $z^\rho = w$. Let $k > 1$, and let

$$A(X, k) = \text{upper bound}_{X/k \leqslant x \leqslant kX} \frac{\log|f(x)|}{x^{\rho(x)}}.$$

Since $h(0) = A$, and since (3.21) holds, we can apply Lemma 3. Take $X_m = 2^m$; then, given $\epsilon_1 > 0$, we can choose $k_0 = k_0(\epsilon_1)$ so that

$$A(2^m, k^4) - A(2^m, k) < \epsilon_1$$

for $k > k_0$, $m > m_0(\epsilon_1, k)$. Hence

$$\log|f(x)| < \{A(2^m, k)+\epsilon_1\} x^{\rho(x)}$$

for $2^m k^{-4} \leqslant x \leqslant k^4 2^m$. We also have

$$\log|f(re^{\pm i\frac{1}{2}\pi})| < (\pm B+\epsilon_2) r^{\rho(r)}$$

for every $\epsilon_2 > 0$ and every $r > r_0(\epsilon_2)$. It follows from Lemma 2 that, given any $\epsilon_3 > 0$, we can choose ϵ_1, ϵ_2, and $k_0(\epsilon_1, \epsilon_2)$ so that

$$\log|f(re^{i\theta})| < \{A(2^m, k)\cos\theta + B\sin\theta + \epsilon_3\} r^{\rho(r)} \tag{3.61}$$

for $|\theta| \leqslant \frac{1}{2}\pi$, $2^m k^{-2} \leqslant r \leqslant k^2 2^m$, provided that $k > k_0$, $m > m_0(\epsilon_3, \epsilon_1, k)$.

Let X_m' be the point in the interval $2^m k^{-1} \leqslant x \leqslant 2^m k$ at which

$$\log|f(x)| = A(2^m, k) x^{\rho(x)},$$

and let $X_m'' = X_m' - 2^m k^{-2}$. Then $X_m' < k^2 X_{m-1}'$, and the circle

$$|z - X_m'| \leqslant X_m''$$

lies in the region $|\arg z| \leqslant \frac{1}{2}\pi$, $2^m k^{-2} \leqslant |z| \leqslant k^2 2^m$. Apply Lemma 1 to

$$F_m(z) = f(z) \exp[-\{A(2^m, k) - iB\} V(z)]$$

* Here, and in the following paragraph, we merely prove that $\underline{\lim}\, A(R) \geqslant -HK$; the more precise value given in (3.14) will be found in § 3.7.

in this circle, and use (3.61). It follows that, given any $k' > 1$, we can find $H_1 = H_1(k')$ such that

$$n(|z-X_m'| = X_m''/k') < \epsilon_3 H_1 V(X_m'),$$

where $n(|z-X_m'| = X_m''/k')$ denotes the number of zeros of $f(z)$ for which $|z-X_m'| \leqslant X''/k'$, for $m > m_0(\epsilon_3, \epsilon_1, k)$. Similarly, given $k' > 1$, $\zeta' > 0$, we can choose $H_2 = H_2(k', \zeta')$ so that

$$\log|F_m(z)| > -\epsilon_3 H_2 V(X_m') \tag{3.62}$$

for $|z-X_m'| \leqslant X_m''/k'$, except in circles the sum of whose radii is less than $\zeta' X_m'$. The circle $|z-X_m'| \leqslant X_m''/k'$ covers the region $|\arg z| \leqslant \frac{1}{2}\pi-\delta$, $\eta X_m' \sin\delta \leqslant r \leqslant X_m' \sin\delta$, provided that $k = k(\epsilon_3, \delta, \eta)$ is sufficiently large, and $k' = k'(\delta, \eta)$ is chosen sufficiently near 1. Put $R_m = X_m' \sin\delta$; and, since $X_m' < k^2 X_{m-1}$, where $k = k(\epsilon_3, \delta, \eta)$, we have the result stated at the beginning of the paragraph.

3.7. *Proof of Theorem I* (continued). Suppose first that, given any $\epsilon > 0$, $\delta > 0$, we can find a sequence R_1', R_2', ..., $R_m' \to \infty$, such that $R_m' < 2R_{m-1}'$, and

$$n(R_m', -\tfrac{1}{2}\pi+\delta, \tfrac{1}{2}\pi-\delta) < \epsilon V(R_m'), \tag{3.71}$$

and also part (2)* of Theorem I holds for $R = R_m'$. The required results follow from this. For part (1) of Theorem I follows at once from (3.71) and (2.11); and, given any $\epsilon > 0$, and any R', however large, we can choose ϵ' so that $\epsilon'(2R')^{\rho(2R')} < \epsilon(R')^{\rho(R')}$. Also, given any η, δ, $\zeta > 0$, we can find a sequence R_1', R_2', ..., $R_m' \to \infty$, such that part (2) of Theorem I holds with ϵ_1, δ, $\frac{1}{2}\eta$, $\frac{1}{2}\zeta$, R_m' in place of ϵ, δ, η, ζ, R, respectively. Since $R_m' < 2R_{m-1}'$, we have $R' \leqslant R_m' \leqslant 2R'$ for some R_m', provided that R' is sufficiently large, and the result for $R = R_m'$ includes that for $R = R'$.

Using the notation of §3.6, we have $R_m = X_m' \sin\delta \leqslant 2^m k \sin\delta$. It will be sufficient to show that, given any ϵ_1, $\delta_1 > 0$, we have

$$n(2^{m-1} k \sin\delta, -\tfrac{1}{2}\pi+\delta, \tfrac{1}{2}\pi-\delta) < \epsilon_1 V(2^{m-1} k \sin\delta) \tag{3.711}$$

for $k > k_0(\epsilon_1, \delta_1, \delta)$, $m > m_0(\epsilon_1, \delta_1, k)$. Suppose that $X_m' < 2^{m-1}k$; for if not, the result follows from §3.6, as we have just shown. Choose l so that $\frac{1}{2}X_m' \sin\delta < l \leqslant X_m' \sin\delta$, and put $R = 2^m k \sin\delta$, $F = F_m$ in Carleman's formula. We denote by I_1, I_2, I_3, the three integrals on the

* Except as regards $\underline{\lim}\, A(R)$. See foot-note, §3.6.

right-hand side, and we have

$$\sum_{l<r_n<R} \frac{\cos\theta_n}{r_n}\left(1-\frac{r_n^2}{R^2}\right) = I_1+I_2+I_3. \tag{3.72}$$

It follows from (3.61) that, if $k = k(\epsilon_3, \delta)$ is sufficiently large,

$$\log|F_m(re^{i\theta})| < \epsilon_3 r^{\rho(r)} \tag{3.73}$$

for $|\theta| \leqslant \frac{1}{2}\pi$, $l \leqslant r \leqslant R$, and so

$$I_1 < \epsilon_3 R^{\rho(R)-1}.$$

It follows from Lemma 1, as in (3.62), that, given any $\delta > 0$, we can choose $H_2 = H_2(\delta)$ so that

$$\log|F_m(le^{i\theta})| > -\epsilon_3 H_2 l^{\rho(l)}, \quad |\theta| \leqslant \tfrac{1}{2}\pi-\delta, \tag{3.74}$$

$$\left|\int_{-\frac{1}{2}\pi+\delta}^{\frac{1}{2}\pi-\delta} d\{\arg F_m(le^{i\theta})\}\right| < \epsilon_3 H_2 l^{\rho(l)} \tag{3.75}$$

for some l such that $\frac{1}{2}X_m' \sin\delta < l < X_m' \sin\delta$.

Also, it follows from Theorem II, and the second part of Lemma 1, that

$$\log|F_m(le^{i\theta})| > -H_2 K l^{\rho(l)}, \quad |\theta| \leqslant \tfrac{1}{2}\pi, \tag{3.76}$$

$$\begin{cases} \left|\int_{-\frac{1}{2}\pi}^{-\frac{1}{2}\pi+\delta} d\{\arg F_m(le^{i\theta})\}\right| < H_2 K l^{\rho(l)}, \\ \left|\int_{\frac{1}{2}\pi-\delta}^{\frac{1}{2}\pi} d\{\arg F_m(le^{i\theta})\}\right| < H_2 K l^{\rho(l)}, \end{cases} \tag{3.77}$$

for some l such that $\frac{1}{2}X_m' \sin\delta < l < X_m' \sin\delta$ and (3.74) holds. We suppose that the determination of $\arg F(z)$ is chosen so that $|\arg F(l)| \leqslant \pi$. Then, using (3.73) and (3.74) for $|\theta| \leqslant \frac{1}{2}\pi-\delta$, and (3.76) and (3.77) for the remaining portions of the integral, we have

$$I_3 < 4(\epsilon_3+\delta K) H_2 l^{\rho(l)-1}.$$

Now, given $\epsilon_4 > 0$, $\delta_1 > 0$, we can choose $\delta < \delta_1$, and k so that

$$4(\epsilon_3+\delta K) H_2 < \tfrac{1}{3}\epsilon_4,$$

and also $\epsilon_3 < \frac{1}{3}\epsilon_4$.

Also, we have

$$\log|F_m(re^{\pm\frac{1}{2}\pi i})| < \tfrac{1}{3}\pi\epsilon_5 r^{\rho(r)}$$

for every $\epsilon_5 > 0$, $r > r_0(\epsilon_5)$. Hence, choosing $\epsilon_5 < \epsilon_4 3k^2$, we have

$$I_2 < \frac{\epsilon_5}{3}\int_l^R r^{\rho(r)-2}\,dr < \frac{\epsilon_5}{3}\frac{R^{\rho(R)}}{l} < \epsilon_4 R^{\rho(R)-1}.$$

For $X_m' \geqslant 2^m k^{-1}$, and so $R < 2k^2 l$. It follows that

$$\sum_{l<r_n<R} \frac{\cos\theta_n}{r_n}\left(1-\frac{r_n^2}{R^2}\right) < \epsilon_4 \max\left(R^{\rho(R)-1};\ l^{\rho(l)-1}\right).$$

But

$$\tfrac{3}{4}\,\frac{n(\frac{1}{2}R,\ -\frac{1}{2}\pi+\delta_1,\ \frac{1}{2}\pi-\delta_1)}{\frac{1}{2}R}\sin\delta_1 < \sum_{l<r_n<R} \frac{\cos\theta_n}{r_n}\left(1-\frac{r_n^2}{R^2}\right).$$

It follows that

$$n(\tfrac{1}{2}R,\ -\tfrac{1}{2}\pi+\delta_1,\ \tfrac{1}{2}\pi-\delta_1) < \tfrac{2}{3}\operatorname{cosec}\delta_1\ \epsilon_4 \max\left(R^{\rho(R)};\ Rl^{\rho(l)-1}\right),$$

where $\frac{1}{2}X_m' \sin\delta < l < X_m' \sin\delta$, $R = 2^m k \sin\delta$.

Choosing $\epsilon_4 < \frac{2}{3}\epsilon \sin\delta_1$, we have (3.71), provided that $R^{\rho(R)}$ is not less than $Rl^{\rho(l)-1}$. In any case $R < 2k^2 l$, and so it follows from (2.11) that

$$\frac{l^{\rho(l)-1}}{R^{\rho(R)-1}} \to 1$$

as $R \to \infty$ for all k. Hence we have part (1) of Theorem I in both cases.

Since $I_3 < \frac{1}{3}\epsilon_4 l^{\rho(l)-1}$, and $I_2 < \frac{1}{3}\epsilon_4 R^{\rho(R)-1}$, and the left-hand side of (3.72) is positive, we have

$$I_1 > -\tfrac{2}{3}\epsilon_4(\max R^{\rho(R)-1};\ l^{\rho(l)-1}) > -\epsilon_6 R^{\rho(R)-1}$$

for every $\epsilon_6 > 0$ and every $R > R_0(\epsilon_6)$. But, since (3.73) holds*, given $\epsilon_7 > 0$, $\delta_2 > 0$, we have

$$\log|F_m(Re^{i\theta})| > -\epsilon_7 R^{\rho(R)} \tag{3.78}$$

except in a set of θ of measure less than δ. Choosing convenient points $Re^{i\psi}$ at which (3.78) holds, and applying Lemma 1, we have the conclusions of Theorem I complete except for $\underline{\lim}\, A(R)$.

3.8. We have now proved that (3.12) and (3.13) hold, and it only remains to show that

$$\underline{\lim_{R\to\infty}}\ \rho A(R) \geqslant \max\{-h'(-\tfrac{1}{2}\pi/\rho-);\ -h'(\tfrac{1}{2}\pi/\rho+)\}.$$

Suppose for convenience that $\rho = 1$, and that

$$C = h'(\tfrac{1}{2}\pi,\rho+) \geqslant h'(-\tfrac{1}{2}\pi,\rho-).$$

* Compare E. C. Titchmarsh, *Theory of functions* (Oxford, 1932), 166, § 5.11.

We show first that, given $\epsilon > 0$, $\delta > 0$, we can choose $r_0(\epsilon, \delta)$ so that

$$\log|f(re^{i\theta})| > (-C\cos\theta + B\sin\theta - \epsilon)\, r^{\rho(r)} \tag{3.81}$$

for some θ such that $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi + \delta$, and every $r > r_0(\epsilon, \delta)$. Consider

$$F(z) = f(z)\exp\{(C + iB + \zeta e^{\frac{1}{2}\delta i})\, V(z) - \xi\, V(r_0)\}.$$

where

$$\zeta > 0,\ \xi > A + |C| + |B| + \zeta,$$

in the angle $-\frac{1}{2}\pi \leqslant \arg z \leqslant \frac{1}{2}\pi + \delta$. We have

$$|F(z)| < 1 \tag{3.82}$$

for $|z| = r_0$, $-\frac{1}{2}\pi/\rho \leqslant \arg z \leqslant \frac{1}{2}\pi/\rho + \delta$, provided that $r_0 = r_0(\xi)$ is sufficiently large. Also (3.82) holds for

$$\arg z = -\tfrac{1}{2}\pi,\ \arg z = \tfrac{1}{2}\pi + \delta,\ |z| \geqslant r_0,$$

provided that $\delta = \delta(\zeta)$ is sufficiently small and $r_0 = r_0(\xi, \zeta, \delta)$ is sufficiently large. For

$$\log|f(re^{i\theta})| < \{h(\theta) + \epsilon_1\} r^{\rho(r)}$$

for every $\epsilon_1 > 0$ and every $r > r_0(\epsilon_1)$. Hence, if $\epsilon_1 < \zeta \sin\frac{1}{2}\delta$, we have

$$\log|F(re^{-\frac{1}{2}\pi i})| < (\epsilon_1 - \zeta\sin\tfrac{1}{2}\delta)\, r^{\rho(r)} < 0,$$

and

$$\log|F(re^{i(\frac{1}{2}\pi+\delta)})| < \{h(\tfrac{1}{2}\pi + \delta) + \epsilon_1 - C\sin\delta - B\cos\delta - \zeta\sin\tfrac{1}{2}\delta\}\, V_2$$

$$< \{h(\tfrac{1}{2}\pi) + \delta h'(\tfrac{1}{2}\pi +) + \delta\epsilon(\delta) + \epsilon_1 - C\delta - B - \tfrac{1}{2}\zeta\delta\}\, V(r), \tag{3.83}$$

where $\epsilon(\delta) \to 0$ as $\delta \to 0$. Since $h(\frac{1}{2}\pi) = B$, $h'(\frac{1}{2}\pi +) = C$, we can make the right-hand side negative by choosing $\delta = \delta(\zeta)$ sufficiently small. But, given any $\epsilon_2 > 0$, we can find a sequence of values of x tending to infinity such that

$$\log|F(x)| > \{h(0) - \epsilon_2 + C + \zeta\cos\tfrac{1}{2}\delta\}\, x^{\rho(x)},$$

and we have already seen in (3.15) that $h(0) \geqslant -C$. Hence, choosing $\epsilon_2 < \zeta\cos\frac{1}{2}\delta$, we have $\log|F(x)| > 0$ for a sequence of values of x tending to infinity. It follows from the maximum modulus principle that there is a point on every arc $|z| = r > r_0(\epsilon_2, \epsilon_3, \xi)$, $-\frac{1}{2}\pi < \arg z < \frac{1}{2}\pi + \delta$, at which

$$\log|F(z)| > 0, \tag{3.84}$$

which gives (3.81). For ζ may be chosen as small as we please.

We next show that points at which $|F(x)|$ is not too small occur fairly frequently on the real axis. Suppose that this is not so; that is, suppose that, given any $\eta > 0$, $\zeta' > 0$, we can find a sequence $X_1, X_2, \ldots, X_m \to \infty$ such that

$$\log|F(x)| < -\zeta' V(X_m)$$

for $\eta^2 X_m \leqslant x \leqslant X_m/\eta^2$. Apply Lemma 2 to $F(z)$ in the angle

$$0 \leqslant \arg z \leqslant \tfrac{1}{2}\pi + \delta.$$

Since, by (3.83),

$$\log|F(re^{i(\frac{1}{2}\pi+\delta)})| < 0$$

for all sufficiently large r, given any $\epsilon_4 > 0$, we can choose $\eta_0(\epsilon_4)$ so that

$$\log|F(re^{i\theta})| < \{-\zeta' \cos(\theta-\delta)\sec\delta + \epsilon_4\} r^{\rho(r)}$$

for $0 \leqslant \theta \leqslant \frac{1}{2}\pi+\delta$, $\eta X_m \leqslant r \leqslant X_m/\eta$, provided that $m > m_0(\epsilon_4, \eta)$. Choosing $\epsilon_4 < \zeta' \sec\delta$, we obtain a contradiction of (3.84). Hence, given $\epsilon_5 > 0$, we can choose $\eta_0(\epsilon_5)$ so that

$$\log|F(x)| > -\epsilon_5 x^{\rho(x)}$$

for some x in every interval $\eta X \leqslant x \leqslant X/\eta$, provided that $\eta < \eta_0(\epsilon_5)$, and $X > X_0(\epsilon_5, \eta)$. But we can choose ζ as small as we please; and so, given ϵ_6, we can choose $\eta_0(\epsilon_6)$ so that

$$\log|f(x)| > -(C+\epsilon_6) x^{\rho(x)}$$

for some x in every interval $\eta X \leqslant x \leqslant X/\eta$, provided that $\eta < \eta_0(\epsilon_6)$, $X > X_0(\epsilon_6, \eta)$. Hence

$$\lim_{R\to\infty} A(R) \geqslant -C = -h'(\tfrac{1}{2}\pi+),$$

which is the required result.

4.1. We now have to prove the converse theorem.

THEOREM III. *Suppose that $f(z)$ is regular and of proximate order $\rho(r)$ for $|z| \geqslant l$, $\alpha \leqslant \arg z \leqslant \beta$, and suppose that $f(z)$ has no directions of Borel of proximate order $\rho(r)$ for which $\alpha < -\frac{1}{2}\pi/\rho < \arg z < \frac{1}{2}\pi/\rho < \beta$. Then*

$$h(\theta) = h(0)\cos\theta\rho$$

for $|\theta| \leqslant \frac{1}{2}\pi/\rho$.

We first observe that $h(\theta)$ is finite and continuous for $\alpha < \arg z < \beta$. Further, either $h(\theta) > 0$ for $|\theta| < \frac{1}{2}\pi/\rho$, or $h(\theta) < 0$ for $|\theta| < \frac{1}{2}\pi/\rho$, or $h(\theta) \equiv 0$

for $|\theta| \leqslant \frac{1}{2}\pi/\rho$. For every interval in which $h(\theta) < 0$ is bounded on either side by intervals in which $h(\theta) > 0$; and, as Valiron has shown, the arms of the angles in which $h(\theta) > 0$ determine directions of Borel of proximate order $\rho(r)$. If $h(\theta) \leqslant 0$ for $|\theta| < \frac{1}{2}\pi/\rho$, then it must necessarily be of the required form*, apart from any consideration of directions of Borel. Hence we may suppose that $h(\theta) > 0$ for $|\theta| < \frac{1}{2}\pi/\rho$.

4.2. In virtue of the following lemma we need only consider the case in which $f(z)$ has very few zeros for which $|\arg z| \leqslant \frac{1}{2}\pi/\rho - \delta$, $|z| \geqslant r_0(\delta)$.

LEMMA 4. *If $f(z)$ is regular and of proximate order $\rho(r)$ for*

$$\alpha \leqslant \arg z \leqslant \beta, \quad |z| > l,$$

and if

$$\overline{\lim_{r\to\infty}}\, n(r, \gamma_1, \gamma_2, a)/r^{\rho(r)} = \zeta > 0, \tag{4.21}$$

where $\alpha < \gamma_1 < \gamma_2 < \beta$, for every value of a, then $f(z)$ has at least one direction of Borel of proximate order $\rho(r)$, $\arg z = \gamma$, such that $\gamma_1 \leqslant \gamma \leqslant \gamma_2$.

We prove this by means of Milloux's theorem†.

MILLOUX'S THEOREM. *Suppose that $F(z)$ is meromorphic for $|z| < 1$, and let $d < 1$, $d' < 1$. Suppose that, in $|z| < 1$, $F(z)$ does not take the three values a_1, a_2, a_3, whose spherical distances are greater than d, more than n times, Then the number of zeros of $F(z) - a$ for which $|z| < \frac{1}{20}$ is less than*

$$K_1 n + K_2 \log(1/d) + K_3 \log(1/d') + K_4,$$

where d' denotes the spherical distance of a from a possible exceptional value a_0, and K_1, K_2, K_3, K_4 are numerical constants.

Proof of Lemma 4. Let p be an integer such that

$$20(\gamma_2 - \gamma_1)/(p-1) = \delta < \min(\beta - \gamma_2;\ \gamma_1 - \alpha), \tag{4.22}$$

and divide the angle $\gamma_1 \leqslant \arg z \leqslant \gamma_2$ into $p-1$ equal parts by

$$\arg z = \theta_2, \theta_3, \ldots, \theta_{p-1},$$

and put $\theta_1 = \gamma_1$, $\theta_p = \gamma_2$. Then $\theta_s - \theta_{s-1} = \frac{1}{20}\delta$. Let $R_m = R_0 k^m$, where $R_0 > l$ and $k > 1$. Suppose that, given any $\epsilon > 0$, $k > 1$, and any p satisfying (4.22), we can choose $m_0 = m_0(\epsilon, k, p)$ so that $f(z)$ does not

* See E. Phragmén and E. Lindelöf, *Acta Math.*, 31 (1908), 403.

† H. Milloux, *Acta Math.*, 52 (1928), 202.

take the values $a_1(m, s)$, $a_2(m, s)$ more than $\epsilon V(R_m)$ times in the circles $|z-R_m e^{i\theta_s}| \leqslant \sin\delta\, R_m$ for $m > m_0$, $s = 1, 2, \ldots, p$. Suppose also that

$$-\log|a_1(m, s)-a_2(m, s)| < \epsilon V(R_m),\ \log|a_1(m, s)| < \epsilon V(R_m),$$

$$\log|a_2(m, s)| < \epsilon V(R_m)$$

for $m > m_0$, $s = 1, 2, \ldots, p$. Then, applying Milloux's theorem to $f(R_m e^{i\theta_s}+z\sin\delta R_m)$ with

$$a_1 = a_1(m, s),\ a_2 = a_2(m, s),\ a_3 = \infty,$$

we have

$$n\left(|z-R_m e^{i\theta_s}| = \frac{\sin\delta}{20} R_m, a\right) < K_5 \epsilon V(R_m), \tag{4.23}$$

where $n(|z-Z| = R, a)$ denotes the number of zeros of $f(z)-a$ for which $|z-Z| \leqslant R$, and K_5 is a numerical constant, except perhaps for values of a such that $\log|a-a_0(m, s)| < -\epsilon V(R_m)$, provided that $m > m_0(\epsilon, \delta, k)$.

Now, given any $r > R_0 k^{m_0}$, we can choose k and m so that the circles $|z-R_q e^{i\theta_s}| \leqslant \sin\delta/20\, R_q$, $s = 1, 2, \ldots, p$, $q = m_0, m_0+1, \ldots, m$, cover the whole region $\gamma_1 \leqslant \arg z \leqslant \gamma_2$, $R_0 k^{m_0} \leqslant |z| \leqslant r$, and $R_{m-1} \leqslant r \leqslant R_m$. It is, in fact, sufficient to put $k-1 = \frac{1}{20}\sin\delta$. Hence

$$\begin{aligned} n(r, \gamma_1, \gamma_2, a) &< K_5 p\epsilon \sum_{q=m_0}^{m} V(R_q) \\ &< 2K_5 p\epsilon V(R_m)\{1+k^{-\rho}+k^{-2\rho}+\ldots+k^{-(m-m_0)\rho}\}. \\ &< \frac{2K_5 p\epsilon}{1-k^{-\rho}} V(R_m) < \frac{3K_5 p\epsilon k^{\rho} V(r)}{1-k^{-\rho}}, \end{aligned} \tag{4.24}$$

except perhaps for values of a such that

$$\log|a-a_0(q, s)| < -\epsilon V(Rq),\ s = 1, 2, \ldots, p,\ q = m_0, m_0+1, \ldots, m.$$

It follows from (4.24) that (4.21) is false unless a belongs to the excepted set. The total measure of the excepted sets is less than

$$\begin{aligned} p \sum_{q=m_0}^{m} \exp\{-\epsilon V(R_q)\} &< \frac{p\exp\{-\frac{1}{2}\epsilon V(R_{m_0})\}}{1-\exp\{-\frac{1}{2}\epsilon(k^{\rho}-1)V(R_{m_0})\}} \\ &< \frac{2p\exp\{-\frac{1}{2}\epsilon V(R_{m_0})\}}{\epsilon(k^{\rho}-1)V(R_{m_0})} \\ &< 1, \end{aligned}$$

provided that m_0 is sufficiently large. But by our hypothesis (4.21) holds for every value of a, and we have a contradiction. Hence, given $\epsilon > 0$,

$\delta > 0$, we can find a sequence $Z'_{m'} = R_{m'} e^{i\theta(m')}$ such that $\theta(m')$ is one of $\theta_1, \theta_2, \ldots, \theta_p$,

$$n(|z - Z'_{m'}| = \sin\delta R_{m'}, a) > \zeta(\delta) V(R_{m'}),$$

except, perhaps, for values of a such that $\log|a| > \epsilon V(R_{m'})$, or $\log|a - a_0(m')| < -\epsilon V(R_{m'})$. We can divide this sequence into at least one, and not more than p, subsequences such that $\theta(m')$ remains the same throughout each subsequence for all $\epsilon > 0$. Now make $p \to \infty$ and the $\theta(m')$ associated with each subsequence will tend to a limit γ, say, and $\arg z = \gamma$, $\gamma_1 \leqslant \gamma \leqslant \gamma_2$, is obviously a direction of Borel of proximate order $\rho(r)$.

4.3. When $f(z)$ has very few zeros, the position of the *cercles de remplissage*, and therefore of the directions of Borel, is determined by the following lemmas*. Let $n(|z - Z_m| = \lambda R_m) = n(|z - Z_m| = \lambda R_m, a)$, and $P_m(z, \lambda) = \prod_n (1 - z/z_n)$, where z_n is a zero of $f(z)$ and the product is taken over zeros for which $|z - Z_m| \leqslant \lambda R_m$.

LEMMA 5. *Suppose that there is a number $\lambda_1 > 0$, and a sequence of points $Z_1, Z_2, \ldots, |Z_m| = R_m \to \infty$ such that $f(z)$ is regular for $|z - Z_m| \leqslant \lambda_1 R_m$, and that*

$$\log|f(Z_m)/P_m(Z_m, \lambda_1)| > -K V(R_m), \tag{4.31}$$

$$\log|f(z)| < K V(R_m) \text{ for } |z - Z_m| \leqslant \lambda_1 R_m. \tag{4.32}$$

Suppose also that

$$\lim_{m \to \infty} n(|z - Z_m| = \lambda_1 R_m)/V(R_m) = 0. \tag{4.33}$$

Then the sequence $\{Z_m\}$ consists of two kinds of subsequences $\{Z_{m'}\}$, $\{Z_{m''}\}$ having the following properties:

(1) *There is a $\lambda_2 > 0$ such that, given any $\epsilon > 0$, $\delta > 0$, we can choose $m_0(\epsilon, \delta)$ so that*

$$\left|\log|f(z)| - \log|f(Z_{m'})/P_{m'}(Z_{m'}, \lambda_1)|\right| < \epsilon V(R_{m'}) \tag{4.34}$$

for $|z - Z_{m'}| \leqslant \lambda_2 R_{m'}$, except in circles the sum of whose radii is less than $\delta R_{m'}$, provided that $m' > m_0(\epsilon, \delta)$.

* See M. L. Cartwright, *Proc. London Math. Soc.* (2), 38 (1935), 426.

(2) *There are positive numbers Q and A such that, given $\epsilon > 0$, $\delta > 0$, $\eta > 0$, we can choose $\lambda_0 = \lambda_0(\epsilon) < \lambda_1$, $m_0(\epsilon, \delta, \eta, \lambda_0)$ so that*

$$(4.35)\quad \Big|\log|f(Z_{m''}+\lambda R_{m''}e^{i\theta})| - \log|f(Z_{m''})/P_{m''}(Z_{m''}, \lambda_1)| - (\lambda/\lambda_1)^{\mu(m'')} A(m'') \cos\{\mu(m'')\theta + \sigma(m'')\} V(R_{m''})\Big| < \epsilon(\lambda/\lambda_1)^{\mu(m'')} V(R_{m''}),$$

where $\mu(m'')$ is a positive integer less than Q, $A < A(m'') < 6K$, for $\eta\lambda_0 < \lambda < \lambda_0$, except perhaps for values of $\lambda R_{m''}e^{i\theta}$ lying in circles the sum of whose radii is less than $\delta\lambda_0 R_{m''}$, provided that $m > m_0(\epsilon, \delta, \eta, \lambda_0)$.

LEMMA 6. *Suppose that the conditions of Lemma 5 hold. Suppose also that*

$$(4.36)\quad \lim_{m\to\infty} \frac{\log|f(Z_m)/P_m(Z_m, \lambda_1)|}{V(R_m)} = 0$$

and that all the points Z_m belong to the second kind of subsequence, $Z_{m''}$, considered in Lemma 5.

Then, for every $\lambda > 0$,

$$\lim_{m\to\infty} \frac{n(|z-Z_m| = \lambda R_m, a)}{V(R_m)} = \zeta(\lambda) > 0$$

for every $a \neq 0$.

The results of Lemma 5 may be summed up very roughly by saying that

$$\frac{\log|f(Z_m+\lambda R_m e^{i\theta})/P_m(Z_m+\lambda R_m e^{i\theta}, \lambda_1)|}{V(R_m)}$$

behaves like a regular harmonic function of λ for small λ and large m.

The result of Lemma 5 holds equally well for a sequence of functions $F_m(z)$ each regular in the corresponding circle $|z-Z_m| \leqslant \lambda_1 R_m$, provided that (4.33) holds where $n(|z-Z_m| = \lambda_1 R_m)$ now denotes the number of zeros of $F_m(z)$ in the circle $|z-Z_m| \leqslant \lambda_1 R_m$.

4.4. We also need the following result. It is practically included in the proof* of Lemma 1, but I have never seen it stated.

LEMMA 7. *Let*

$$(4.41)\quad P(z) = \prod_{n=1}^{N} \left(1-\frac{z}{z_n}\right),$$

* Compare G. Valiron, *General theory of integral functions* (1923), 79, 80.

where $0<|z_1|\leqslant|z_2|\leqslant\ldots\leqslant|z_N|\leqslant R$. *Then*

$$\int_E \log|P(re^{i\theta})|\,dr>-2NR,$$

where E denotes any set of values in the interval $0\leqslant r\leqslant R$.

Proof. We have

$$\int_E \log\left|1-\frac{re^{i\theta}}{z_n}\right|dr>\int_E \log\left|1-\frac{r}{r_n}\right|dr,$$

where $r_n=|z_n|$. Since $|1-r/r_n|>1$ outside the interval $0<r<2r_n$, and $|1-r/r_n|<1$ inside the interval, we have

$$\int_E \log\left|1-\frac{r}{r_n}\right|dr>\int_0^{2r_n}\log\left|1-\frac{r}{r_n}\right|dr=2r_n\int_0^1\log|1-2x|\,dx$$

$$=-2r_n.$$

Hence

$$\int_E \log|P(re^{i\theta})|\,dr>-2\sum_1^N r_n>-2NR.$$

4.4. *Proof of Theorem III.* We need only consider the case $\rho=1$; and it follows from Lemma 4 that we may suppose that

$$\lim_{r\to\infty} n(r,\,-\tfrac{1}{2}\pi+\delta,\,\tfrac{1}{2}\pi-\delta,\,a)/r^{\rho(r)}=0$$

for some a. We have also seen in § 4.1 that we may suppose that $h(\theta)>0$ for $|\theta|<\frac{1}{2}\pi$. Hence $h(\theta)$ is the same for $f(z)-a$ as for $f(z)$, and we may suppose, without loss of generality, that

$$\lim_{r\to\infty} n(r,\,-\tfrac{1}{2}\pi+\delta,\,\tfrac{1}{2}\pi+\delta)/r^{\rho(r)}=0. \tag{4.41}$$

Let $P(z,\,R_m,\,k,\,\delta)=\Pi(1-z/z_n)$, where z_n is a zero of $f(z)$ and the product is taken over zeros for which $|\arg z_n|\leqslant\frac{1}{2}\pi-\delta$, $R_m k^{-4}\leqslant|z_n|\leqslant R_m k^4$. It follows from (4.41) that, given any $\epsilon_1>0$, $k>1$, $\delta>0$, we can choose $m_0(\epsilon_1,\,\delta,\,k)$ so that

$$\log|P(re^{i\theta},\,R_m,\,k,\,\delta)|<\epsilon_1 V(R_m) \tag{4.42}$$

for $r\leqslant 2R_m k^4$, $m>m_0(\epsilon_1,\,\delta,\,k)$. Hence, by Lemma 1, given $\epsilon_1>0$, $\eta>0$, $\delta>0$, $k>1$, we can choose $m_0(\epsilon_1,\,\delta,\,\eta,\,k)$ so that

$$\log|P(re^{i\theta},\,R_m,\,k,\,\delta)|>-\epsilon_1 V(R_m) \tag{4.43}$$

for $r \leqslant R_m k^4$, except for values of $re^{i\theta}$ lying in circles the sum of whose radii is less than ηR_m. These inequalities hold, *a fortiori*, when the product is only taken over some of the zeros in the region.

Consider any sequence of circles $|z-Z_m| \leqslant \lambda_1 |Z_m|$ such that $\lambda_1 < \sin\delta$, $|\arg Z_m| \leqslant \frac{1}{2}\pi - 2\delta$, $R_m k^{-4} \leqslant (1-\lambda_1)|Z_m| < (1+\lambda_1)|Z_m| \leqslant R_m k^4$. Each of these circles lies entirely in the corresponding region $R_m k^{-4} \leqslant |z| \leqslant R_m k^4$, $|\arg z| \leqslant \frac{1}{2}\pi - \delta$; and it follows from Theorem II and (4.42) that (4.31) is satisfied with KH in place of K. The inequality (4.32) is obviously satisfied and (4.33) is also satisfied in virtue of (4.41). Since $f(z)$ has no directions of Borel of proximate order $\rho(r)$ for which $|\arg z| \leqslant \frac{1}{2}\pi - 2\delta$, the case considered in Lemma 6 is excluded. We can therefore divide the sequence $\{Z_m\}$ into three kinds of subsequences $\{Z_{m'}\}$, $\{Z_{m''}\}$, $\{Z_{m'''}\}$, having the following properties:

(1) $$\varliminf_{m\to\infty} \frac{\log|f(Z_{m'})/P_{m'}(Z_{m'}, \lambda_1)}{V(R_{m'})} = \zeta > 0.$$

(2) $$\varlimsup_{m\to\infty} \frac{\log|f(Z_{m''})/P_{m''}(Z_{m''}, \lambda_1)|}{V(R_{m''})} = -\zeta > 0.$$

(3) There is a $\lambda_2 > 0$ such that, given any $\epsilon > 0$, we can choose $m_0(\epsilon)$ so that

$$\left|\log|f(z)/P_{m'''}(z, \lambda_1)|\right| < \epsilon V(R_{m'''})$$

for $|z-Z_{m'''}| < \lambda_2 R_{m'''}$. For in case (3) (4.36) is satisfied, the points are all of the first kind, and the exceptional values mentioned in Lemma 5 only arise from the zeros of $f(z)$ and do not occur if we consider $f(z)/P_m(z, \lambda_1)$ which is free from zeros.

Since we have supposed that $h(\theta) > 0$ for $|\theta| < \frac{1}{2}\pi/\rho$, the sequences of points, Z_m, for which the last two alternatives hold do not affect the values of $h(\theta)$. Let $R_m = R_0 2^m$, and let $\{R_{m'}\}$ denote a sequence for which each of the regions $R_{m'} \leqslant |z| \leqslant R_{m'+1}$, $|\arg z| \leqslant \frac{1}{2}\pi - 2\delta$ contains a point belonging to the first kind of subsequence. It will be sufficient to prove that, given $\epsilon > 0$, $\delta > 0$, we have

(4.44) $$\left|\log|f(re^{i\theta})/P(re^{i\theta}, R_{m'}, k, \delta)| - A(R_{m'})\cos\theta\rho\, r^{\rho(r)}\right| < \epsilon V(R_{m'}),$$

where $A(R_{m'})$ is independent of r and θ, $k > 1$ for

$$|\theta| \leqslant \tfrac{1}{2}\pi - 2\delta, \quad R_{m'} \leqslant r \leqslant R_{m'+1},$$

provided that $m' > m_0(\epsilon, \delta, k,)$. For if (4.44) holds, it follows from (4.42) and (4.43) that $h(\theta) = \varlimsup_{m'\to\infty} A(R_{m'})\cos\theta\rho$. Now it follows from

Lemmas 5 and 6 that, given any $\delta > 0$, $k > 1$,

$$\log|f(z)/P_{m'}(z, \lambda_1)| > \zeta(\delta, k)\, V(R_{m'}), \tag{4.45}$$

where $\zeta(\delta, k) > 0$, for $R_{m'} k^5 \leqslant |z| \leqslant R_{m'+1} k^5$, $|\arg z| \leqslant \frac{1}{2}\pi - 2\delta$, $m' > m_0(\delta, k)$. For if not, there will be a sequence of points lying in those regions satisfying the conditions of Lemma 6, and therefore $f(z)$ will have a direction of Borel of proximate order $\rho(r)$ for which $|\arg z| \leqslant \frac{1}{2}\pi - 2\delta$. The rest of the proof will consist in deducing (4.44) from (4.45) by methods similar to those which we used in Theorem I. Put

$$F_{m'}(z) = P(z, R_{m'}, k, \delta)/f(z),$$

and let

$$\underset{R_{m'} k^{-1} \leqslant x \leqslant R_{m'} k}{\text{upper bound}} \frac{\log|F_{m'}(x)|}{x^{\rho(x)}} = A(R_{m'}, k). \tag{4.46}$$

Then $F_{m'}(z)$ is regular for $R_{m'} k^{-4} \leqslant |z| \leqslant R_{m'} k^4$, $|\arg z| \leqslant \frac{1}{2}\pi - 2\delta$, and it follows from (4.42), (4.43), and (4.46) that*

$$-h(0) - \epsilon_2 < A(R_{m'}, k^4) < -\zeta(\delta, k) + \epsilon_2$$

for every $\epsilon_2 > 0$ and every $m' > m_0(\epsilon_2, \delta, k)$. Hence, by Lemma 3, we have

$$A(R_{m'}, k^4) - A(R_{m'}, k) < \epsilon_3$$

for every $\epsilon_3 > 0$, provided that $k > k_0(\epsilon_3)$, $m' > m_0(\epsilon_3, k)$. It follows from (4.45) that

$$\log|F_{m'}(re^{\pm i(\frac{1}{2}\pi - 2\delta)})| < \{-\zeta(\delta, k) + \epsilon_4\} V(r),$$

where $R_{m'} k^{-4} \leqslant r \leqslant R_{m'} k^4$, for every $\epsilon_4 > 0$, $m' > m_0(\epsilon_4, \delta, k)$. Hence, applying Lemma 2 in the angles

$$-\tfrac{1}{2}\pi + 2\delta \leqslant \arg z \leqslant 0, \quad 0 \leqslant \arg z \leqslant \tfrac{1}{2}\pi - 2\delta,$$

we have

$$\log|F_{m'}(re^{i\theta})| < [\{A(R_{m'}, k^4)\cos(\theta - 2\delta) + \zeta(\delta, k)\sin\theta\}\sec 2\delta + \epsilon_5] V(r),$$

where $$-\tfrac{1}{2}\pi + 2\delta \leqslant \theta \leqslant 0, \quad R_{m'} k^{-4} \leqslant r \leqslant R_{m'} k^4,$$

$$\log|F_{m'}(re^{i\theta})| < [\{A(R_{m'}, k^4)\cos(\theta + 2\delta) - \zeta(\delta, k)\sin\theta\}\sec 2\delta + \epsilon_5] V(r),$$

* In virtue of Lemmas 2 and 3, $A(R_{m'}, k)$ has more or less the same properties as $h(\theta)$. The following argument is substantially the same as that which shows that, if $h(\theta) \leqslant 0$ for $|\theta| \leqslant \frac{1}{2}\pi/\rho$, then $h(\theta) = h(0)\cos\theta\rho$.

where $$0 \leqslant \theta \leqslant \tfrac{1}{2}\pi - 2\delta, \quad R_{m'} k^{-2} \leqslant r \leqslant R_{m'} k^2,$$

for every $\epsilon_5 > 0$, provided that $k > k_0(\epsilon_5)$ and $m' > m_0(\epsilon_5, \delta, k)$. Applying the lemma again in the angle $|\theta| \leqslant \tfrac{1}{4}\pi$, we have

$$\begin{aligned}\log|F_{m'}(re^{i\theta})| &< [\{A(R_{m'}, k^4) - \zeta(\delta, k)\sec 2\delta + A(R_{m'}, k^4)\tan 2\delta\}\cos\theta + \epsilon_6]\, V(r) \\ &< [\{A(R_{m'}, k) - \zeta(\delta, k)\sec 2\delta + A(R_{m'}, k)\tan 2\delta\}\cos\theta + \epsilon_7]\, V(r),\end{aligned}$$

where $|\theta| \leqslant \tfrac{1}{4}\pi$, $R_{m'} k^{-1} \leqslant r \leqslant R_{m'} k$, for every $\epsilon_6 > 0$, $\epsilon_7 > 0$, provided that $k > k_0(\epsilon_7)$, $m' > m_0(\epsilon_7, \delta, k)$. But this gives a contradiction of (4.46) unless $\zeta(\delta, k)\sec 2\delta \leqslant A(R_{m'}, k)\tan 2\delta + \epsilon_7$ for every $\epsilon_7 > 0$, $\delta > 0$, $k > 1$, $m' > m_0(\epsilon_1, \delta, k)$. Hence

$$\log|F_{m'}(re^{i\theta})| < \{A(R_{m'}, k)\cos\theta\rho + 2\epsilon_7\}\, V(r) \tag{4.461}$$

for $|\theta| \leqslant \tfrac{1}{2}\pi - 2\delta$, $R_{m'} k^{-2} \leqslant r \leqslant R_{m'} k^2$. Taking the point, $X_{m'}$, at which $\log|F_{m'}(x)|/x^{\rho(x)}$ attains its upper bound in the interval $R_{m'} k^{-1} \leqslant x \leqslant R_{m'} k$, as centre, we apply Lemma 1 to

$$\Phi_{m'}(z) = F_{m'}(z)\exp\{-A(R_{m'}, k)\, V(z)\}$$

in the circle $|z - X_{m'}| \leqslant X_{m'} - R_{m'} k^{-2}$, as in Theorem I. By this means we show that, given any $\epsilon_8 > 0$, $\delta > 0$, $\eta > 0$, we can choose $k > k_0(\epsilon_8, \delta)$ so that

$$\left|\log|F_{m'}(re^{i\theta})| - A(R_{m'}, k)\cos\theta\, V(r)\right| < \epsilon_8 V(r) \tag{4.47}$$

for $$|\theta| \leqslant \tfrac{1}{2}\pi - 3\delta, \quad \eta X_{m'}\sin\delta \leqslant r \leqslant X_{m'}\sin\delta,$$

$$k > k_0(\epsilon_8, \delta, \eta), \quad m' > m_0(\epsilon_8, \delta, \eta, k).$$

If we may put $X_{m'} = 2R_{m'}\operatorname{cosec}\delta$, this gives the required result. For in that case, given $\epsilon_9 > 0$, $\delta > 0$, we have

$$\log|f(re^{i\theta})| < \{h(0)\cos\theta\rho + \epsilon_9\}\, V(r),$$

where $h(0) = -\varliminf_{m\to\infty} A(R_m, k)$ for $|\theta| \leqslant \tfrac{1}{2}\pi - 3\delta$, $r > r_0(\epsilon_9, \delta)$.

It remains to construct the part of the proof corresponding to §3.6. We first show that, in virtue of (4.45),

$$\int\{\log|f(ir)| + \log|f(-ir)|\}\, dr,$$

where the integral is taken over any part of the interval $R_{m'} k^{-2} \leqslant r \leqslant R_{m'} k^2$, is not too small.

We now use Carleman's formula in a way similar to that of § 3.6; but since we are working with $F_m(z)$, which is substantially $1/f(z)$, the inequalities are reversed.

Consider first the positive imaginary axis. Let $\{Z_{m'}\}$ be any sequence of points on the positive imaginary axis* such that $R_{m'}k^{-2} \leqslant |Z_{m'}| \leqslant R_{m'}k^2$, and suppose that $n(|z-Z_{m'}| = \lambda_1 |Z_{m'}|) > H_1 V(|Z_{m'}|)$, where H_1 is independent of λ_1†, for every $\lambda_1 > 0$, $m' > m_0(\lambda_1)$. Then we have

$$\log|P_{m'}(z, \lambda_1)| < H_1 V(|Z_{m'}|) \log\frac{2\lambda_1}{1-\lambda_1}$$

for $|z-Z_{m'}| \leqslant \lambda_1 |Z_{m'}|$. Let

$$f(z) = P_{m'}(z, \lambda_1)\Phi_{m'}(z, \lambda_1);$$

it is easy to see from Theorem II that

$$\log|\Phi_{m'}(z, \lambda_1)| < H_2 K V(|Z_{m'}|),$$

where H_2 is independent of λ_1 and m, for $|z-Z_{m'}| < \lambda_1 |Z_{m'}|$, and so

$$\log|f(z)| < \left(H_1 \log\frac{2\lambda_1}{1-\lambda_1} + H_2 K\right) V(|Z_{m'}|) < 0,$$

provided that $2\lambda_1/(1-\lambda_1) < \exp(-H_2 K/H_1)$. Hence, choosing $\delta < \lambda_1$ in (4.45), we have a contradiction, and so

$$n(|z-Z_{m'}| = \lambda_1 |Z_{m'}|) < \epsilon(\lambda_1) V(|Z_{m'}|), \tag{4.48}$$

where $\epsilon(\lambda_1) \to 0$ as $\lambda_1 \to 0$, for $m' > m_0(\lambda_1)$.

Now apply Lemma 5 to $\Phi_{m'}(z, \lambda_1)$. It is easy to see that (4.31) and (4.32) hold in virtue of Theorem II, and $\Phi_{m'}(z, \lambda_1)$ has no zeros in $|z-Z_{m'}| \leqslant \lambda_1 |Z_{m'}|$. Suppose that, corresponding to some subsequence $\{Z_{m''}\}$, we can find a sequence $\{Z'_{m''}\}$ such that

$$(1-\tfrac{1}{2}\lambda_1)|Z_{m''}| \leqslant |Z'_{m''}| \leqslant (1+\tfrac{1}{2}\lambda_1)|Z_{m''}|,$$

and

$$\log|\Phi_{m''}(Z'_{m''}, \lambda_1)| < -\xi V(|Z_{m''}|),$$

where ξ is independent of λ_1, for every $\lambda_1 > 0$, $m'' > m_0(\lambda_1)$. Then it follows from Lemma 5 that we can find a number λ_0 such that

$$\log|\Phi_{m''}(z, \lambda_1)| < -\tfrac{1}{2}\xi V(|Z_{m''}|)$$

for

$$|z-Z'_{m''}| \leqslant \lambda_0 |Z_{m''}|;$$

* There need not be a point corresponding to each $R_{m'}$.

† $\lambda_1 < \max\{\sin(\beta-\tfrac{1}{2}\pi);\ \sin(-\tfrac{1}{2}\pi-\alpha)\}$.

and it follows from (4.48) that

$$\log|P_{m''}(z, \lambda_1)| < \epsilon(\lambda_1) HV(|Z_{m''}|),$$

where H is independent of λ_1, for $|z-Z_{m''}| \leqslant \lambda_1 |Z_{m''}|$. Choosing λ_1 so that $\epsilon(\lambda_1) H < \frac{1}{2}\xi$, and then $\delta < \lambda_0$, we have a contradiction of (4.45). Hence

$$|\Phi_{m''}(ir, \lambda_1)| > -\epsilon(\lambda_1) HV(|Z_{m'}|)$$

for $|r-Z_{m'}| \leqslant \frac{1}{2}\lambda_1 |Z_{m'}|$, $m' > m_0(\lambda_1)$; and so, by Lemma 7, we have

$$\begin{aligned}\int_E \log|f(ir)|\,dr &= \int_E \{\log|P_{m'}(ir, \lambda_1)| + \log|\Phi_{m'}(ir, \lambda_1)|\,dr \\ &> -\epsilon(\lambda_1)\, 2\lambda_1 |Z_{m'}| V(|Z_{m'}|) - \epsilon(\lambda_1) H\lambda_1 |Z_{m'}| V(|Z_{m'}|) \\ &> -\epsilon(\lambda_1) H\lambda_1 |Z_{m'}| V(|Z_{m'}|),\end{aligned}$$

where E denotes any part of the interval $(1-\frac{1}{2}\lambda_1)|Z_{m'}| \leqslant r \leqslant (1+\frac{1}{2}\lambda_1)|Z_{m'}|$. Let $R'_{m'}$ and $R''_{m'}$ denote any two numbers in the interval $R_{m'} k^{-2} \leqslant r \leqslant R_{m'} k^2$, and let

$$Z^{(1)}_{m'} = iR'_{m'},\quad Z^{(2)}_{m'} = (1+\lambda_1) Z^{(1)}_{m'},\quad \ldots,\quad Z^{(j)}_{m'} = (1+\lambda_1) Z^{(j-1)}_{m'},\quad \ldots,\quad |Z^{J}_{m'}| \leqslant R''_{m'}.$$

Then
$$\int_{R_{m'}}^{R''_{m'}} \log|f(ir)|\,dr = \sum_{j=1}^{J} \int_{E_j} \log|f(ir)|\,dr,$$

where E_j denotes the interval $(1+\frac{1}{2}\lambda_1)|Z^{(j-1)}_{m'}| \leqslant r \leqslant (1+\frac{1}{2}\lambda_1)|Z^{(j)}_{m'}|$, or (if $j = 1$ or $j = J$) the appropriate part of the interval; and this interval is included in $(1-\frac{1}{2}\lambda_1)|Z^{(j)}_{m'}| \leqslant r \leqslant (1+\frac{1}{2}\lambda_1)|Z^{(j)}_{m'}|$.

We have

$$\begin{aligned}(4.49)\quad &\int_{R'_{m'}}^{R''_{m}} \log|f(ir)|\,dr \\ &> -\epsilon(\lambda_1) H\lambda_1 \sum_{j=1}^{J} |Z^{(j)}_{m'}| V(|Z^{(j)}_{m'}|) \\ &> -\epsilon(\lambda_1) H\lambda_1 R''_{m'} V(R''_{m'}) \{1 + (1+\lambda_1)^{-2} + (1+\lambda_1)^{-4} + \ldots\} \\ &> -\epsilon(\lambda_1) H R''_{m'} V(R''_{m'}) (1+\lambda_1) \\ &> -\epsilon(\lambda_1) H R''_{m'} V(R''_{m'}),\end{aligned}$$

and a similar result holds for $\log|f(-ir)|$.

Now apply Carleman's formula to $F(z) = f(z) \exp\{A(R_{m'}, k) V(z)\}$, putting $l = R'_{m'}$ and $R = R''_{m'}$, where $\frac{1}{2} X_{m'} \sin\delta \leqslant R_{m'} \leqslant X_{m'} \sin\delta$, $R''_{m'} = R_{m'+1}$, and $R'_{m'}$ is chosen so that (4.47) holds on it, and also (4.43).

We have

$$\sum_{R'_{m'}\leqslant r_n\leqslant R''_{m'}} \frac{\cos\theta_n}{r_n}\left\{1-\left(\frac{r_n}{R''_{m'}}\right)^2\right\} = I_1+I_2+I_3, \tag{4.491}$$

and, as in §3.7, we have

$$|I_3|<\epsilon_{10}V(R'_{m'})/R'_{m'}$$

for every $\epsilon_{10}>0$ and every $m'>m_0(\epsilon_{10})$. It follows from (4.49) that

$$I_2=\frac{1}{2\pi}\int_{R'_{m'}}^{R''_{m'}}\{\log|f(ir)|+\log|f(-ir)|\}\left(\frac{1}{r^2}-\frac{1}{(R''_{m'})^2}\right)dr$$

$$+\frac{1}{2\pi}\int_{R''_{m'}}^{R''_{m'}}A(R_{m'},k)\Re\{V(ir)+V(-ir)\}\left(\frac{1}{r^2}-\frac{1}{(R''_{m'})^2}\right)dr$$

$$>\frac{1}{2\pi}\frac{1}{(R'_{m'})^2}\int_{R'_{m'}}^{R''_{m}}\{\bar{\log}|f(ir)|+\bar{\log}|f(-ir)|\}dr-\tfrac{1}{2}\epsilon_{10}\frac{V(R''_{m'})}{R''_{m'}}$$

$$>-\frac{1}{2\pi}\epsilon(\lambda_1)H\frac{R''_{m'}V(R''_{m'})}{(R'_{m'})^2}>-\epsilon(\lambda_1)Hk^4\frac{V(R''_{m'})}{R''_{m'}},$$

where $\bar{\log}|f|=\log|f|$ if $|f|<1$, and 0 otherwise, and so, given $\epsilon_{10}>0$, $\delta>0$, $k>1$, we can choose λ_1 so that

$$I_2>-\epsilon_{10}V(R''_{m'})/R''_{m'} \quad \text{for} \quad m>m_0(\epsilon_{10}).$$

Since (4.41) holds, we have

$$\sum\frac{\cos\theta_n}{r_n}\left\{1-\left(\frac{r_n}{R''_{m'}}\right)^2\right\}<\sum\frac{1}{r_n}$$

$$<\frac{n(R''_{m'},\,-\frac{1}{2}\pi+\delta,\,\frac{1}{2}\pi-\delta)-n(R'_{m'},\,-\frac{1}{2}\pi+\delta,\,\frac{1}{2}\pi-\delta)}{R'_{m'}}$$

$$<\epsilon_{11}V(R''_{m'})/R'_{m'},$$

where the sum is taken over zeros for which $R'_{m'}<r_n<R''_{m'}$, $|\theta|\leqslant\frac{1}{2}\pi-\delta$, for every $\epsilon_{11}>0$, $\delta>0$, $k>1$, provided that $m>m_0(\epsilon_{11},\,\delta,\,k)$. Since $R''_{m'}<2k^4R'_{m'}$, we can choose ϵ_{11} so that the sum is less than $\epsilon_{10}V(R''_{m'})/R''_{m'}$. The remaining part of the sum on the left of (4.491) is less than

$$\frac{\sin\delta}{R'_{m'}}\{n(R''_{m'},\,-\tfrac{1}{2}\pi,\,\tfrac{1}{2}\pi)-n(R'_{m'},\,-\tfrac{1}{2}\pi,\,\tfrac{1}{2}\pi)\}<K\delta\frac{V(R''_{m'})}{R_{m'}}$$

$$<\epsilon_{10}V(R''_{m'})/R''_{m'},$$

provided that $\delta<\delta_0(\epsilon_{10},\,k)$, $m'>m_0(\epsilon_{10},\,\delta)$.

Combining these results, we have

$$(4.492)\qquad I_1 = \frac{1}{2\pi R''_{m'}}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \log|F(R''_{m'}e^{i\theta})|\cos\theta\, d\theta$$

$$= \sum_{R'_{m'}\leqslant r_n\leqslant R''_{m'}} \frac{\cos\theta_n}{r_n}\left\{1-\left(\frac{r_n}{R''_{m'}}\right)^2\right\} - I_2 - I_3$$

$$< \epsilon_{10}\left\{\frac{V(R''_{m'})}{R''_{m'}} + 2\frac{V(R''_{m'})}{R''_{m}}\right\} < 4\epsilon_{10}\frac{V(R''_{m'})}{R''_{m'}}.$$

But it follows from (4.461) that, given $\epsilon_{10}>0$, $\delta>0$, $k>1$, we have

$$(4.493)\qquad \log|f(R''_{m'}e^{i\theta})| + A(R_{m'}, k)\cos\theta\, V(R''_{m'}) > -\epsilon_{10}V(R''_{m'})$$

for $|\theta|\leqslant\frac{1}{2}\pi-3\delta$, except in a set of θ of measure δ, $m>m_0(\epsilon_{10}, k, \delta)$. Hence, given any $\epsilon_{12}>0$, $\delta>0$, we have

$$(4.494)\qquad \log|f(R''_{m'}e^{i\theta})| < \{-A(R'_{m'}, k)\cos\theta+\epsilon_{12}\}V(R''_{m'})$$

for $|\theta|\leqslant\frac{1}{2}\pi-3\delta$ except a set of measure 2δ, provided that $m>m_0(\epsilon_{12}, \delta, k)$. For if not we can obtain a contradiction* of (4.492). Choosing a convenient one of the points at which (4.494) holds as centre and applying Lemma 1 to $F_{m'}(z)\exp A(R_{m'}, k)\ V(z)$, we see that, given $\epsilon>0$, $\delta>0$, $k>1$, we have

$$(4.495)\qquad \log|f(re^{i\theta})| < \{-A(R_m, k)\cos\theta+\epsilon\}V(r)$$

for $|\theta|\leqslant\frac{1}{2}\pi-4\delta$, $R_{m'}\leqslant r\leqslant R_{m'+1}$ without excepting any set of θ. For, since $F_m(z)\exp\{A(R_m, k)V(z)\}$ has no zeros, there are no excepted circles in the result obtained from the lemma. This completes the result. For (4.493) and (4.495) with (4.42) and (4.43) give (4.44).

Applications.

5.1. We now apply Theorems I and III to integral functions of order $\rho\geqslant 1$ whose directions of Borel of the maximum kind lie in an angle π/ρ. We write

$$f(z) = z^m\exp\{c_0+c_1z+\ldots+c_pz^p\}\prod_{n=1}^{\infty}\left(1-\frac{z}{z_n}\right)\exp\left\{\frac{z}{z_n}+\frac{z^2}{2z_n}+\ldots+\frac{z^p}{pz_n{}^p}\right\},$$

where $p=[\rho]$, unless otherwise stated.

* Compare (3.78).

If $\rho > 1$, the functions whose directions of Borel lie in an angle* π/ρ behave very much like Mittag-Leffler's function

$$E_a(z) = \Sigma z^n/\Gamma(1+an), \text{ where } a = 1/\rho < 1.$$

The following theorem is the key to the situation.

THEOREM IV. *Suppose that $f(z)$ is an integral function of proximate order $\rho(r)$, where $\rho > 1$, and that*

$$h(\theta) = h(0)\cos\theta\rho \text{ for } |\theta| \leqslant \tfrac{1}{2}\pi/\rho,$$

$$h(\theta) = 0 \qquad \text{otherwise.}$$

Then (1) $\lim_{r\to\infty} \dfrac{n(r, -\frac{1}{2}\pi/\rho+\delta, \frac{1}{2}\pi/\rho-\delta)}{r^{\rho(r)}} = 0$ *for every* $\delta > 0$,

(2) $\lim_{r\to\infty} \dfrac{n(r, \frac{1}{2}\pi/\rho+\delta, 2\pi-\frac{1}{2}\pi/\rho-\delta)}{r^{\rho(r)}} = 0$ *for every* $\delta > 0$,

(3) *given any* $\epsilon, \delta, \eta > 0$, *we can choose* $R_0 = R_0(\epsilon, \delta, \eta)$ *so that*

$$|n(r, \pm\tfrac{1}{2}\pi/\rho-\delta, \pm\tfrac{1}{2}\pi/\rho+\delta) - A(R)r^{\rho(r)}/2\pi| < \epsilon r^{\rho(r)},$$

where $\overline{\lim_{R\to\infty}} A(R) = 1$ *and* $\underline{\lim_{R\to\infty}} A(R) \geqslant 0$ *for* $\eta R \leqslant r \leqslant R$, $R > R_0$,

(4) *if ρ is an integer, we have, in addition, for every* $\epsilon, \eta > 0$,

$$\left| c_\rho + \frac{1}{\rho}\sum_{r_n \leqslant r} \frac{1}{z_n{}^\rho} - \frac{A(R)}{2\rho} r^{\rho(r)-\rho} \right| < \epsilon R^{\rho(R)-\rho},$$

for $\eta R \leqslant r \leqslant R$, $R > R_0$ (ϵ, η).

COROLLARY 1. *If $f(z)$ satisfies the hypotheses of Theorem IV, then* $\arg z = \pm\frac{1}{2}\pi/\rho$ *are the only directions of Borel of the maximum kind, and there is no exceptional value in these directions.*

COROLLARY 2. *If $f(z)$ is an integral function of order $\rho > 1$, and if all the directions of Borel of the maximum kind lie in the angle* $|\arg z| \leqslant \frac{1}{2}\pi/\rho$, *then the conclusions of Theorem IV hold,* $\arg z = \pm\frac{1}{2}\pi/\rho$ *are the only directions of Borel of the maximum kind, and there is no exceptional value in these directions.*

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* See H. Milloux, *Acta Math.* 52 (1928), Ch. V, and M. L. Cartwright, *Proc. London Math. Soc.* (2), 38 (1935), 176, 177.

It is sometimes convenient to express parts (3) and (4) of Theorem IV as follows : *There is a non-negative function $A(r)$ such that*

$$\lim_{r\to\infty} A(r) = 1, \quad \lim_{r\to\infty} \{A(r) - A(\eta r)\} = 0$$

for every $\eta > 0$, and, as $r \to \infty$,

$$n(r, \pm\tfrac{1}{2}\pi/\rho - \delta, \pm\tfrac{1}{2}\pi/\rho + \delta) \sim A(r)\, r^{\rho(r)}/(2\pi)$$

for every $\delta > 0$. If ρ is an integer, we have, in addition,

$$c_\rho + \frac{1}{\rho} \sum_{r_n \leqslant r} \frac{1}{z_n{}^\rho} \sim \frac{A(r)}{2\rho}\, r^{\rho(r)-\rho}$$

as $r \to \infty$.

Corollary 1 follows immediately from the theorem when we observe that $f(z) - a$ satisfies the hypotheses for every value of a. For, obviously,

$$h(\theta, a) = \overline{\lim_{r\to\infty}} \frac{\log|f(re^{i\theta}) - a|}{r^{\rho(r)}} = \cos\theta\rho \text{ for } |\theta| \leqslant \tfrac{1}{2}\pi/\rho,$$

and $h(\theta, a) \leqslant 0$ otherwise. Since the remaining angle is greater than π/ρ, it follows from the properties of $h(\theta)$ that $h(\theta, a) = 0$ in the remaining angle, and therefore

$$\lim_{r\to\infty} n(r, -\tfrac{1}{2}\pi/\rho + \delta, \tfrac{1}{2}\pi/\rho - \delta, a)/r^{\rho(r)} = 0,$$

$$\lim_{r\to\infty} n(r, +\tfrac{1}{2}\pi/\rho + \delta, 2\pi - \tfrac{1}{2}\pi/\rho - \delta, a)/r^{\rho(r)} = 0,$$

for every a, and a similar result corresponding to (3).

To prove Corollary 2, we observe that, if $f(z)$ satisfies the hypotheses of Corollary 2, it follows from the theorem quoted in § 1.1 that $h(\theta) = 0$ for $+\tfrac{1}{2}\pi/\rho \leqslant \theta \leqslant 2\pi - \tfrac{1}{2}\pi/\rho$. Since the remaining angle is exactly equal to π/ρ, it follows from the properties of $h(\theta)$ that $h(\theta) = h(0)\cos\theta\rho$ for $|\theta| \leqslant \tfrac{1}{2}\pi/\rho$. By the definition of the proximate order of an integral function, we have $h(0) = 1$, and the result now follows from Theorem IV and Corollary 1.

5.2. Part (1) of Theorem IV follows immediately from Theorem I. Part (2) also follows from Theorem I, or from a much easier theorem* in another paper. For parts (3) and (4) we require two lemmas. The one

* See M. L. Cartwright, *Proc. London Math. Soc.* (2), 38 (1935), 166.

required for part (4) has been proved elsewhere*, and is as follows:

LEMMA 8. *Suppose that $F(z)$ is regular for $|z| \leqslant 1$, $F(0) = 1$, and let $z_1, z_2, \ldots, z_n = r_n e^{i\theta_n}$ denote the zeros of $F(z)$ for which $|z| \leqslant R$. Then*

$$\left[\frac{d^p}{dz^p} \log F(z)\right]_{z=0} + \frac{1}{p} \sum_{r_n < R} \frac{1}{z_n^p} \left(1 - \frac{r_n^{2p}}{R^{2p}}\right) = \frac{1}{\pi R^p} \int_{-\pi}^{\pi} \log |F(Re^{i\theta})| e^{-i\theta_p} d\theta.$$

where p is an integer

It is possible to deduce part (3) from Lemma 8, but it is more direct to deduce it from Lemma 9, which we shall use again later.

LEMMA 9. *Suppose that $f(z)$ is regular for $|\arg z| \leqslant \alpha$, $|z| \geqslant l$, and let $R_1, R_2, \ldots, R_m \to \infty$ be a given sequence of positive numbers. Suppose that, given $\epsilon, \delta, \eta, \zeta > 0$, $\delta < \alpha$, we can choose $m_0(\epsilon, \delta, \eta, \zeta)$ so that, if $m > m_0(\epsilon, \delta, \eta, \zeta)$, we have*

$$\log |f(re^{i\theta})| < \{B_1(R_m) \sin \theta\rho + \epsilon\} r^{\rho(r)} \tag{1}$$

for $0 \leqslant \theta \leqslant \alpha$, $\eta R_m \leqslant r \leqslant 2R_m$,

$$\log |f(re^{i\theta})| < \{B_2(R_m) \sin \theta\rho + \epsilon\} r^{\rho(r)} \tag{2}$$

for $-\alpha \leqslant \theta \leqslant 0$, $\eta R_m \leqslant r \leqslant 2R_m$,

$$\log |f(re^{i\theta})| > \{B_1(R_m) \sin \theta\rho - \epsilon\} r^{\rho(r)} \tag{3}$$

for $\delta \leqslant \theta \leqslant \alpha - \delta$, $\eta R_m \leqslant r \leqslant 2R_m$,

$$\log |f(re^{i\theta})| > \{B_2(R_m) \sin \theta\rho - \epsilon\} r^{\rho(r)} \tag{4}$$

for $-\alpha + \delta \leqslant \theta \leqslant -\delta$, $\eta R_m \leqslant r \leqslant 2R_m$, except perhaps in (3) and (4) for circles the sum of whose radii is less than ζR_m.

Then, given any positive numbers $\epsilon_1, \eta_1, \delta_1$ less than 1 ($\delta_1 < \frac{1}{2}\alpha$), we can choose $\epsilon, n, \delta, \zeta$, and m_0 so that

$$\left| n(R_m \sin \delta_1, -\delta_1, -\delta_1) - n(\eta_1 R_m \sin \delta_1, -\delta_1, \delta_1) - \frac{1}{2\pi} \{B_1(R_m) + B_2(R_m)\} \{V(R_m \sin \delta_1) - V(\eta_1 R_m \sin \delta_1)\} \right| < \epsilon_1 V(R_m \sin \delta_1),$$

where $n(r, -\delta, \delta)$ denotes the number of zeros of $f(z)$ for which $|\arg z| \leqslant \delta$, $l \leqslant |z| \leqslant r$, provided that $m > m_0$.

It may be observed that $B_1(R_m) > -B_2(R_m)$. For if not, applying Lemma 2 with $\alpha = \gamma$, we have a contradiction.

* See M. L. Cartwright, *loc. cit.*, 33 (1932), 220, Lemma C.

Proof. We may, without loss of generality, suppose that $\alpha = \pi$. For if not, we can reduce the lemma to this form by putting $z = w^{\pi/\alpha}$. We can choose a point $z = iR_m'$ such that (3) is satisfied and also $0 < R_m - R_m' < \zeta R_m$; and, if $k < 1 - 2\eta - 2\zeta$, the circle $|z - iR_m'| \leqslant kR_m'$ lies in the region $0 \leqslant \arg z \leqslant \pi$, $\eta R_m \leqslant |z| \leqslant 2R_m$. In this region (1) holds; and so, applying Lemma 1, parts (1) and (3), to

$$F(z) = f(z) \exp\{iB_1(R_m) V(z)\}$$

in this circle, we have

$$n(|z - iR_m'| \leqslant k^2 R_m') < H(k)\, \epsilon V(R_m), \tag{5.21}$$

$$\left| \int d \arg F(z) \right| < H(k)\, \epsilon V(R_m), \tag{5.22}$$

where the integral is taken along any arc of $|z - iR_m'| = k^2 R_m'$ of length less than $2\pi k^2 R_m'$.

Now k^2 may be chosen as near to 1 as we please, provided that η and ζ are sufficiently small. If $k^2 > \cos\delta_1$, the circle $|z - iR_m'| = k^2 R_m'$ cuts $\arg z = \delta_1$, where $|z| = R_m' \{\sin\delta_1 - \sqrt{(k^4 - \cos^2\delta_1)}\} = \eta_1 \sin\delta_1 R_m'$, say; and, by choosing k sufficiently near 1, we can make η_1 as small as we please. The straight line $\arg z = \delta_1$ from $|z| = \eta_1 R_m' \sin\delta_1$ to $|z| = R_m \sin\delta_1$ lies inside the circle $|z - iR_m'| \leqslant k^2 R_m'$; and so, in virtue of (5.21), we can deform the arc into this straight line without altering the form of (5.22). Hence

$$\left| \int d \arg f(z) + B_1(R_m) \cos\delta_1 \rho\{V(R_m \sin\delta_1) - V(\eta_1 R_m \sin\delta_1)\} \right| < \epsilon H(k)\, V(R_m), \tag{5.23}$$

where the integral is taken along $\arg z = \delta_1$ from $|z| = \eta_1 R_m \sin\delta_1$ to $|z| = R_m \sin\delta_1$. Applying a similar argument to $f(z)$ in the region $-\pi \leqslant \arg z \leqslant 0$, we have

$$\left| \int d \arg f(z) - B_2(R_m) \cos\delta_1 \rho\{V(R_m \sin\delta_1) - V(\eta_1 R_m \sin\delta_1)\} \right| < \epsilon H(k)\, V(R_m), \tag{5.24}$$

where the integral is taken along $\arg z = -\delta_1$ from $|z| = \eta_1 R_m \sin\delta_1$ to $|z| = R_m \sin\delta_1$.

Now choose $\delta \leqslant \delta_1$ and R_m'' so that (3) is satisfied at $e^{i\delta} R_m''$, and $0 < R_m'' - R_m \sin\delta_1 < \zeta R_m$, and consider the circle

$$|z - R_m'' e^{i\delta}| \leqslant 3 \sin\delta_1 R_m''.$$

This circle lies in the region $|\arg z| \leqslant 4\delta_1$, $\eta R_m \leqslant |z| \leqslant R_m$, provided that η and ζ are sufficiently small. Hence we may apply Lemma 1 to $f(z)$, and we have

$$(5.25)\quad n(|z-R_m''e^{i\delta}| \leqslant 2\sin\delta_1 R_m'') < H[\epsilon+\{|B_1(R_m)|+|B_2(R_m)|\}\sin(4\delta_1\rho)]V(R_m'\sin\delta_1),$$

$$(5.26)\quad \left|\int d\arg f(z)\right| < H[\epsilon+\{|B_1(R_m)|+|B_2(R_m)|\}\sin(4\delta_1\rho)]V(R_m'\sin\delta_1),$$

where the integral is taken along any arc of $|z-R_m''e^{i\delta}| = 2\sin\delta_1 R_m''$. This circle covers the arc $|z| = R_m\sin\delta_1$, $|\arg z| \leqslant \delta_1$, provided that ζ is sufficiently small; and, in virtue of (5.25), we may deform the arc of integration into the arc $|z| = R_m\sin\delta_1$, $|\arg z| \leqslant \delta_1$, without altering the form of (5.26). A similar argument shows that (5.26) holds when the integral is taken along $|z| = \eta_1 R_m\sin\delta_1$, $|\arg z| = \delta_1$, provided that η and ζ are chosen so small that the appropriate circle lies inside the region $|\arg z| \leqslant 4\delta_1$, $\eta R_m \leqslant |z| \leqslant R_m$. The result now follows if we combine (5.23) and (5.24) and (5.26) with the formula

$$n(R_m\sin\delta_1, -\delta_1, \delta_1) - n(\eta_1 R_m\sin\delta_1, -\delta_1, \delta_1) = \frac{1}{2\pi}\int d\arg f(z),$$

where the integral is taken round the boundary of the region $|\arg z| \leqslant \delta_1$, $\eta_1 R_m\sin\delta_1 \leqslant |z| \leqslant R_m\sin\delta_1$. For, since k depends only on δ_1, η, and ζ, we can choose ϵ so that $\epsilon H(k)V(R_m) < \frac{1}{4}\epsilon_1 V(R_m\sin\delta_1)$ in (5.23) and (5.24); and the right-hand side of (5.26) is less than $\frac{1}{4}\epsilon_1 V(R_m\sin\delta_1)$, provided that δ_1 and ϵ are sufficiently small. Since the result holds for small δ_1, it follows from (5.21) that it holds, *a fortiori*, for $\delta_1 < \frac{1}{2}\alpha$.

5.3. *Proof of Theorem IV, parts* (3) *and* (4). It follows from Theorem II of my former paper* that, given $\epsilon, \eta, \zeta, \delta > 0$, we can choose $R_0(\epsilon, \delta, \eta, \zeta)$ so that

$$(5.31)\quad \left|\log|f(re^{i\theta})|\right| < \epsilon r^{\rho(r)}$$

for $\frac{1}{2}\pi/\rho+\delta \leqslant |\theta| \leqslant \pi$, $\eta R \leqslant r \leqslant R$, $R > R_0(\epsilon, \delta, \eta, \zeta)$, except perhaps in circles the sum of whose radii is less than ζR. Similarly, it follows from Theorem I of this paper that

$$(5.32)\quad \left|\log|f(re^{i\theta})| - A(R)\cos\theta\rho\, r^{\rho(r)}\right| < \epsilon r^{\rho(r)},$$

* M. L. Cartwright, *Proc. London Math. Soc.* (2), 38 (1935), 166. This is the same as Theorem II of this paper with ϵ in place of K, or Theorem I with $h'(\frac{1}{2}\pi/\rho) = -A$.

where $A(R)$ satisfies the required conditions, for $|\theta| \leqslant \frac{1}{2}\pi/\rho$, $\eta R \leqslant r \leqslant R$, $R > R_0(\epsilon, \delta, \eta, \zeta)$, except in circles the sum of whose radii is less than ζR. Part (3) follows at once on applying Lemma 9 to $f(z)$ in the angles

$$|\arg z \mp \tfrac{1}{2}\pi/\rho| \leqslant \pi/\rho.$$

We have $B_1(R_m) = 0$, $B_2(R_m) = -A(R_m)$ in $|\arg z - \frac{1}{2}\pi/\rho| \leqslant \pi/\rho$, and $B_1(R_m) = A(R_m) = B_2(R_m) = 0$ in $|\arg z + \frac{1}{2}\pi/\rho| \leqslant \pi/\rho$.

We divide $f(z)$ by z^m; and then it follows from (5.31), (5.32), and Lemma 8 that, if ρ is an integer, given ϵ, $\eta > 0$, we have

$$\left| c_\rho + \frac{1}{\rho} \sum_{r_n \leqslant r} \frac{1}{z_n^{\rho}} \left(1 - \frac{r_n^{2\rho}}{r^{2\rho}}\right) - \frac{A(R)}{2\pi} r^{\rho(r)-\rho} \right| < \epsilon R^{\rho(r)-\rho}$$

for $\eta R \leqslant r \leqslant R$, $R > R_0(\epsilon)$. And it follows from parts (1), (2), and (3) of the theorem that

$$\frac{1}{\rho} \sum_{r_n \leqslant r} \frac{r_n^{2\rho}}{z_n^{\rho}} = o(r^{\rho(r)+\rho})$$

as $r \to \infty$. Hence we have part (4).

5.4. By using only Theorem III of this paper, instead of combining it with the theorem for an angle greater than π/ρ, we can obtain similar results for functions of order 1 whose directions of Borel lie in an angle π; but in this case we cannot tell whether the functions have exceptional values or not. The key theorem now is as follows*:

THEOREM V. *Suppose that $f(z)$ is an integral function of order 1, and that*

$$h(\theta) = \max\,(A\cos\theta;\ B\cos\theta),$$

where $B \geqslant A$, $B > 0$. Then

(1) $$\lim_{r\to\infty} n(r, -\tfrac{1}{2}\pi+\delta, \tfrac{1}{2}\pi-\delta)/r^{\rho(r)} = 0,$$

(2) $$\lim_{r\to\infty} n(r, \tfrac{1}{2}\pi+\delta, \tfrac{3}{2}\pi-\delta)/r^{\rho(r)} = 0,$$

(3) *given ϵ, δ, $\eta > 0$, we can choose $R_0(\epsilon, \delta, \eta)$ so that*

$$|n(r, \pm\tfrac{1}{2}\pi-\delta, \pm\tfrac{1}{2}\pi+\delta) - \tfrac{1}{2}\{B(R)-A(R)\}r/\pi| < \epsilon R^{\rho(R)},$$

where $$B(R) \geqslant A(R), \quad \overline{\lim_{R\to\infty}}\, B(R) = B, \quad \underline{\lim_{R\to\infty}}\, A(R) = A,$$

* Compare M. L. Cartwright, *Proc. London Math. Soc.* (2), 38 (1935), 178–179.

for $\eta R \leqslant r \leqslant R$, $R > R_0(\epsilon, \delta, \eta)$,

$$\left| c_1 + \sum_{r_n \leqslant r} \frac{1}{z_n} - \tfrac{1}{2}\{A(R)+B(R)\} r^{\rho(r)-1} \right| < \epsilon R^{\rho(r)-1}, \tag{4}$$

where ϵ, η, $A(R)$, $B(R)$ *have the same meanings as before, for* $\eta R \leqslant r \leqslant R$, $R > R_0(\epsilon, \delta, \eta)$.

COROLLARY 1. *If* $f(z)$ *satisfies the hypotheses of Theorem V, then* $\arg z = \pm\frac{1}{2}\pi$ *are the only directions of Borel of the maximum kind.*

COROLLARY 2. *If* $f(z)$ *is an integral function of order* 1 *and has no directions of Borel of the maximum kind for which* $|\arg z| < \frac{1}{2}\pi$, *then* $f(z)$ *satisfies the hypotheses of Theorem V for some* A *and* B, *and the lines* $\arg z = \pm\frac{1}{2}\pi$ *are the only directions of Borel of the maximum kind.*

It is sometimes convenient to express parts (3) and (4) in the alternative form: *As* $r \to \infty$,

$$n(r, \pm\tfrac{1}{2}\pi-\delta, \pm\tfrac{1}{2}\pi+\delta) \sim \tfrac{1}{2}\{B(r)-A(r)\} r^{\rho(r)}/\pi,$$

$$c_1 + \sum_{r_n < r} \frac{1}{z_n} \sim \tfrac{1}{2}\{A(r)+B(r)\} r^{\rho(r)-1},$$

where $A(r)$ *and* $B(r)$ *are real functions such that* $B(r) \geqslant A(r)$,

$$\underline{\lim_{r\to\infty}}\, A(r) = A, \quad \overline{\lim_{r\to\infty}}\, B(r) = B,$$

$$\lim_{r\to\infty} \{A(r)-A(\eta r)\} = 0, \quad \lim_{r\to\infty} \{B(r)-B(\eta r)\} = 0,$$

for every $\eta > 0$.

Unlike the functions of order $\rho > 1$ which we considered in Theorem IV, these functions may have an exceptional value, as may easily be seen by considering the function e^z. Exceptional values necessarily arise whenever $h(\theta) < 0$, but they may also arise when $h(\theta) > 0$ for every value of θ except $\pm\frac{1}{2}\pi$, as may be seen from examples which I have considered elsewhere*. In these examples, as $r \to \infty$, $n(r) = o(r)$, and

$$\log|f(re^{i\theta})| \sim S(r) \cos\theta\rho\, r,$$

where $S(r) = \sum_{r_n \leqslant r} \frac{1}{z_n}$, oscillates between ± 1, in a set of linear density 1.

* M. L. Cartwright, *Proc. London Math. Soc.* (2), 33 (1932), §2.4, 218, 219.

We have $h(\theta) = |\cos\theta|$, and from Lemma 9 it is easy to see that

$$\overline{\lim_{r\to\infty}}\, n(r, \pm\tfrac{1}{2}\pi-\delta, \tfrac{1}{2}\pi-\delta, a)/r^{\rho(r)} = \frac{1}{\pi}$$

for every $\delta > 0$ and every $a \neq 0$.

To prove Corollary 1, we observe that

$$h(\theta, a) = \overline{\lim_{r\to\infty}} \frac{\log|f(re^{i\theta})-a|}{r^{\rho(r)}} \leqslant \max(A\cos\theta;\ B\cos\theta;\ a),$$

and is of the same form as $h(\theta)$ although A may not be the same in both cases. Hence

$$\lim_{r\to\infty} n(r, -\tfrac{1}{2}\pi+\delta, \tfrac{1}{2}\pi-\delta, a)/r^{\rho(r)} = 0$$

for every $\delta > 0$ and every a, and a similar result for the left half-plane.

It follows from Theorem III that, if $f(z)$ satisfies the hypotheses of Corollary 2, then $h(\theta) = h(0)\cos\theta\rho$ for $|\theta| \leqslant \frac{1}{2}\pi$. Since

$$h(\tfrac{1}{2}\pi) = h(-\tfrac{1}{2}\pi) = 0,$$

by the properties of $h(\theta)$, we have $h(\theta) = h(-\pi)\cos\theta\rho$ for $|\theta+\pi| \leqslant \frac{1}{2}\pi$, and the result follows from Theorem V and Corollary 1.

The theorem itself follows from Theorem I by the methods used in Theorem IV. For by Theorem I we have parts (1) and (2) immediately, and

$$\log|f(re^{i\theta})| \sim B(r)\cos\theta r, \quad |\theta| < \tfrac{1}{2}\pi,$$

$$\log|f(re^{i\theta})| \sim A(r)\cos\theta r, \quad |\theta+\pi| < \tfrac{1}{2}\pi,$$

where $A(r)$ and $B(r)$ satisfy the required conditions, as $r\to\infty$ in a set of linear density 1. We can now apply Lemmas 8 and 9 as before.

5.5. It should be observed that, since $A(R)$ and $B(R)$ are real,

$$\Sigma \frac{\sin\theta_n}{r_n}$$

is convergent. It follows from Theorem V that $n(r)$ and $\Sigma 1/z_n$ are connected. I have proved elsewhere that, if $n(r) \sim \frac{1}{2}(B-A)r/\pi$, then

$$\sum_{n=1}^{\infty} \frac{1}{z_n} = \tfrac{1}{2}(A+B).$$

THEOREM VI. *If $f(z)$ satisfies the hypotheses of Theorem V and*

$$c_1+\sum_{r_n\leq r}\frac{1}{z_n}$$

is convergent, then

$$\overline{\lim_{r\to\infty}}\frac{n(r)}{r}=\frac{B-A}{\pi}.$$

Proof. It follows from parts (3) and (4) of Theorem V that

$$\frac{n(r)}{r}\sim\frac{B(r)-A(r)}{\pi},$$

where $$\lim_{R\to\infty}\{A(R)+B(R)\}=C=c_1+\sum_{n=1}^{\infty}\frac{1}{z_n}.$$

Since $B(r)\leq B+\epsilon$ for every $\epsilon>0$, $r>r_0(\epsilon)$, and $\underline{\lim}\, A(r)=A$, we have

$$C\leq A+B;$$

and since $$A(R)\geq A-\epsilon,\ \overline{\lim}\, B(R)=B,$$

we have $$C\geq A+B.$$

Hence $C=A+B$; and it follows that, if $A(R)\to A$ through a certain sequence of values of R, then $B(R)\to B$ through that sequence. Hence

$$\overline{\lim_{R\to\infty}}\frac{n(r)}{r}=\frac{B-A}{\pi}.$$

5.6. Suppose that $$f(z)=\sum_{n=0}^{\infty}a_n z^n$$

is an integral function of order 1 and mean type. Then there is a power series

$$\phi(z)=\sum_{n=0}^{\infty}n!\,a_n z^n$$

with a finite radius of convergence, and the Borel polygon of $\phi(z)$ determines the value of $h(\theta)$ to some extent. For the Borel polygon is defined by*

$$r\leq\frac{1}{\overline{h^+(\theta)}},$$

where $h^+(\theta)=h(\theta)$ if $h(\theta)>0$ and 0 otherwise. Hence we have the following theorem.

* See E. Borel, *Leçons sur les séries divergentes* (Paris, 1928), 152.

THEOREM VII. *Suppose that $f(z)$ is an integral function of order 1 and mean type and that the Borel polygon of $\phi(z)$ consists of one or two lines parallel to the imaginary axis. Then* $\arg z = \pm\frac{1}{2}\pi$ *are the only directions of Borel of the maximum kind.*

It is easy to see that

$$h^+(\theta) = \max(A\cos\theta,\ B\cos\theta),$$

where $B > A$, $B > 0$, and $A \leqslant 0$, and we can apply Theorem V, Corollary 2.

Theorem VII can also be stated in terms of the indicator diagram* of $f(z)$. *If the indicator diagram of $f(z)$ is a portion of the real axis, then* $\arg z = \pm\frac{1}{2}\pi$ *are the only directions of Borel of the maximum kind.*

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* See G. Pólya, *Math. Zeitschrift*, 29 (1929), 550–640.

THE DUAL INTEGRAL REPRESENTATIONS OF KUMMER'S SERIES ${}_1F_1(a;\ c;\ x)$

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1. *Introduction.*

An interesting point in connection with the hypergeometric integrals of Euler's type was raised in the last paragraph of Riemann's fundamental memoir on the hypergeometric function. The equation

$$x(1-x)\,D^2y+[c-(a+b+1)\,x]\,Dy-aby=0 \tag{1}$$

is one of twenty-four equivalent canonical forms to which any Riemann P-equation can be reduced. If we proceed to solve (1) as it stands by Euler's transformation

$$y=\int(1-xt)^{\lambda}\,f(t)\,dt, \tag{2}$$

we find two distinct solutions, with $\lambda=-a$ and $\lambda=-b$. Thus, if a, b, $c-a$, $c-b$ are all positive, and $|x|<1$, we find Euler's formulae

$$F(a,\,b;\ c;\ x)=\frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\,dt, \tag{3}$$

$$=\frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b}\,dt. \tag{4}$$

These are two out of forty-eight different Eulerian representations of $F(a,\,b;\ c;\ x)$. Now these forty-eight are divided into two sets of twenty-four, which can all be derived from either (3) or (4) by a change of

variable under the integral sign and an appropriate adjustment of the limits. We cannot, in general, thus transform (3) into (4); but Riemann indicated that both could be derived from a multiple integral, which was subsequently constructed by Wirtinger.

The purpose of the present note is to look for the analogues of these properties in the confluent hypergeometric functions. These include Bessel's functions, Sonine's and Hermite's polynomials, Airy's integral and others, which can all be reduced to Kummer's confluent hypergeometric function; they have been very lucidly systematized by means of Whittaker's $W_{k,m}(x)$-function and its integral representations; but, since our sole object at present is to emphasize a certain analogy, it is more convenient to work with Kummer's form

$$xD^2y+(c-x)Dy-ay=0. \tag{5}$$

When c is not an integer, the solutions are

$$\left.\begin{aligned} y_1 &\equiv {}_1F_1(a;\ c;\ x)\equiv e^x\,{}_1F_1(c-a;\ c;\ -x),\\ y_2 &\equiv x^{1-c}\,{}_1F_1(a-c+1;\ 2-c;\ x)\equiv e^x x^{1-c}\,{}_1F_1(1-a;\ 2-c;\ -x),\end{aligned}\right\} \tag{6}$$

and these relations embody Kummer's linear transformation, with the usual notation

$${}_1F_1(a;\ c;\ x)=\lim_{b\to\infty}F(a,b;\ c;\ x/b)=1+\frac{a}{1.c}x+\frac{a(a+1)}{1.2c(c+1)}x^2+\ldots. \tag{7}$$

The equation (5) is one of a set of four equivalent confluent hypergeometric equations; the other three are obtained by putting

$$\left.\begin{aligned} y&=e^xy', && x=-x', && a=c'-a', && c=c',\\ y&=x^{1-c}y', && x=x', && a=a'-c'+1, && c=2-c',\\ y&=e^xx^{1-c}y', && x=-x', && a=1-a', && c=2-c'.\end{aligned}\right\} \tag{8}$$

The equation (5) is of Laplace's linear type, and can be solved by putting

$$y=\int_\alpha^\beta e^{xt}f(t)\,dt, \tag{9}$$

between suitable limits of integration. From the equation as it stands,

and from its three equivalent forms, we obtain the representations

$$\int e^{xt} t^{a-1}(1-t)^{c-a-1}\,dt, \tag{10a}$$

$$e^x \int e^{-xt} t^{c-a-1}(1-t)^{a-1}\,dt, \tag{10b}$$

$$x^{1-c} \int e^{xt} t^{a-c}(1-t)^{-a}\,dt, \tag{10c}$$

$$e^x x^{1-c} \int e^{-xt} t^{-a}(1-t)^{a-c}\,dt. \tag{10d}$$

This set represents the analogue of the forty-eight Eulerian hypergeometric integrals, and corresponds exactly to the following set of four:

$$\int_0^1 (1-xt)^{-b} t^{a-1}(1-t)^{c-a-1}\,dt, \tag{11a}$$

$$(1-x)^{c-a-b} \int_0^1 (1-xt)^{b-c} t^{c-a-1}(1-t)^{a-1}\,dt, \tag{11b}$$

$$x^{1-c} \int_{1/x}^{\infty} (1-xt)^{c-b-1} t^{a-c}(1-t)^{-a}\,dt, \tag{11c}$$

$$(1-x)^{c-a-b} x^{1-c} \int_{1/x}^{\infty} (1-xt)^{b-1} t^{-a}(1-t)^{a-c}\,dt. \tag{11d}$$

Now we can transform (11a) into (11b), and (11c) into (11d), by putting $xtt'-t-t'+1=0$; we can also obtain (11c) from (3) by putting $xtt'-1=0$; thus the first pair of representations (11) belong to the type (4) and the last pair to the type (3). Now evidently (10a) is transformed into (10b), and (10c) into (10d), by putting $t+t'-1=0$; but there is no elementary method of transforming the first pair into the last.

A similar problem arises in connection with Kummer's second equation,

$$xD^2y+cDy-y=0, \tag{12}$$

which is a transformation of Bessel's equation of order $n=\pm\frac{1}{2}(1-c)$, and is satisfied by

$$y = A_0 F_1(c;\ x)+Bx^{1-c}\,{}_0F_1(2-c;\ x), \tag{13}$$

where

$${}_0F_1(c;\ x) = \lim_{(a,\ b)\to\infty} F\left(a, b;\ c;\ \frac{x}{ab}\right). \tag{14}$$

The formal solutions given by Laplace's rule are

$$y = \int e^{xt+1/t} t^{c-2}\, dt, \tag{15}$$

$$y = x^{1-c} \int e^{xt+1/t} t^{-c}\, dt, \tag{16}$$

according as we treat the equation (12) as it stands or transform it by putting $y = x^{1-c} y'$. But the problem of the equivalence of the two forms is now trivial, since we need only put $xtt' - 1 = 0$ to pass from one to the other.

2. *Standard paths.*

The ordinary hypergeometric function has six principal branches (of which any three are linearly connected), corresponding in each of the forty-eight Eulerian representations to six standard double loop paths. The confluent hypergeometric function has two principal branches (6); we can assign two standard paths, corresponding to these branches, from the classical theory of Laplace's linear equation. It will be found that, in general, when we pass from one type of representation to the other of the same branch, the two standard paths are interchanged; the case where the solution involves a logarithm constitutes an exception.

In accordance with the rules, the limits of integration in (9) must be so chosen that

$$\left[e^{xt} t(1-t) f(t) \right]_\alpha^\beta = 0, \tag{17}$$

or, with the form (10a),

$$\left[e^{xt} t^a (1-t)^{c-a} \right]_\alpha^\beta = 0. \tag{17a}$$

I. *General case.* If a, c, $c-a$ are none of them integers, two independent solutions not identically zero are given by simple loops from infinity, in a direction such that $R(xt) = -\infty$, about $t = 0$ and $t = 1$. We denote this point at infinity by I and write

$$L_0 = \int_I^{(0+)} e^{xt} f(t)\, dt, \quad L_1 = \int_I^{(1+)} e^{xt} f(t)\, dt. \tag{18}$$

From these we form the combinations [in the case (10a)]

$$\left.\begin{aligned} \int_\Phi &= \int_I^{(0+,\ 1+,\ 0-,\ 1-)} = (1 - e^{2\pi i(c-a)}) L_0 - (1 - e^{2\pi i a}) L_1, \\ \int_\Omega &= \int_I^{(0+,\ 1+)} = L_0 + e^{2\pi i a} L_1. \end{aligned}\right\} \tag{19}$$

Since $e^{2\pi i c} \neq 1$, and since L_0 and L_1 are linearly independent solutions (as is proved, for example, in Picard's treatise), the integral about the double loop Φ interlacing both points $t = 0$ and $t = 1$, and the integral about the loop from infinity Ω enclosing them, are linearly independent.

When t describes the double loop Φ, the amplitude of the integrand returns to its initial value; we may thus regard the path as closed and deform it into a fixed path (independent of x) lying entirely in the finite part of the plane, but clear of the points $t = 0$ and $t = 1$. Since $f(t)$ is continuous on such a path, we may expand e^{xt} in ascending powers of x; for any fixed value of x, we have a uniformly convergent series of functions of t. On evaluating term by term, we have, for the representation (10a),

$$\int_\Phi e^{xt} t^{a-1}(1-t)^{c-a-1}\,dt = {}_1F_1(a;\ c;\ x)\int_\Phi t^{a-1}(1-t)^{c-a-1}\,dt. \tag{20}$$

Thus the finite path yields the solution which is a uniform integral function of x.

On the other hand, the integral along the loop Ω is to be evaluated by expanding the loop, so that $|t| > 1$ at every point of the contour. On expanding $f(t)$ in descending powers of t, we have a uniformly convergent series of functions of the type

$$\begin{aligned}\int_\Omega e^{xt} t^{a-1}(t-1)^{c-a-1}\,dt &= x^{1-c}\int_{-\infty}^{(0+)} e^u u^{c-2}\left(1-\frac{x}{u}\right)^{c-a-1} du \\ &= x^{1-c}\sum_{n=0}^{\infty}\binom{c-a-1}{n}(-x)^n\int_{-\infty}^{(0+)} e^u u^{c-2-n}\,du \\ &= x^{1-c}\sum_{n=0}^{\infty}\binom{c-a-1}{n}(-x)^n\frac{2\pi i}{\Gamma(n+2-c)},\end{aligned} \tag{21}$$

by Hankel's formula. Thus the path Ω gives that solution which belongs to the exponent $1-c$, or, with the previous form of $f(t)$,

$$\int_\Omega e^{xt} t^{a-1}(1-t)^{c-a-1}\,dt = A\,x^{1-c}\,{}_1F_1(a-c+1;\ 2-c;\ x). \tag{22}$$

Now when we pass from the representations (10ab) to (10cd), by putting $y = x^{1-c}y'$, the rôles of the uniform solution, y_1, and the non-uniform solution, y_2, are interchanged. Thus the scheme of correspondence of paths is

$$\left.\begin{aligned} y_1 &\propto \int_\Phi e^{xt} t^{a-1}(1-t)^{c-a-1}\,dt \propto x^{1-c}\int_\Omega e^{xt} t^{a-c}(1-t)^{-a}\,dt, \\ y_2 &\propto \int_\Omega e^{xt} t^{a-1}(1-t)^{c-a-1}\,dt \propto x^{1-c}\int_\Phi e^{xt} t^{a-c}(1-t)^{-a}\,dt. \end{aligned}\right\} \tag{23}$$

II. *Sonine polynomials.* If c is not an integer, but either a or $c-a$ is an integer, one of the four series (6) is terminated and gives a Sonine polynomial. By comparing the four equivalent forms of the equation, we see that it is sufficient to discuss the case where a is a positive integer.

We find now, in (10a), that the solution L_0 vanishes identically, but may be replaced by

$$L_0^* = \int_0^I e^{xt} t^{a-1}(1-t)^{c-a-1}\,dt, \tag{24}$$

while the path Ω is deformable into the loop about $t=1$. As before, Ω gives the solution belonging to the exponent $1-c$; to replace Φ, which gives a zero solution, we may take a simple loop from $t=0$ about $t=1$. Thus we find now

$$\int_0^{(1+)} e^{xt} t^{a-1}(1-t)^{c-a-1}\,dt = {}_1F_1(a;\ c;\ x)\int_0^{(1+)} t^{a-1}(1-t)^{c-a-1}\,dt; \tag{25}$$

$$\begin{aligned}
&\int_\Omega e^{xt} t^{a-1}(1-t)^{c-a-1}\,dt \\
&\quad = \left(\frac{d}{dx}\right)^{a-1}\int_\Omega e^{xt}(1-t)^{c-a-1}\,dt \\
&\quad = B\left(\frac{d}{dx}\right)^{a-1}[e^x x^{a-c}] \\
&\quad = Be^x\left(1+\frac{d}{dx}\right)^{a-1} x^{a-c} \\
&\quad = B(a-c)(a-c-1)\ldots(2-c)\,e^x x^{1-c}\,{}_1F_1(1-a;\ 2-c;\ -x),
\end{aligned} \tag{26}$$

in accordance with the classical theory of Sonine's polynomials.

Now, in the alternative representation (10c), the integrand has a pole at $t=0$, so that the loop L_0 may be contracted to a small contour about $t=0$ and the solution evaluated as a Cauchy residue; thus we have

$$\begin{aligned}
&x^{1-c}e^x\int^{(0+)} e^{-xt}t^{-a}(1-t)^{a-c}\,dt \\
&\quad = x^{1-c}e^x\,\frac{2\pi i}{(a-1)!}\left[\left(\frac{d}{dt}\right)^{a-1} e^{-xt}(1-t)^{a-c}\right]_{t=0} \\
&\quad = (-)^{a-1}x^{1-c}e^x\,\frac{2\pi i}{(a-1)!}\,(a-c)(a-c+1)\ldots(2-c)\,{}_1F_1(1-a;\ 2-c;\ -x).
\end{aligned} \tag{27}$$

The contour Φ is found now to reduce to the loop L_0 and so gives the solution (27); the solution y_1, given by the finite loop in the representation (25), is now given by the infinite loop Ω. Thus the general character of the correspondence of finite and infinite paths is essentially the same as in the general case (23).

III. *Apparent singularities.* If c is an integer other than unity, the solution may nevertheless be free from logarithms, provided that a, $c-a$ are both integers not less than unity, or both integers not greater than zero. Suppose, to fix the ideas, that $a \geqslant 1, c-a \geqslant 1$; then, in the representation (10a), $f(t)$ is a polynomial, and both loops (18) give zero solutions. They may be replaced by

$$L_0^* = \int_0^I e^{xt} t^{a-1}(1-t)^{c-a-1}\,dt, \quad L_1^* = \int_1^I e^{xt} t^{a-1}(1-t)^{c-a-1}\,dt, \qquad (28)$$

and these can both be evaluated in finite terms. Since $c \geqslant 2$, the solution y_2 has a pole at $x = 0$; the holomorphic solution y_1 is given by the finite path from $t = 0$ to $t = 1$, which plays the rôle of Φ, and is the difference of the solutions (28).

In the alternative form (10c), both loops may be contracted to small circuits about $t = 0$ and $t = 1$ respectively, and both solutions can be evaluated as residues.

In this case, Kummer's relations (6) no longer hold as they stand; the appropriate amendment is given in Watson's treatise.

3. *The logarithmic case.*

We have now dealt with every case except the one where the solution requires a logarithm, c being an integer and a either not an integer or one such that $a(a-c) \geqslant 0$. Since $e^{2\pi i c} = 1$, the solutions (19) are linearly connected, and another pair must be chosen.

If a is not an integer, Ω is equivalent to a closed finite contour Γ enclosing $t = 0$ and $t = 1$, since the integrand now resumes its value after describing such a contour, and so the condition (17) is satisfied. This contour gives the holomorphic solution, belonging to the higher of the exponents 0, $1-c$. An independent solution is given by L_0 or L_1, and is logarithmic.

Since c can be replaced by $c' = 2-c$, we may assume that $c \geqslant 1$ without loss of generality. If $c = 1$, the representations (10ab) and (10cd) are

identical; we therefore consider $c \geqslant 2$. Now if we integrate $c-1$ times by parts, with any of the contours Γ, L_0, L_1, we find that the integrated parts vanish, so that we have

$$\int_C e^{xt} t^{a-1}(1-t)^{c-a-1}\,dt = (-x)^{1-c} \int_C e^{xt} \left[\left(\frac{d}{dt}\right)^{c-1} \{t^{a-1}(1-t)^{c-a-1}\} \right] dt. \tag{29}$$

Now, in Jacobi's identity

$$\left(\frac{d}{dt}\right)^n \{t^{n+\alpha}(1-t)^{n+\beta}\} = \frac{\Gamma(n+1+\alpha)}{\Gamma(1+\alpha)}\, t^{\alpha}(1-t)^{\beta} F(-n,\, n+1+\alpha+\beta;\, 1+\alpha;\, t), \tag{30}$$

let us write $\alpha = a-c$, $\beta = -a$, $n = c-1$; then we have

$$\left(\frac{d}{dt}\right)^{c-1} \{t^{a-1}(1-t)^{c-a-1}\} = \frac{\Gamma(a)}{\Gamma(a-c+1)}\, t^{a-c}(1-t)^{-a}, \tag{31}$$

since $n+1+\alpha+\beta = 0$, and so the hypergeometric function in (30) is unity. Thus

$$\int_C e^{xt} t^{a-1}(1-t)^{c-a-1}\,dt = \frac{\Gamma(a)(-x)^{1-c}}{\Gamma(a-c+1)} \int_C e^{xt} t^{a-c}(1-t)^{-a}\,dt. \tag{32}$$

The *same* branch is thus given by the *same* contour in the two representations (10a) and (10c); it should be observed that

$$\frac{\Gamma(a)}{\Gamma(a-c+1)} = (a-1)(a-2)\ldots(a-c+1) \neq 0, \tag{33}$$

because the values of a which make this vanish are the excluded values giving an apparent singularity.

If a is an integer, say $a \geqslant c$, the contour may be replaced by a small circuit about $t = 1$ and an open path from $t = 0$ to infinity. Since $f(t)$ has now a zero of order $a-1 \geqslant c-1$ at $t = 0$, we can still integrate by parts $c-1$ times and equate the integrated parts to zero. If $a \leqslant 0$, the rôles of the points $t = 0$ and $t = 1$ are interchanged.

4. *The analogue of Wirtinger's transformation.*

As a formal illustration, we confine ourselves to the case where x, a, $a-c+1$ are all positive; we do not then require to use double loop

integrals. We begin with the representation of ${}_1F_1(a;\ c;\ x)$ given by (10c) and (23), transform it into a double integral, and re-evaluate it in the form (10a). We have first

$$2\pi i\,{}_1F_1(a;\ c;\ x)/\Gamma(c) = x^{1-c}\int_{-\infty}^{(0+,\ 1+)} e^{xt}t^{a-c}(t-1)^{-a}\,dt$$

$$= \int_{-\infty}^{(0+,\ x+)} e^{u}\,u^{a-c}(u-x)^{-a}\,du$$

$$= \int_{-\infty}^{(0+,\ x+)} e^{u}\,u^{-c}\left(1-\frac{x}{u}\right)^{-a} du. \tag{34}$$

We may now suppose the loop opened out, so that we have $|u| > x(1+\eta)$ at every point of the contour; then the real part of $1-x/u$ is positive, and we may put

$$\left(1-\frac{x}{u}\right)^{-a} = \frac{1}{\Gamma(a)}\int_0^\infty e^{-v(1-x/u)}v^{a-1}\,dv \tag{35}$$

in (34), and obtain an absolutely convergent double integral

$$2\pi i\,\Gamma(a)\,{}_1F_1(a;\ c;\ x)/\Gamma(c)$$

$$= \int_{-\infty}^{(0+,\ x+)} du\int_0^\infty e^{u-v+(xv/u)}u^{-c}v^{a-1}\,dv$$

$$= \int_{-\infty}^{(0+,\ x+)} du\int_0^\epsilon P\,dv + \int_{-\infty}^{(0+,\ x+)} du\int_\epsilon^{2x} P\,dv + \int_{-\infty}^{(0+,\ x+)} du\int_{2x}^\infty P\,dv$$

$$= J_1+J_2+J_3 \quad \text{say,} \tag{36}$$

where P denotes the integrand in the first of these integrals.

(i) Since $a>0$, we have $J_1 = O(\epsilon^a)\to 0$, as $\epsilon\to 0$.

(ii) In the finite range $0<\epsilon<v<2x$, the condition $|u|>x$ is not required to ensure absolute convergence, and the loop may be deformed into any loop from $u=-\infty$ about $u=0$; if now we put $u=vw$, the corresponding path may be any loop in the w-plane, from $w=-\infty$ about $w=0$; to fix the ideas, we take the negative real axis from $w=-\infty$ to $w=-\frac{3}{4}$, then the circle $|w|=\frac{3}{4}$, followed by a return along the negative

axis, and we call this path C. Thus we find

$$J_2 = \int_C dw \int_\epsilon^{2x} e^{v(w-1)+x/w} w^{-c} v^{a-c} dv. \tag{37}$$

(iii) If we make the same change of variable in J_3, the condition $|u| > x$ requires $|w| > |x/v|$; but, since $v \geqslant 2x$, this is certainly satisfied if $|w| > \frac{1}{2}$, and in particular by the contour C. Adding the result to J_2, we have

$$J_2 + J_3 = \int_C dw \int_\epsilon^{\infty} e^{v(w-1)+x/w} w^{-c} v^{a-c} dv. \tag{38}$$

This converges as $v \to \infty$, because $R(w-1) \leqslant -\frac{1}{4}$; and also as $\epsilon \to 0$, because $a-c+1$ is assumed to be positive. Hence, in the limit, we have

$$2\pi i\, \Gamma(a)\, {}_1F_1(a;\ c;\ x)/\Gamma(c) = \int_C dw \int_0^{\infty} e^{-v(1-w)+x/w} w^{-c} v^{a-c} dv. \tag{39}$$

This is another form of the double integral representation (36); we can now perform the integration with respect to v and obtain the form

$$\Gamma(a-c+1) \int_C e^{x/w} w^{-c} (1-w)^{c-a-1} dw. \tag{40}$$

The contour C encloses $w = 0$ and excludes $w = 1$; on putting $w = 1/s$, we have a path from $s = 0$, enclosing $s = 1$ but lying in the finite part of the plane; hence

$$2\pi i\, \Gamma(a)\, {}_1F_1(a;\ c;\ x)/\Gamma(c)\, \Gamma(a-c+1) = -\int_0^{(1-)} e^{sx} s^{a-1} (s-1)^{c-a-1} ds. \tag{41}$$

When a is positive, we can deform the double loop Φ so that the parts encircling $t = 0$, in the representation (10a) of ${}_1F_1(a;\ c;\ x)$, are infinitely small. With appropriate initial values of the integrand, we have then, from (19),

$$\begin{aligned} \int_\Phi e^{xt} t^{a-1} (1-t)^{c-a-1} dt &= (e^{2\pi i a} - 1) \int_0^{(1+)} e^{xt} t^{a-1} (1-t)^{c-a-1} dt \\ &= (e^{2\pi i a} - 1)\, {}_1F_1(a;\ c;\ x) \int_0^{(1+)} t^{a-1} (1-t)^{c-a-1} dt. \end{aligned} \tag{42}$$

We see that, but for notation and numerical multipliers, the form (41) is the representation of the type (10a), which was required.

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CORRIGENDUM.

THE LATTICE-POINTS IN A CIRCLE.

E. C. TITCHMARSH.

[*Proc. London Math. Soc.* (2), 38 (1935), 96–115.]

Dr. Jarnik has pointed out to me that the proof of Lemma ϵ of my paper, "The lattice-points in a circle", on pp. 96–115 of this volume, is incorrect. It may be corrected by reading from the top of p. 103 to the end of §6 as follows:

Divide the range of integration into intervals throughout which

$$\text{(i)} \quad |\lambda f_x + f_y| \geqslant r/\sqrt{R},$$

or

$$\text{(ii)} \quad |\lambda f_x + f_y| < r/\sqrt{R}, \quad |\lambda f_{xx} + f_{xy}| \geqslant \tfrac{1}{2}r,$$

or

$$\text{(iii)} \quad |\lambda f_x + f_y| < r/\sqrt{R}, \quad |\lambda f_{xx} + f_{xy}| < \tfrac{1}{2}r,$$

the arguments of each function being $\lambda y+\mu$, y. The number of intervals of each type is bounded.

For an interval of type (i),

$$\int e^{2\pi i f(\lambda y+\mu,\, y)}\, dy = O\left(\frac{\sqrt{R}}{r}\right),$$

and the second mean-value theorem gives

$$\int \frac{e^{2\pi i f(\lambda y+\mu,\, y)}}{f_x(\lambda y+\mu,\, y)}\, dy = O\left(\frac{1}{r}\right).$$

For an interval of type (ii),

$$\left|\int \frac{e^{2\pi i f(\lambda y+\mu,\, y)}}{f_x(\lambda y+\mu,\, y)}\, dy\right| \leqslant \frac{2}{r}\left|\int \frac{\lambda f_{xx}+f_{xy}}{f_x}\, dy\right| = \frac{2}{r}\Big|[\log f_x]\Big| = O\left(\frac{|\log R|+|\log e|}{r}\right).$$

For an interval of type (iii), put $w = \lambda f_x + f_y$. Then

$$\left|\frac{dw}{dy}\right| = \left|\lambda^2 f_{xx} + 2\lambda f_{xy} + f_{yy}\right| = \left|\frac{(\lambda f_{xx}+f_{xy})^2 + f_{xx}f_{yy} - f_{xy}^2}{f_{xx}}\right| \geqslant \frac{r^2 - \frac{1}{4}r^2}{R} = \frac{3r^2}{4R}.$$

Hence

$$\left|\int \frac{e^{2\pi i f(\lambda y+\mu,\, y)}}{f_x(\lambda y+\mu,\, y)}\, dy\right| \leqslant \frac{1}{\sqrt{R}}\int_0^{r/\sqrt{R}} \left|\frac{dy}{dw}\right| dw \leqslant \frac{4}{3r}.$$

This proves the lemma.

A similar correction is needed in Lemma ϵ of my previous paper, "On Epstein's zeta-function", *Proc. London Math. Soc.* (2), 36 (1934), 485–500.

I must also refer to a paper by I. Vinogradov, *Bull. Acad. Sci. URSS*, 7 (1932), 313–336 [see *Zentrallblatt für Math.*, 5 (1933), 247], in which the error-term $O(x^{\frac{13}{53}+\epsilon})$ in the circle problem is said to be obtained. There is a good deal in common between his method and mine; while mine is an extension to two variables of van der Corput's method, Vinogradov's appears to be a similar extension of the parallel Hardy-Littlewood-Walfisz method. But I cannot follow the details of Vinogradov's analysis.

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